HOMOGENEOUS AND ISOTROPIC COSMOLOGY IN GAUGE GRAVITATIONAL THEORY WITH GRAVITATIONAL LAGRANGIAN

$$L_g = \frac{c^4}{16\pi G}(\Omega^i_{.k}\wedge\eta_i{}^k+\Theta^i\wedge *\Theta_i)+\frac{\hbar c}{16\pi}\Omega^i_{.k}\wedge *\Omega^k_{.i}\ *$$

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The paper is devoted to the singularity problem in the spatially homogeneous and isotropic cosmology in the framework of the gauge gravitational theory with quadratic gravitational Lagrangian

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1. Introduction

The homogeneous and isotropic cosmology in the framework of gravitational theories with quadratic Lagrangians was recently examined by many authors [1, 2, 3], see also References given in [6]. In this paper we wish to study this cosmology in the framework of gauge gravitational theory with the following, quadratic gravitational Lagrangian

$$L_g = \alpha(\Omega^i_{.k}\wedge\eta_i{}^k+\Theta^i\wedge *\Theta_i)+\beta\Omega^i_{.k}\wedge *\Omega^k_{.i},$$

$$\alpha = \frac{c^4}{16\pi G},\quad \beta = \alpha\mathscr{A} = \frac{\alpha\hbar G}{c^3} = \frac{\hbar c}{16\pi},$$

$$i, j, b, k, l, m, n, p, r, s, t = 0, 1, 2, 3. \tag{1}$$

In the Lagrangian (1) $\hbar$ is the Planck constant, c is the value of the velocity of light in vacuum and G denotes the Newtonian gravitational constant. $\Omega^i_{.k}$ is the curvature two-form, Θ^i is the torsion two-form and η_{ik} means the pseudotensorial two-form introduced by Trautman [4]; $*$ denotes the Hodge-star-operator.

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The theory with gravitational Lagrangian (1) was presented in [5] as the most satisfactory model of a gauge gravitational theory because, among other things, the Lagrangian (1) has the best physical and geometrical motivation.

In [6] we presented simple cosmological models with torsion and with the O(3) isotropy group existing in the framework of the theory.

The present paper is mainly devoted to examination of the existence of geometrical singularities in the cosmological solutions of the theory having the O(3) isotropy group and the SO(3) isotropy group.

The notation used in this paper is the same as that used in [6].

At the end of the Introduction we give a new form of the Criterion "*C*" originally formulated in [6]. The Criterion "*C*" gives the necessary conditions under which solutions with torsion having a symmetric energy-momentum tensor may exist in vacuum and inside of matter. (Inside of matter solutions having asymmetric energy-momentum tensor may exist only for solutions with torsion.)

The Criterion "*C*"

Solutions to the field equations having dynamical torsion may exist when:

(*i*) The torsion constraints (see [5, 6])

$$\nabla_k Q_{[bp]}{}^k + Q_{[bp]}{}^k Q_k = \tfrac{1}{2} \nabla_k Q^k{}_{.bp} + \nabla_{[b} Q_{p]} + \tfrac{1}{2} Q_n Q^n{}_{.bp}, \tag{2}$$

where

$$Q_n := Q^k{}_{.nk}$$

are a consequence of the field equations of the theory and of some additional compatibility conditions having, in general, a form of differential equations simpler than the constraints and the resulting system of equations consisting of the field equations and compatibility conditions is not overdetermined,

or

(*ii*) The constraints (2) are identically satisfied (reduce to the form $0 = 0$) and the system of the field equations (with not entirely vanishing torsion) to be solved is not overdetermined,

or

(*iii*) The constraints (2) immediately follow from the field equations of the theory (the trivial consistency of the constraints with the field equations) and the system of the field equations is not overdetermined.

If the Criterion "*C*" is not fulfilled, then there exist only torsionless solutions.

2. *Spatially homogeneous and isotropic cosmology with the O(3) isotropy group*

The field equations in this case have the following form (see [6])

$$(-)\frac{9h\dot{a}}{ac} - \tfrac{15}{2}h^2 - \frac{3\dot{a}^2}{a^2c^2} - \frac{3k}{a^2} + \frac{3\beta}{\alpha}\left\{\frac{\left[\left(\frac{\dot{a}}{c}+ah\right)^{\cdot}\right]^2}{a^2c^2}\right.$$

$$-\left[\left(\frac{\dot a}{ac}+h\right)^2+\frac{k}{a^2}\right]^2\Bigg\} = (-)\frac{\varepsilon}{2\alpha},$$

$$(-)\frac{3\dot h}{c}-\frac{6h\dot a}{ac}-\tfrac{9}{2}h^2-\frac{\dot a^2}{a^2c^2}-\frac{2\ddot a}{a^2c^2}-\frac{k}{a^2}$$

$$-\frac{\beta}{\alpha}\left\{\frac{\left[\left(\frac{\dot a}{c}+ah\right)^{\cdot}\right]^2}{a^2c^2}-\left[\left(\frac{\dot a}{ac}+h\right)^2+\frac{k}{a^2}\right]^2\right\} = \frac{p}{2\alpha},$$

$$\frac{1}{c^2}\left[\frac{\left(\frac{\dot a}{c}+ah\right)^{\cdot}}{a}\right]^{\cdot}-2\left(\frac{\dot a}{ac}+h\right)^3-\frac{2k}{a^2}\left(\frac{\dot a}{ac}+h\right)+\frac{2\dot a}{a^2c^2}\left(\frac{\dot a}{c}+ah\right)^{\cdot}$$

$$+\frac{4h}{ac}\left(\frac{\dot a}{c}+ah\right)^{\cdot}+\frac{\alpha}{2\beta}h = 0, \tag{3}$$

where $k = 0, \pm 1$.
We denote by dot differentiation with respect to the cosmic time t.

In each case, $k = 0, \pm 1$, we have a system of three ordinary, nonlinear differential equations for three unknown functions: the scale factor $a(t)$, the torsion component $h(t)$ and the energy density $\varepsilon(t)$ or pressure $p(t)$. The classical spin is completely eliminated from these equations. The Criterion "C" is satisfied in this case and, therefore, the systems (3) may have solutions with torsion (and with a symmetric energy-momentum tensor (see [6]).

Discussion of solutions of such a kind will be given later. Now we consider the differential conservation laws having the form

$$\nabla_l S^l_{.ij}+S^l_{.ij}Q_l = t_{ij}-t_{ji},$$

$$\nabla_j t^j_{.i}+t^j_{.i}Q_j = Q^j_{.il}t^l_{.j}+\tfrac{1}{2}R^{jk}_{..il}S^l_{.jk}. \tag{4}$$

In the framework of isotropic and homogeneous cosmology with the O(3) (or SO(3)) isotropy group these conservation laws reduce, in the orthonormal tetrad determined by the Robertson-Walker line element (see [6]), to a single equation

$$\dot\varepsilon+3\frac{\dot a}{a}(\varepsilon+p) = (-)6hca^3(\varepsilon+p) \tag{5}$$

or, equivalently

$$(\varepsilon a^3)^{\cdot} = (-)6hca^3(\varepsilon+p)-p(a^3)^{\cdot}. \tag{6}$$

We see from (6) that the mass contained inside of a "sphere" of radius a is not conserved unless $h = p = 0$.

To solve the system (3) we need an appropriate Ansatz. A suitable Ansatz is determined by the form of the system (3) and, first of all, by demanded macroscopic behaviour of the model.

An Ansatz suitable for equations (3) has the following form

$$\frac{\dot{a}}{ac}+h=\left(\frac{\dot{a}}{ac}\right) \quad \text{calculated from a solution to the macroscopic cosmological equations (Friedmann equations)} \tag{7}$$

Calculating $\frac{\dot{a}}{ac}$ on the right hand side of equation (7) from different solutions to the Friedmann equations we get different cosmological solutions to the equations (3) with the a priori given macroscopic behaviour.

The procedure of solving the system (3) using the Ansatz of the form (7) is the following:
(*a*) We substitute the Ansatz into the system (3) and calculate from the last equation of this system $h = h(a, \dot{a}, \ddot{a}, \dddot{a})$.
(*b*) We substitute $h = h(a, \dot{a}, \ddot{a}, \dddot{a})$ into the Ansatz (7) and get an ordinary differential equation for the scale factor $a = a(t)$. Solving this equation we get the function $a = a(t)$.
(*c*) Having $a = a(t)$ we calculate $h(t) = h[a(t), \dot{a}(t), \ddot{a}(t), \dddot{a}(t)]$.
(*d*) Having $a(t)$ and $h(t)$ we calculate $\varepsilon(t)$ from the first equation of the system (3) and $p(t)$ from the second equation of the system. Thus, the whole procedure is practically reduced to solving of one ordinary, nonlinear differential equation for the scale factor $a = a(t)$.

Interesting, continually expanding solutions to the system (3) obtained with the help of the Ansatz

$$\frac{\dot{a}}{ac}+h=\frac{1}{a}\equiv\frac{\dot{a}}{c}+ah=1 \tag{8}$$

were presented in [6].

These solutions have the following macroscopic behaviour

$$a = ct+\text{const}, \quad h=0, \quad p=(-)\frac{\varepsilon}{3}, \quad \text{const}\geqslant 0, \tag{9}$$

and they are without any singularities (in metric and in torsion). At the cost of work performed by negative pressure and torsion during expansion the continual "creation" of mass takes place in these models. However, these models are evolutionary models, not steady-state models, because the expansion dominates over "creation" [6].

An Ansatz suitable for the Friedmann-like solutions in the macroscopic domain has the following form

$$\frac{\dot{a}}{ac}+h=\left(\frac{\dot{a}}{ac}\right) \quad \text{calculated from the Friedmann solutions} \tag{10}$$

In the case $k = p = 0$ we have the Friedmann solution of the form (see, e.g., [7])

$$a = At^{2/3}, \quad \text{vanishing torsion,} \tag{11}$$

where $A = \text{const} > 0$.

Therefore

$$\left(\frac{\dot a}{ac}\right) \quad \text{calculated from the Friedmann solution (11)} \quad = \frac{2}{3ct} \tag{12}$$

and the Ansatz (10) takes the following form[1]

$$\frac{\dot a}{ac} + h = \frac{2}{3ct} \equiv \frac{\dot a}{c} + ah = \frac{2a}{3ct}. \tag{13}$$

The equations (3) give for this Ansatz

$$h = \frac{\mathscr{A}}{tc^3} \frac{\left(\frac{2\dot a t}{a} - \frac{2\ddot a t^2}{3a} - \frac{20}{27} - \frac{2\dot a^2 t^2}{3a^2}\right)}{\left(\frac{t^2}{2} + \frac{8\mathscr{A}\dot a t}{3ac^2} - \frac{8\mathscr{A}}{3c^2}\right)}. \tag{14}$$

Substituting (14) into the Ansatz equation (13) we get the following differential equation for the scale factor

$$a\ddot a - 3\dot a^2 - \tfrac{3}{2}\left(\frac{c^2 t}{2\mathscr{A}} - \frac{22}{9t}\right) a\dot a + \tfrac{3}{2}\left(\frac{c^2}{3\mathscr{A}} - \frac{28}{27t^2}\right) a^2 = 0. \tag{15}$$

Putting $u(t) := \frac{\dot a}{a}$ we transform this nonlinear equation of second order onto the Riccati equation

$$\dot u - 2u^2 - \tfrac{3}{2}\left(\frac{c^2 t}{2\mathscr{A}} - \frac{22}{9t}\right) u + \tfrac{3}{2}\left(\frac{c^2}{3\mathscr{A}} - \frac{28}{27t^2}\right) = 0. \tag{16}$$

We cannot solve this equation by direct integration [8] but we can associate with it the following second order differential equation (see [8])

$$\ddot y - \tfrac{3}{2}\left(\frac{c^2 t}{2\mathscr{A}} - \frac{22}{9t}\right)\dot y - \left(\frac{c^2}{\mathscr{A}} - \frac{28}{9t^2}\right) y = 0. \tag{17}$$

This equation belongs to the so-called "Fuchs class". Each non-zero solution $y \neq 0$ determines a solution

$$u(t) = (-)\frac{\dot y}{2y} \tag{18}$$

of the Riccati equation (16) (see [8]). We can solve the equation (17) in the form of an infinite series and get

$$y(t) = \sum_{\lambda=0}^{\infty} a_\lambda t^{\lambda+s}, \tag{19}$$

[1] The Ansatz (13) leads to a solution to the equations (3) with $k = 0$ having $\varepsilon, p \neq 0$.

where

$$s = (-)\tfrac{4}{3} \mp \frac{2\sqrt{3}\, i}{3},$$

$$a_{\lambda+2} = \frac{[\tfrac{3}{4}(\lambda+s)+1]c^2}{[(\lambda+s+2)(\lambda+s+\tfrac{14}{3})+\tfrac{28}{9}]\mathscr{A}}\, a_\lambda,$$

$$a_0 \neq 0, \quad a_1 = 0, \quad \lambda = 0, 1, 2, 3, \ldots \tag{20}$$

Thus, solutions to the Riccati equation (16) given by (18) will be quotients of two suitable series and $a = a(t)$ will be given by

$$a(t) = (-)\tfrac{1}{2}\,\mathrm{const}\cdot \mathrm{Re}\,[y(t)], \quad (a(t) \geqslant 0), \tag{21}$$

where Re $[y(t)]$ means the real part of the solution (19)–(20) (this real part satisfies also the equation (17)).

The real part Re $[y(t)]$ has the form

$$\mathrm{Re}\,[y(t)] = \sum_{\lambda=0}^{\infty} t^{-4/3+\lambda}\left[\mathrm{Re}\,(a_\lambda)\cos\left(\frac{2\sqrt{3}}{3}\ln t\right) \pm \mathrm{Im}\,(a_\lambda)\sin\left(\frac{2\sqrt{3}}{3}\ln t\right)\right] \tag{22}$$

and never vanishes.

Summing up we can say that solutions of equation (15) obtained in the above described manner will be rather complicated in the microscopic domain of $a(t)$ but they are without any metric singularity.

However, these solutions (and all other solutions) must develop singularities in torsion when t goes to zero. It is easily seen from the formula (14).

In the macroscopic domain ($\hbar = 0$, $a(t)$ sufficiently large) the behaviour of the solutions of the equation (15) will be Friedmann-like and given with very good accuracy by (11).

In the cases $k = \pm 1$ we have serious complications caused by the fact that the solutions of the Friedmann equations with $p = 0$ (or $p \neq 0$) are given in parametric form (see, e.g., [7]).

However, for sufficiently small values of the parameter (this corresponds to sufficiently small values of the scale factor $a(t)$) we can give the approximate form of the function $a(t)$ (see, e.g., [7]). It is also of the form (11).

Keeping in mind this fact we can try to solve the system (3) for sufficiently small values of $a(t)$ using the Ansatz (13). This enables us to investigate the cosmological singularity problem at $t = 0$ in macroscopically Friedmannian cosmological models with $k = \pm 1$.

Proceeding in this way we get the following ordinary differential equation on $a(t)$, valid for sufficiently small values of the scale factor a:

$$a\ddot{a} - 3\dot{a}^2 - \tfrac{3}{2}\left(\frac{c^2 t}{2\mathscr{A}} - \frac{22}{9t}\right)a\dot{a} + \tfrac{3}{2}\left(\frac{c^2}{3\mathscr{A}} - \frac{28}{27t^2}\right)a^2 = 2kc^2, \tag{23}$$

where $k = \pm 1$.

The equation (23) differs from the equation (15) only by the constant term $2kc^2$. Therefore, it admits only such metric singularities at the point $t = 0$ as the singularities admitted by the equation (15).

The solutions to the equation (15) may be without any metric singularities. Thus, metric singularities in the solutions to the equation (23) are also not necessary, i.e., there may exist solutions without singularities at the point $t = 0$.

On the other hand the solutions to the equations (23) must develop singularity in torsion because we have for them

$$h = \frac{\mathscr{A}}{t}\,\frac{\left(-\dfrac{20}{27c^3} - \dfrac{2\dot{a}^2t^2}{3a^2c^3} - \dfrac{2\ddot{a}t^2}{3ac^3} + \dfrac{2\dot{a}t}{ac^3} + \dfrac{4t^2}{3a^2c}k\right)}{\left(\dfrac{t^2}{2} + \dfrac{8\mathscr{A}\dot{a}t}{3ac^2} - \dfrac{8\mathscr{A}}{3c^2}\right)}, \tag{24}$$

where $k = \pm 1$.

The analysis of the expression (24) shows that in both cases, $k = \pm 1$, we have $\lim_{t\to 0} h(t) = \infty$.

Apart from the Ansatz method developed up to now and applied to the field equations (3), we may use from the beginning the spherical symmetry Ansatz [5, 9]

$$* \Omega^{ji} = (-)\tfrac{1}{2}\,\eta^{ji}{}_{lm}\Omega^{lm} + \frac{\mathscr{B}}{2\mathscr{A}}\,\eta^{ji} + \frac{\mathscr{C}}{2\mathscr{A}}\,V^j \wedge V^i \tag{25}$$

with $\mathscr{B} = \text{const}$, $\mathscr{C} = \text{const}$, $\mathscr{A} = \dfrac{\hbar G}{c^3}$. V^i denotes the coreper field.

This Ansatz is compatible with the O(3) isotropy and with the third order part of the field equations (see [5]) if and only if $\mathscr{C} = 0$ and $\mathscr{B} = \frac{1}{2}$.

It leads to the following system of equations

$$\frac{2\ddot{a}}{ac^2} - \frac{2h\dot{a}}{ac} + \frac{2\dot{h}}{c} + 6h^2 + \frac{3}{4\mathscr{A}} = (-)\frac{2\varepsilon}{3\alpha},$$

$$\frac{2\ddot{a}}{ac^2} + \frac{10h\dot{a}}{ac} + \frac{6\dot{h}}{c} + 14h^2 - \frac{5}{4\mathscr{A}} = (-)\frac{2p}{\alpha},$$

$$\frac{\ddot{a}}{ac^2} + \frac{\dot{a}^2}{a^2c^2} + \frac{3h\dot{a}}{ac} + \frac{\dot{h}}{c} + h^2 + \frac{1}{4\mathscr{A}} + \frac{k}{a^2} = 0, \tag{26}$$

where $k = 0, \pm 1$.

The Criterion "C" is satisfied here. Therefore, there may exist (and really do exist) solutions with torsion.

We can solve the system (26) in the following way: we take arbitrarity the function $a = a(t)$ and then find $h = h(t)$ from the third equation of the system. Afterwards, having $a(t)$ and $h(t)$ we calculate $\varepsilon = \varepsilon(t)$ and $p = p(t)$ and the state equation $p = p(\varepsilon)$ from the remaining equations of the system.

Proceeding in this way we have examined the simplest case $a = e^{Ht}$, $H = \text{const} > 0$ without metric singularity for finite values of the cosmic time t. The results are as follows.

1. $k = 0$

The equation for $h(t)$ has the form

$$\dot{h} = (-)ch^2 - 3Hh - \frac{c}{4\mathscr{A}} - \frac{2H^2}{c}. \tag{27}$$

This equation has the solution[2]

$$h = \frac{1}{2c}\sqrt{\frac{c^2}{\mathscr{A}} - H^2}\,\mathrm{tg}\left(\text{const} - \tfrac{1}{2}\sqrt{\frac{c^2}{\mathscr{A}} - H^2}\,t\right) - \frac{3H}{2c} \tag{28}$$

with

$$(-)\frac{\pi}{2} < \text{const} - \tfrac{1}{2}\sqrt{\frac{c^2}{\mathscr{A}} - H^2}\,t < \frac{\pi}{2}. \tag{29}$$

The solution exists only in the bounded interval of the cosmic time t, determined by (29), and it has singularities at both ends of the interval: $h \to \pm\infty$. The solution (28) does not correspond to any cosmological solution of the macroscopic gravitational theory (see [6]).

2. $k = \pm 1$

In this case we have the following Riccati's equation for $h(t)$

$$\dot{h} = (-)ch^2 - 3Hh - \frac{c}{4\mathscr{A}} - \frac{2H^2}{c} - kce^{-2Ht}, \tag{30}$$

where $k = \pm 1$.

We can associate with equation (30) the following second order linear differential equation

$$\ddot{u} + 3H\dot{u} + c\left(kce^{-2Ht} + \frac{2H^2}{c} + \frac{c}{4\mathscr{A}}\right)u = 0. \tag{31}$$

Every nonzero solution $u(t) \neq 0$ of the linear equation (31) determines a solution $h(t) = \dfrac{\dot{u}}{uc}$ to the Riccati equation (30).

For our purposes it is sufficient to study the solutions of the asymptotic equation obtained from (31) for large, positive values of t.

The asymptotic equation has the form

$$\ddot{u} + 3H\dot{u} + \left(2H^2 + \frac{c^2}{4\mathscr{A}}\right)u = 0 \tag{32}$$

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[2] We have put here $\dfrac{c^2}{\mathscr{A}} - H^2 > 0$ because $\dfrac{c^2}{\mathscr{A}} \sim 10^{87}$ s^{-2} and $H = \dfrac{\dot{a}}{a}\,\mathrm{s}^{-1}$ is usually very small.

and has the following general solution

$$u = e^{(-)\frac{3}{2}Ht}\mathscr{B}\sin\left(\sqrt{\frac{c^2}{\mathscr{A}} - H^2}\, t + \gamma\right), \tag{33}$$

where $\mathscr{B}$ and γ are constants.

The solution (33) determines the zero points of solutions to the equation (31) for large, positive values of t. The function $h(t) = \frac{\dot{u}}{uc}$ with $u(t)$ given by (33) has infinitely many singularities.

The above analysis shows the existence of infinitely many singularities in the solutions of the equations (30) for large but finite values of the cosmic time t. Thus, the solutions of the equation (30) must develop singularities for finite values of the cosmic time.

We get analogical results by studying of the approximate solutions to the equation (31) in a small vicinity of the moment $t = 0$.

3. Spatially homogeneous and isotropic cosmology with the SO(3) isotropy group

In the case of the SO(3) isotropy group we have the following cosmological equations in the orthonormal tetrad determined by the Robertson-Walker line element (see [6])

$$(-)\frac{9h\dot{a}}{ac} - \frac{15}{2}h^2 - \frac{3\dot{a}^2}{a^2c^2} - \frac{3k}{a^2} - 3Q^2 + \frac{3\beta}{\alpha}\left\{\left[\frac{\left(\frac{\dot{a}}{c} + ah\right)^{\cdot}}{ac}\right]^2\right.$$

$$\left. - \left[\frac{(aQ)^{\cdot}}{ac}\right]^2 + 4Q^2\left(\frac{\dot{a}}{ac} + h\right)^2 - \left[\left(\frac{\dot{a}}{ac} + h\right)^2 + \frac{k}{a^2} - Q^2\right]^2\right\} = (-)\frac{\varepsilon}{2\alpha},$$

$$(-)\frac{3\dot{h}}{c} - \frac{6h\dot{a}}{ac} - \frac{9}{2}h^2 - Q^2 - \frac{k}{a^2} - \frac{\dot{a}^2}{a^2c^2} - \frac{2\ddot{a}}{ac^2}$$

$$- \frac{\beta}{\alpha}\left\{\left[\frac{\left(\frac{\dot{a}}{c} + ah\right)^{\cdot}}{ac}\right]^2 + 4Q^2\left(\frac{\dot{a}}{ac} + h\right)^2 - \left[\frac{(aQ)^{\cdot}}{ac}\right]^2\right.$$

$$\left. - \left[\left(\frac{\dot{a}}{ac} + h\right)^2 + \frac{k}{a^2} - Q^2\right]^2\right\} = \frac{p}{2\alpha},$$

$$\frac{1}{c^2}\left[\frac{(aQ)^{\cdot}}{a}\right]^{\cdot} - 2Q^3 + \frac{2Qk}{a^2} + 2\left(\frac{\dot{a}}{ac} + h\right)\frac{(aQ)^{\cdot}}{ac} - 2Q\left(\frac{\dot{a}}{ac} + h\right)^2$$

$$+ \frac{2h}{ac}(aQ)^{\cdot} - \frac{\alpha}{\beta}Q = 0,$$

$$\frac{1}{c^2}\left[\frac{\left(\frac{\dot a}{c}+ah\right)^{\boldsymbol\cdot}}{a}\right]^{\boldsymbol\cdot}-2Q^2\left(\frac{\dot a}{ac}+\quad\right)+\frac{2}{ac}\left(\frac{\dot a}{ac}+h\right)\left(\frac{\dot a}{c}+ah\right)^{\boldsymbol\cdot}$$

$$-2\left(\frac{\dot a}{ac}+h\right)^3+\frac{2h}{ac}\left(\frac{\dot a}{c}+ah\right)^{\boldsymbol\cdot}-\frac{2k}{a^2}\left(\frac{\dot a}{ac}+h\right)+\frac{\alpha}{2\beta}h=0, \tag{34}$$

where $k = 0, \pm 1$.

In each case, $k = 0, \pm 1$, we have a system of four ordinary, nonlinear differential equations for four unknown functions: the scale factor $a(t)$, the torsion components $h(t)$, $Q(t)$ and the energy density $\varepsilon(t)$ or pressure $p(t)$.

As in the case of the O(3) isotropy group, the classical spin is eliminated from these equations.

The Criterion "C" is satisfied in the case and, therefore, there may exist cosmological solutions with torsion. We can obtain these solutions using a suitable Ansatz which reduces the cosmological problem to the problem of solving only one nonlinear, ordinary differential equation of the third order for the scale factor $a(t)$.

We obtain this equation by combination of the Ansatz equation with the two last equations of the system (34).

In this paper we consider only expanding cosmological models with $\dot a > 0$ given by the simplest Ansatz (8). This Ansatz leads to the macroscopic behaviour of the models determined by

$$a = ct+\text{const}, \qquad h = Q = 0, \qquad p = (-)\frac{\varepsilon}{3}, \ \text{const} \geqslant 0. \tag{35}$$

The procedure of solving the system (34) using the Ansatz (8) is this:

1. First of all we solve the differential equations of the third order for $a(t)$. These equations are essentially different for the different cases $k = 0, \pm 1$ and they are too complicated to be considered here.
2. Having $a = a(t)$ we can calculate

$$Q^2 = Q^2(a, \dot a) \qquad \text{and} \qquad h = h(Q, a).$$

3. Having $a(t)$, $Q(t)$ and $h(t)$ we determine $\varepsilon(t)$ and $p(t)$ from the first two field equations (34).

We have worked out the above procedure because the equations for the scale factor $a(t)$ are too complicated. However, from the expression on Q^2

$$Q^2 = \frac{1}{4\mathscr{A}} - \frac{\dot a}{4\mathscr{A}c} - \frac{1}{a^2} - \frac{k}{a^2}, \qquad k = 0, \pm 1 \tag{36}$$

and from the fact that the solutions to the equation for $a(t)$ must be of the asymptotic form $a = ct + \text{const}$, we get a very important conclusion: in the cases $k = 0, 1$ the condition $Q^2 > 0$ limits the possible values of the scale factor $a = a(t)$ to the bounded interval

(a_0, a_1) in which $Q^2 > 0$. The extreme points $0 < a_0$ and $a_0 < a_1 < \infty$ of this interval are determined, for a given function $a(t)$, by the solutions to the equation $Q^2 = 0$.

For the values of $a \geqslant a_1$ the macroscopic solution determined by $a = ct + \text{const}$, $h = Q = 0$ is valid.

Summing up, we can say that in the cases $k = 0, 1$ the Ansatz (8) leads to cosmological solutions (with $\dot{a} > 0$) without singularity in metric and in torsion. In the case $k = -1$ the condition $Q^2 > 0$ does not give any lower limit on the scale factor $a(t)$ as it tends to zero.

The special case of cosmology with the SO(3) isotropy group: $h = 0$, $Q \neq 0$ is very interesting. In this case we can determine algebraically $Q^2 = f(a, \dot{a}, \ddot{a}, \dddot{a})$ from the last equation of the system (34). Substituting $Q^2 = f(a, \dot{a}, \ddot{a}, \dddot{a})$ to the third equation of the system we get an ordinary, nonlinear differential equation of the fifth order for the scale factor $a(t)$.

Finding $a = a(t)$ from this equation and, afterwards, $Q^2(t) = f[a(t), \dot{a}(t), \ddot{a}(t), \dddot{a}(t)]$, we can calculate $\varepsilon = \varepsilon(t)$ from the first equation of the system (34) and $p = p(t)$ from the second equation of this system. Thus, the cosmological problem is reduced in this case to solving the equation for $a(t)$.

This equation has the following form

$$\frac{1}{c^2}\left(\frac{\ddot{a}P}{a} + \frac{3\dot{a}\dot{P}}{2a} - \frac{\dot{a}^2 P}{a^2} + \frac{\ddot{P}}{2} - \frac{\dot{P}^2}{4P}\right) - 2P^2$$
$$- \frac{P}{\mathscr{A}} - \frac{2Pk}{a^2} = 0, \tag{37}$$

where

$$P \equiv Q^2 = \frac{\dddot{a}}{2\dot{a}c^2} + \frac{\ddot{a}}{2ac^2} - \frac{\dot{a}^2}{a^2c^2} - \frac{k}{a^2} \tag{38}$$

and $k = 0, \pm 1$.

The nonlinear equation (37) is difficult to integrate. Nevertheless, in the case $k = 1$, the solutions may be without singularities in metric and in torsion. (The metric singularities are excluded by the condition $Q^2 > 0$.)

The macroscopic limit of the solution to the equation (37) is given (see [6]) by the condition $P \equiv Q^2 = 0$.

In the case $k = 0$ the equation $P = 0$ has the form

$$\frac{\dddot{a}}{2\dot{a}} + \frac{\ddot{a}}{2a} - \frac{\dot{a}^2}{a^2} = 0. \tag{39}$$

The equation (39) has an exact solution without singularity (the de Sitter model)

$$a = e^{bt}, \quad \varepsilon = \frac{6\alpha b^2}{c^2} = \text{const}, \quad p = (-)\varepsilon \tag{40}$$

where $b = \frac{\dot{a}}{a} =: H$ is the Hubble constant and another exact solution given by

$$a = \text{const}\, t^{1/2}, \quad \varepsilon = \frac{3\alpha}{2c^2 t^2}, \quad p = \frac{\varepsilon}{3}. \tag{41}$$

For sufficiently large values of the scale factor a, the equation (39) is satisfied, with very good accuracy, by the solution to the macroscopic cosmological equations (Friedmann equations) with $k = p = 0$ given by (11).

The spherical symmetry Ansatz (25) is consistent with the SO(3) isotropy and with the third order part of the field equations (see [5]) in the following two cases:

(*i*)
$$\mathscr{B} = -1, \quad \mathscr{C} = 0, \quad h = 0, \quad Q(t) \neq 0, \tag{42}$$

(*ii*)
$$\mathscr{B} = 0, \quad \mathscr{C} = \frac{\sqrt{2}}{2}, \quad h = (-)\sqrt{2}\, Q \neq 0 \tag{43}$$

The Criterion "C" is satisfied in these two cases too. Therefore there may exist the solution with torsion. However, in the case (i) there is no solution with torsion: the equations which remain to integrate imply vanishing torsion. There may exist only torsionless solutions satisfying the following system of equations

$$\frac{\ddot{a}}{ac^2} + \frac{\dot{a}^2}{a^2c^2} + \frac{k}{a^2} = \frac{1}{2\mathscr{A}},$$

$$\frac{\dot{a}^2}{a^2c^2} - \frac{1}{8\mathscr{A}} + \frac{k}{a^2} = \frac{\varepsilon}{12\alpha}, \quad \frac{\ddot{a}}{ac^2} + \frac{1}{8\mathscr{A}} = (-)\frac{p}{4\alpha}, \tag{44}$$

where $k = 0, \pm 1$.

The above system of equations leads to the state equation

$$p = \frac{\varepsilon}{3} - \frac{2\alpha}{\mathscr{A}}, \tag{45}$$

the same for all cases $k = 0, \pm 1$.

In the case $k = 1$ there exists a static solution given by

$$a = \sqrt{2\mathscr{A}}, \quad \varepsilon = \frac{9\alpha}{2\mathscr{A}}, \quad p = (-)\frac{\alpha}{2\mathscr{A}} = (-)\frac{\varepsilon}{9}.$$

We can solve the system (44) in the following way: we find $a = a(t)$ from the first equation of this system and then calculate $\varepsilon(t)$ and $p(t)$ from the remaining equations.

The equations for the scale factor $a = a(t)$ are easily transformed into Bernoulli's equations and have the following solutions:

1. $k = 0$

$$\frac{\sqrt{\mathscr{A}}}{c}\ln\left(\frac{ca^2}{2\sqrt{\mathscr{A}}}+\sqrt{\frac{c^2a^4}{4\mathscr{A}}+\mathscr{B}}\right) = \pm t+E, \tag{46}$$

where $\mathscr{B} > 0$ and E are the integration constants.

2 · $k = 1$

$$\frac{\sqrt{\mathscr{A}}}{c}\ln\left(\frac{c}{\sqrt{\mathscr{A}}}\sqrt{\frac{c^2a^4}{4\mathscr{A}}-c^2a^2+\mathscr{B}}+\frac{c^2a^2}{2\mathscr{A}}-c^2\right) = \pm t+E, \tag{47}$$

where $\mathscr{B} = \mathscr{A}c^2$ and E are the integration constants.

3. $k = (-)1$

$$\frac{\sqrt{\mathscr{A}}}{c}\ln\left(\frac{c}{\sqrt{\mathscr{A}}}\sqrt{\frac{c^2a^4}{4\mathscr{A}}+c^2a^2+\mathscr{B}}+\frac{c^2a^2}{2\mathscr{A}}+c^2\right) = \pm t+E, \tag{48}$$

where $\mathscr{B} = \mathscr{A}c^2$ and E are the integration constants.

Eqs (46)–(48) represent pairs of solutions which are transformed into each other by time reflection. In the cases $k = 0$, 1 and with the sign $(+)$ before t, the solutions are continuously expanding from $a = 0$ at the moment $t = (-)\infty$ to $a = \infty$ at the moment $t = \infty$ and with the sign $(-)$ before the cosmic time t, the solutions are continuously imploding from $a = \infty$ at $t = (-)\infty$ to $a = 0$ at $t = \infty$. These solutions are without any singularity for finite values of the cosmic time t.

Solution (48) with the sign $(+)$ before t is expanding from $a = 0$ at the moment $t = \frac{\sqrt{\mathscr{A}}}{c}\ln 2c^2-E$ to $a = \infty$ as the cosmic time t goes to infinity and with the sign $(-)$ before t it is continuously imploding from $a = \infty$ at $t = (-)\infty$ to $a = 0$ at $t = (-)\frac{\sqrt{\mathscr{A}}}{c}\ln 2c^2+E$.

Therefore, the solution (48) has a metric singularity for finite time

$$t = \pm\frac{\sqrt{\mathscr{A}}}{c}\ln 2c^2\mp E.$$

The singularity can be removed by a suitable choice of the constant E.

Solutions (46)–(48) do not correspond to any cosmological solution of the macroscopic gravitational theory.

In the second case (*ii*) we have the following system of equations to integrate

$$\frac{\ddot{a}}{ac^2}+\frac{\dot{a}^2}{a^2c^2}-3\sqrt{2}\,Q\frac{\dot{a}}{ac}-\sqrt{2}\frac{\dot{Q}}{c}+Q^2+\frac{k}{a^2} = 0,$$

$$3Q\frac{\dot{a}}{ac}+\frac{\dot{Q}}{c}-2\sqrt{2}\,Q^2+\frac{\sqrt{2}}{4\mathscr{A}} = 0,$$

$$\frac{\dot{a}^2}{a^2c^2} - \frac{7\sqrt{2}}{2} Q \frac{\dot{a}}{ac} - \frac{\sqrt{2}\,\dot{Q}}{2c} + 6Q^2 + \frac{k}{a^2} = \frac{\varepsilon}{6\alpha},$$

$$\frac{2\ddot{a}}{ac^2} + \frac{\dot{a}^2}{a^2c^2} - \frac{11\sqrt{2}}{2} Q \frac{\dot{a}}{ac} - \frac{5\sqrt{2}}{2}\frac{\dot{Q}}{c} + 10Q^2$$

$$+ \frac{1}{8\mathscr{A}} + \frac{k}{a^2} = (-)\frac{p}{2\alpha}, \tag{49}$$

where $k = 0, \pm 1$.

In the case $k = -1$ there exists a static solution characterized by

$$a = \sqrt{8\mathscr{A}} = \frac{1}{Q}, \quad \varepsilon = \frac{15\alpha}{4\mathscr{A}}, \quad p = (-)\frac{5\alpha}{2\mathscr{A}} = (-)\tfrac{2}{3}\varepsilon. \tag{50}$$

The system (49) can be solved in the following way:

a) We determine $\frac{\dot{a}}{ac}$ from the second equation and get

$$\frac{\dot{a}}{ac} = \frac{2\sqrt{2}}{3} Q - \frac{\dot{Q}}{3Qc} - \frac{\sqrt{2}}{12\mathscr{A}Q}.$$

b) Substituting $\frac{\dot{a}}{ac}$ into the first equation we get a second order equation for $Q = Q(t)$.

c) We find $Q = Q(t)$ and then calculate

$$a(t) = \exp\left[\int\left(\frac{2\sqrt{2}\,c}{3} Q - \frac{\dot{Q}}{3Q} - \frac{\sqrt{2}\,c}{12\mathscr{A}Q}\right) dt\right]. \tag{51}$$

d) Having $a(t)$ and $Q(t)$ we find $\varepsilon(t)$ and $p(t)$ from the remaining equations of the system.

The equation for $Q(t)$ is

$$\frac{\ddot{Q}}{3Qc^2} - \frac{5}{9}\frac{\dot{Q}^2}{Q^2c^2} - \left(\frac{7\sqrt{2}}{36\mathscr{A}Q^2} - \frac{2\sqrt{2}}{9}\right)\frac{\dot{Q}}{c} + \tfrac{11}{9} Q^2 - \frac{1}{36\mathscr{A}^2Q^2} - \frac{1}{18\mathscr{A}}$$

$$+ k\left\{\exp\left[2\int\left(\frac{2\sqrt{2}\,c}{3} Q - \frac{\dot{Q}}{3Q} - \frac{\sqrt{2}\,c}{12\mathscr{A}Q}\right) dt\right]\right\}^{-1} = 0, \tag{52}$$

where $k = 0, \pm 1$.

In the case $k = 0$ we get, after some transformations (see [8]), the following differential equation for $Q(t)$

$$zz' - \left(\frac{7\sqrt{2}\,c}{20\mathscr{A}} Q^{-5/3} + 2\sqrt{2}\,cQ^{1/3}\right) z' = \frac{c^2}{12\mathscr{A}^2} Q^{-13/3}$$

$$+ \frac{c^2}{6\mathscr{A}} Q^{-7/3} - \frac{11c^2}{3} Q^{-1/3}, \tag{53}$$

where

$$Q^{5/3}\left[z(Q)-\frac{7\sqrt{2}\,c}{20\mathscr{A}}Q^{-5/3}-2\sqrt{2}\,cQ^{1/3}\right]=\dot{Q}. \tag{54}$$

A qualitative analysis shows that if the equation (53) has a real solution $Z = Z(Q)$, then the solution should develop metric and torsion singularities.

4. *Discussion and conclusions*

We have investigated the existence of geometric singularities in the cosmological solutions of the gauge gravitational theory with gravitational Lagrangian (1). The general conclusions which follow from the paper are the following:

1. Addition of terms quadratic in curvature and torsion to the Einstein Lagrangian does not remove cosmological singularities at all.
2. Lack of metric singularities does not ensure that the torsion singularities are also absent.
3. The simplest cosmological models with $k = 0$, 1, and with macroscopic behaviour $\boldsymbol{a} = ct + \text{const}$, $h = Q = 0$, $p = (-)\frac{\varepsilon}{3}$, do not have any cosmological singularity. We think that the Nature realizes the simplest model of this kind with $k = 0$. This model was discussed in details in [6].

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