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SUPERSYMMETRIC ELECTRIC AND MAGNETIC MONOPOLES

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The spherically symmetric Yang-Mills theory with GSU(2) gauge supergroup is analysed. The solution is found consisting of electric and magnetic monopoles coupled with some scalar anticommuting field. Long range terms $\left(\sim \frac{1}{r} \text{ if } r \to \infty\right)$ appear in the electric and magnetic fields.

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Introduction

In the present paper we analyse the generalization of the classical Yang-Mills theory to the case when the gauge group is the GSU(2) supergroup. This supergroup described by three even and two odd (anticommuting) parameters is constructed in Sect. 1. In order to do this we need the complex Grassmann algebra with some special involution [1]. In Sect. 2 we impose the spherical symmetry ansatz on our gauge field (we follow Manton [2]), solve the appropriate constraints and obtain two independent solutions: one with vanishing anticommuting part (this case was analysed in pure SU(2) theory by Witten [3]) and the second one with U(1) commuting part only. We write the equations of motion for the second sector and limit ourselves to the static case. In Sect. 3 we try to solve these equations. We find the simplest solution which describes the electric and magnetic monopoles interacting via some scalar anticommuting field. It is interesting that the asymptotic behaviour of the electric and magnetic fields is different from the classical case for $r \to \infty$ (long range terms appear) and remains classical for $r \to 0$. This is our main result. In the appendix we give some definitions and conventions concerning supermatrices.

1. The gauge supergroup

Let Λ be an even (possibly infinite) dimensional complex Grassmann algebra with involution "—" satisfying:

$$\overline{a\eta} = a^*\bar{\eta},$$

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$$\bar{\bar{\eta}} = (-)^{p(\eta)}\eta,$$

$$\overline{\eta_1\eta_2} = \bar{\eta}_1\bar{\eta}_2, \tag{1}$$

(a is complex number, $p(\eta)$ denotes the parity of $\eta \in \Lambda$). Such Grassmann algebras exist [1]. We need one of them to build the GSU(2) supergroup. This supergroup consists of $(1\times 2)\times(1\times 2)$ dimensional, even supermatrices M with properties:

$$M^{sT}JM = J, \quad M^{s\dagger}M = 1, \tag{2}$$

where

$$J = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right], \tag{3}$$

(definitions and conventions concerning supermatrices are given in the Appendix). It is easy to prove that GSU(2) is a group in the usual sense, i.e.:

— for $A, B \in$ GSU(2): $A \cdot B \in$ GSU(2),

— $1 \in$ GSU(2) and $1 \cdot A = A \cdot 1 = A$,

— A^{-1} exists and $A^{-1}A = AA^{-1} = 1$.

However, we stress that the elements of GSU(2) matrices are not numbers but belong to the abovementioned Grassmann algebra Λ. The maximal Lie subgroup of GSU(2) is isomorphic to SU(2) which — of course — is simple and compact. The possibility of such construction arises owing to the non-standard properties of the involution (1).

The infinitesimal form of (2) can be written as:

$$m^{sT}J + Jm = 0, \quad m^{s\dagger} + m = 0. \tag{4}$$

The $(1\times 2)\times(1\times 2)$ dimensional supermatrices m satisfying (4) constitute the so-called real Grassmann shell of the osp(1|2) superalgebra. This shell (we will denote it gsu(2)) may be viewed as the generalization of the su(2) Lie algebra because gsu(2) involves su(2) as a substructure and in addition we have:

— for $m, m' \in$ gsu(2): $[m, m'] \in$ gsu(2),

— for $m_1, m_2, m_3 \in$ gsu(2): $[[m_1, m_2], m_3] + [[m_2, m_3], m_1] + [[m_3, m_1], m_2] = 0$.

Let us stress that $[\cdot, \cdot]$ denotes the ordinary commutator. Apart from this the elements of gsu(2) may be multiplied by even and real elements of Λ. GSU(2) and gsu(2) will play in our considerations the role of the gauge group and its Lie algebra, respectively. As can easily be shown from (4) gsu(2) may be parametrized in the following way:

$$m = \left[\begin{array}{c|cc} 0 & -\frac{i}{\sqrt{2}}\bar{\psi} & \frac{i}{\sqrt{2}}\psi \\ \hline \frac{i}{\sqrt{2}}\psi & \multicolumn{2}{c}{\multirow{2}{*}{$\frac{i}{2}\sum_{k=1}^{3} A^k\sigma^k$}} \\ \frac{i}{\sqrt{2}}\bar{\psi} & & \end{array}\right], \tag{5}$$

where A^k ($k = 1, 2, 3$) denote even and real elements of Λ and ψ is an arbitrary odd element of Λ (σ^k are he Pauli matrices). To each $m \in \mathrm{gsu}(2)$ there corresponds an $M \in \mathrm{GSU}(2)$:

$$M = \exp(m) \equiv \sum_{n=0}^{\infty} \frac{m^n}{n!}. \tag{6}$$

M may also be parametrized in other ways, for instance:

$$M = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & & \\ & \overline{M} & \\ 0 & & \end{array}\right] \exp \left[\begin{array}{c|cc} 0 & -\frac{i}{\sqrt{2}}\bar{\psi} & \frac{i}{\sqrt{2}}\psi \\ \hline \frac{i}{\sqrt{2}}\psi & & \\ & & 0 \\ \frac{i}{\sqrt{2}}\bar{\psi} & & \end{array}\right], \tag{7}$$

where $\overline{M}$ is a SU(2) matrix (with even Grassmann elements). The form (7) is very suitable because the sum in the exp has only three components.

2. *The gauge field and the equations of motion*

We define the gauge field as a gsu(2)-valued vector field in Minkowski space. Hence, for each $\mu = 0, 1, 2, 3$: A_μ has the form (5). The equations of motion of A_μ follow from the action (written in arbitrary curvilinear coordinates):

$$S = -\frac{1}{16\pi} \int d^4x \sqrt{-g}\, \mathrm{str}\,(F_{\mu\nu}F^{\mu\nu}), \tag{8}$$

where str denotes the supertrace (see appendix) and $F_{\mu\nu}$ is the gsu(2) Yang-Mills tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$

($[\cdot, \cdot]$ is the commutator — see Sect. 1). Thus these equations have the form:

$$\frac{1}{\sqrt{-g}} \{\partial_\mu(\sqrt{-g}\, F^{\mu\nu}) + \sqrt{-g}\, [A_\mu, F^{\mu\nu}]\} = 0, \tag{9}$$

in strict analogy with the classical Yang-Mills theory. We will try to solve (9) assuming A_μ to be spherically symmetric and static. The spherically symmetric gauge field has been considered in the literature many times — the general form of it given by Manton [2] may be applied here. In spherical coordinates this ansatz reads:

$$A_t = f(t, r)\Phi_3,$$

$$A_r = g(t, r)\Phi_3,$$

$$A_\theta = -\Phi_1(t, r),$$
$$A_\varphi = \Phi_2(t, r)\sin\theta + \Phi_3 \cos\theta,$$
$$\Phi_3 = C\left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \frac{i}{2}\sigma^3 & \\ 0 & & \end{array}\right], \tag{10}$$

where f, g are even and real (t, r)-dependent elements of Λ; Φ_1, Φ_2 have the form (5), C is a constant and the following constraints should be fulfilled:

$$[\Phi_3, \Phi_1] = -\Phi_2,$$
$$[\Phi_3, \Phi_2] = \Phi_1. \tag{11}$$

Let us note that according to [2] Φ_3 should be the generator of the Cartan subalgebra of the gauge algebra. In our case this subalgebra is one dimensional and we can choose any (constant) element of gsu(2) as Φ_3 getting of course different Φ_1, Φ_2. However the equations (11) are gauge covariant and — as can easily be checked — any Φ_3 with nontrivial su(2) part can be reduced by some gauge transformation to the form given in (10). If the su(2) part of Φ_3 is trivial then the equations (11) have no solutions except $\Phi_1 = \Phi_2 = 0$. (We say that the gsu(2) supermatrix m of the form (5) has nontrivial su(2) part if $(A^k)^{-1}$ exists for almost one value of k).

The solution of (11) consists of two sectors. In the first ($c^2 = 1$) the anticommuting part of the gauge field is absent. In fact this case is gauge equivalent to the spherically symmetric SU(2)-field analysed by Witten in euclidean space [3]. The second sector ($c^2 = 4$) has the form:

$$\Phi_1 = \left[\begin{array}{c|cc} 0 & -\frac{i}{\sqrt{2}}\bar{\psi} & \frac{i}{\sqrt{2}}\psi \\ \hline \frac{i}{\sqrt{2}}\psi & & \\ \frac{i}{\sqrt{2}}\bar{\psi} & \multicolumn{2}{c}{0} \end{array}\right],$$

$$\Phi_2 = \left[\begin{array}{c|cc} 0 & \frac{1}{\sqrt{2}}\bar{\psi} & \frac{1}{\sqrt{2}}\psi \\ \hline \frac{1}{\sqrt{2}}\psi & & \\ -\frac{1}{\sqrt{2}}\bar{\psi} & \multicolumn{2}{c}{0} \end{array}\right], \tag{12}$$

(we have assumed $C = 2$); ψ is an arbitrary odd (t, r)-dependent element of Λ. The radial component A_r in (10) can be gauged away by the transformation generated by:

$$M = \exp\{\textstyle\int dr g(t, r)\Phi_3\}, \tag{13}$$

(then the forms of A_t, Φ_1 and Φ_2 remain unchanged). The equations (9) reduce in the static case to:

$$r^2f''+2rf'+2f\psi\bar{\psi} = 0,$$

$$\psi''+\left(f^2+\frac{1}{r^2}\right)\psi = 0,$$

$$\bar{\psi}''+\left(f^2+\frac{1}{r^2}\right)\bar{\psi} = 0,$$

$$\psi\bar{\psi}'-\psi'\bar{\psi} = 0 \tag{14}$$

(the prime denotes the differentiation with respect to r). Let us note that the third equation of (14) is the conjugate of the second in the sense of the involution (1).

3. *The simplest solution*

The equations (14) can be solved by perturbation procedure in the number of independent anticommuting parameters (we call two anticommuting "numbers" independent if their product is different from zero). The simplest solution of (14) with only one anticommuting parameter η (and its conjugate $\bar{\eta}$) has the form:

$$f = \frac{e_1+e_2\eta\bar{\eta}}{r}-4e_1\eta\bar{\eta}\ln r,$$

$$\psi = e^{i\beta}(\eta r^{\alpha}+\bar{\eta}r^{\alpha*}),$$

$$\alpha = \tfrac{1}{2}+\frac{i}{2}\sqrt{4e_1^2+3} \tag{15}$$

(e_1, e_2 and β are arbitrary real numbers). To analyse the above solutions let us write the electric and magnetic components of the whole Yang-Mills tensor $F_{\mu\nu}$. From (15) we obtain:

$$E_r = \left(\frac{e_1+e_2\eta\bar{\eta}}{r^2}+\frac{4e_1\eta\bar{\eta}}{r}\right)\left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\sigma^3} \\ 0 & & \end{array}\right],$$

$$E_\theta = \frac{e_1}{\sqrt{2}\,r}\left[\begin{array}{c|cc} 0 & \bar{\psi} & \psi \\ \hline \psi & \multicolumn{2}{c}{0} \\ -\bar{\psi} & & \end{array}\right],$$

$$E_\varphi = \frac{ie_1}{\sqrt{2}\,r}\sin\theta\left[\begin{array}{c|cc} 0 & -\bar{\psi} & \psi \\ \hline \psi & \multicolumn{2}{c}{0} \\ \bar{\psi} & & \end{array}\right], \tag{16}$$

$$H_r = -\left(\frac{1}{r^2} + \frac{2\eta\bar{\eta}}{r}\right)\left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\sigma^3} \\ 0 & & \end{array}\right],$$

$$H_\theta = -\frac{1}{\sqrt{2}}\left[\begin{array}{c|cc} 0 & \bar{\psi}' & \psi' \\ \hline \psi' & \multicolumn{2}{c}{0} \\ -\bar{\psi}' & & \end{array}\right],$$

$$H_\varphi = \frac{-i\sin\psi}{\sqrt{2}}\left[\begin{array}{c|cc} 0 & -\bar{\psi}' & \psi' \\ \hline \psi' & \multicolumn{2}{c}{0} \\ \bar{\psi}' & & \end{array}\right]. \tag{17}$$

In these formulas ψ is that from (15) and prime denotes as earlier the differentiation with respect to r. We see that the commuting sector of our spherically symmetric field is abelian and corresponds to the pure electromagnetic field embedded in SU(2) Yang-Mills theory. So our theory may be viewed as the supersymmetric extension of the spherically symmetric electromagnetic field (the electric and magnetic fields are coupled via some scalar anticommuting field ψ). Let us note that the commuting sector is radial and describes the electric and magnetic monopoles with long range supersymmetric terms $\left(\text{the } \eta\bar{\eta} \text{ terms behave as } \frac{1}{r} \text{ for } r \to \infty\right)$. The anticommuting sector of our solutions (16) and (17) consists of fields with vanishing radial components. The asymptotic behaviour of these fields is also interesting. They behave as $\frac{1}{r^{3/2}}$ for $r \to \infty$ as can easily be checked from (16), (17) and (15). These features do not seem to be specific for the simplest solution (15) only. We have found some other (more complicated) solutions of (14) with long range terms also present. However by now we cannot confirm strictly our results because of the technical difficulties. There also remains the problem of physical interpretation of the anticommuting parameters present in our solutions. We do not touch this question here. Summarizing, we feel there is some hope that the supersymmetric extension of electromagnetism may lead toward confinement.

APPENDIX

A supermatrix is an ordinary matrix whose elements belong to some Grassmann algebra Λ. If the supermatrix M can be written in the block form:

$$M = \left[\begin{array}{c|c} M_1 & M_2 \\ \hline M_3 & M_4 \end{array}\right], \tag{A1}$$

the elements of the $p \times p$ dimensional submatrix M_1 and $q \times q$ dimensional submatrix M_4 are even and the elements of the $p \times q$ dimensional submatrix M_2 and $q \times p$ dimensional submatrix M_3 are odd then we call the supermatrix M even. We say also that it is $(p \times q) \times (p \times q)$ dimensional.

The supertranspose of M: M^{sT} is defined in the following way:

$$M^{sT} = \left[\begin{array}{c|c} M_1^T & -M_3^T \\ \hline M_2^T & M_4^T \end{array}\right], \tag{A2}$$

and the superhermitian conjugate of M is equal to

$$M^{s\dagger} = \left[\begin{array}{c|c} M_1^\dagger & -M_3^\dagger \\ \hline M_2^\dagger & M_4^\dagger \end{array}\right]. \tag{A3}$$

In the last definition the involution (1) is used to define the conjugate Grassmann "numbers".

The supertrace of M can be determined in two possible ways, either:

$$\operatorname{str} M = \operatorname{tr} M_1 - \operatorname{tr} M_4, \tag{A4}$$

or:

$$\operatorname{str} M = -\operatorname{tr} M_1 + \operatorname{tr} M_4. \tag{A5}$$

We use the second possibility in our paper.

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