$d = 2$ CONFORMAL GAUGE THEORY*

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Conformal gravity in $d = 2$ is constructed and discussed with a view to its application in string theory.

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1. Introduction

An important aspect of string theory is local conformal invariance on the 2-dimensional world sheet [1]. This local symmetry plays an important role in the quantization of string models, as has become especially clear from the work on the BRST formulation of string dynamics by Siegel [2]. It was shown by Brink, Di Vecchia and Howe [3] that a first order formulation of string theory, which includes the constraints among its field equations, can be obtained by coupling the string co-ordinates to 2-dimensional gravitational fields defined on the world sheet. Therefore it seems desirable to have a formulation of $d = 2$ gravity which possesses manifest local conformal invariance.

Conformally invariant theories in 2 dimensions are also of interest in a wider context, for example in the theory of critical phenomena, as was emphasized in [4]. The approach adopted there, which makes use throughout of operator product expansion (OPE) techniques, was elaborated for the case of string theory in [5, 6]. In this lecture I will describe a formalism for $d = 2$ conformal gravity that is closely related to this approach and allows easy translation into the language of [4–6]. Still, the structure of the formalism is very close to that of conformal gravity and supergravity in dimensions $d > 2$. In fact almost all results we are going to obtain have a generalization (or in some instances: particularization) to higher dimensions, but lacking the time to digress on this interesting topic, I must refer those wishing to pursue it further to the existing literature, for example the reviews in [7, 8].

2. Conformal transformations and space-time geometry

When a string propagates in time, it sweeps out a 2-dimensional surface called the world sheet of the string. Obviously, this world sheet has one space-like and one time-like direction and may be covered by local co-ordinate systems ($x^1 = \sigma$, $x^2 = i\tau$). In

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fact, we can think of the world sheet as a 2-dimensional space-time, and its geometry is governed accordingly by a 2-dimensional version of general relativity. We describe the local geometry of the world sheet in terms of the line element ds in the surface at a certain point (x^α) by giving its length

$$ds^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta, \tag{2.1}$$

where $g_{\alpha\beta}(x)$ is the metric of the surface. The length defined by ds^2 is a co-ordinate-invariant notion (it is independent of any reparametrizations of the surface). Then it follows that under an infinitesimal general co-ordinate transformation (GCT), $\delta x^\alpha = \xi^\alpha(x)$, the metric must change according to

$$\delta g_{\alpha\beta}(x) = -\xi^\gamma\partial_\gamma g_{\alpha\beta} - \partial_\alpha\xi^\gamma g_{\gamma\beta} - \partial_\beta\xi^\gamma g_{\alpha\gamma} = -(D_\alpha\xi_\beta + D_\beta\xi_\alpha). \tag{2.2}$$

In the last expression we have used the notation

$$D_\alpha\xi_\beta = \partial_\alpha\xi_\beta - \Gamma_{\alpha\beta}{}^\gamma\xi_\gamma, \quad \xi_\alpha = g_{\alpha\beta}\xi^\beta,$$
$$\Gamma_{\alpha\beta}{}^\delta = \tfrac{1}{2}\, g^{\gamma\delta}(\partial_\alpha g_{\gamma\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}). \tag{2.3}$$

$\Gamma_{\alpha\beta}{}^\delta$ is the affine (or Riemann-Christoffel) connection.

In order to discuss possible symmetries of a geometrical space, we must introduce the notion of a Killing vector. A Killing vector is a linear differential operator $\hat{R}(\xi) = \xi^\alpha\partial_\alpha$, the coefficients of which satisfy the Killing equation

$$D_\alpha\xi_\beta + D_\beta\xi_\alpha = 0. \tag{2.4}$$

Therefore, from Eq. (2.2), $\delta x^\alpha = \hat{R}(\xi)x^\alpha = \xi^\alpha(x)$ defines a GCT which leaves the metric invariant: $\delta g_{\alpha\beta} = 0$.

As a simple example, it is instructive to consider the case of Minkowski space-time. Then $g_{\alpha\beta} = \eta_{\alpha\beta}$, and $\Gamma_{\alpha\beta}{}^\gamma = 0$. The Killing equation now becomes

$$\partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha = 0, \tag{2.5}$$

which has the general solution

$$\xi^\alpha = a^\alpha + \omega^{\alpha\beta}x_\beta, \quad \omega^{\alpha\beta} = -\omega^{\beta\alpha}. \tag{2.6}$$

Thus the Killing vectors $\hat{R}(\xi)$ of flat (Minkowski-) space-time generate infinitesimal translations with parameter a^α, and Lorentz transformations with parameter $\omega^{\alpha\beta}$. It is easy to show, (I leave this as an exercise to you), that the commutator of two Killing vectors is again a Killing vector. In our example this can be summarized as follows:

$$\text{let } \hat{R}(\xi) = a^\alpha P_\alpha + \tfrac{1}{2}\,\omega^{\alpha\beta}M_{\alpha\beta}, \quad \text{with} \quad P_\alpha = \partial_\alpha, \quad M_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha. \tag{2.7}$$

Then the commutator algebra of the generators $(P_\alpha, M_{\alpha\beta})$ closes on itself:

$$[M_{\alpha\beta}, M_{\gamma\delta}] = \eta_{\beta\gamma}M_{\alpha\delta} + \eta_{\alpha\delta}M_{\beta\gamma} - \eta_{\beta\delta}M_{\alpha\gamma} - \eta_{\alpha\gamma}M_{\beta\delta},$$
$$[M_{\alpha\beta}, P_\gamma] = \eta_{\beta\gamma}P_\alpha - \eta_{\alpha\gamma}P_\beta, \quad [P_\alpha, P_\beta] = 0. \tag{2.8}$$

This Lie algebra of space-time transformations leaving Minkowski space invariant is often called the Poincaré algebra.

Next we introduce a generalization of the concept of Killing vector which is of great importance in the following. This is the notion of a conformal Killing vector, defined as a linear differential operator $\hat{C}(\xi)$, with $\hat{C}(\xi) = \xi^\alpha \partial_\alpha$ and

$$D_\alpha \xi_\beta + D_\beta \xi_\alpha = \frac{2}{d} g_{\alpha\beta} D \cdot \xi. \tag{2.9}$$

Conformal Killing vectors generate transformations which do not leave the metric invariant, but rescale it by a (generally space-time dependent) factor:

$$\delta g_{\alpha\beta} = \lambda(x) g_{\alpha\beta}, \quad \lambda(x) = -\frac{2}{d} D \cdot \xi. \tag{2.10}$$

Therefore conformal Killing vectors generate a symmetry of the line element (2.1) only if $ds^2 = 0$, i.e. they generate invariance transformations of the light cone.

As an example, take again Minkowski space-time. The conformal Killing equation becomes

$$\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha - \frac{2}{d} \eta_{\alpha\beta} \partial \cdot \xi = 0. \tag{2.11}$$

For its solution we must distinguish between $d = 2$ and $d > 2$. In dimension $d > 2$ the general solution of (2.11) is given by

$$\xi^\alpha = a^\alpha + \omega^{\alpha\beta} x_\beta + \lambda x^\alpha + [2x^\alpha x^\beta - \eta^{\alpha\beta} x^2] b_\beta \tag{2.12}$$

Then the conformal Killing vector becomes

$$\hat{C}(\xi) = a^\alpha P_\alpha + \tfrac{1}{2} \omega^{\alpha\beta} M_{\alpha\beta} + \lambda D + b^\alpha K_\alpha, \tag{2.13}$$

where

$$\begin{aligned} P_\alpha &= \partial_\alpha && \text{generates translations,} \\ M_{\alpha\beta} &= x_\alpha \partial_\beta - x_\beta \partial_\alpha && \text{generates Lorentz transformations,} \\ D &= x \cdot \partial && \text{generates dilatations,} \\ K_\alpha &= 2x_\alpha x \cdot \partial - x^2 \partial_\alpha && \text{generates special conformal transformations.} \end{aligned} \tag{2.14}$$

The commutator algebra of these transformations is

$$\begin{gathered} [M_{\alpha\beta}, M_{\gamma\delta}] = \eta_{\beta\gamma} M_{\alpha\delta} + \eta_{\alpha\delta} M_{\beta\gamma} - \eta_{\beta\delta} M_{\alpha\gamma} - \eta_{\alpha\gamma} M_{\beta\delta}, \\ [M_{\alpha\beta}, P_\gamma] = \eta_{\beta\gamma} P_\alpha - \eta_{\alpha\gamma} P_\beta, \quad [M_{\alpha\beta}, K_\gamma] = \eta_{\beta\gamma} K_\alpha - \eta_{\alpha\gamma} K_\beta, \\ [D, P_\alpha] = -P_\alpha, \quad [D, K_\alpha] = K_\alpha, \\ [P_\alpha, K_\beta] = 2(\eta_{\alpha\beta} D - M_{\alpha\beta}). \end{gathered} \tag{2.15}$$

This is the conformal Lie algebra in $d > 2$.

The solution (2.12)–(2.15) exists also for $d = 2$. However, in this case there are infinitely many other solutions and as a consequence the conformal group in $d = 2$ is an infinite-parameter Lie group. To see this it is convenient to introduce a set of coordinates in $d = 2$ space-time called light-cone co-ordinates:

$$z = \frac{1}{\sqrt{2}}(x^1 + ix^2), \qquad \bar{z} = \frac{1}{\sqrt{2}}(x^1 - ix^2),$$

with

$$\partial_z = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2), \qquad \partial_{\bar{z}} = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2). \tag{2.16}$$

Note, that after a Wick rotation $\tau \to i\tau$ the variables $(z, \bar{z})$ become complex conjugates in a 2-dimensional Euclidean manifold. I will regularly make use of the Euclidean formulation without explicitly stating this every time, because it allows us to use all the tricks of complex analysis, such as contour integrations. However, all these manipulations have their counterparts in $d = 2$ Minkowski space-time. In light-cone co-ordinates the invariant tensors $\eta_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ become:

$$\eta_{\alpha\beta} \to \begin{pmatrix} \eta_{zz} & \eta_{z\bar{z}} \\ \eta_{\bar{z}z} & \eta_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.17}$$

and

$$\varepsilon^{\alpha\beta} \to \begin{pmatrix} \varepsilon^{zz} & \varepsilon^{z\bar{z}} \\ \varepsilon^{\bar{z}z} & \varepsilon^{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.18}$$

Hence for two vectors $(a_z, a_{\bar{z}})$ and $(b_z, b_{\bar{z}})$ we have an inner product $a \cdot b$ and an exterior product $a \wedge b$ given by:

$$a \cdot b = a_{\bar{z}} b_z + a_z b_{\bar{z}} \quad \text{and} \quad a \wedge b = i(a_{\bar{z}} b_z - a_z b_{\bar{z}}). \tag{2.19}$$

In these co-ordinates the conformal Killing equation (2.11) reduces to a set of 2 independent equations:

$$\partial_z \xi_z = 0, \qquad \partial_{\bar{z}} \xi_{\bar{z}} = 0. \tag{2.20}$$

Therefore the general solution is

$$\xi^z = \eta^{z\bar{z}} \xi_{\bar{z}} = f(z), \qquad \xi^{\bar{z}} = \eta^{\bar{z}z} \xi_z = \bar{f}(\bar{z}), \tag{2.21}$$

with $f(z)$ ($\bar{f}(\bar{z})$) arbitrary (anti-)analytic functions of z ($\bar{z}$).

Any solution regular at the origin can be expanded in a Taylor series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$

and depends indeed on infinitely many parameters $(a_0, a_1, \ldots)$. The $d = 2$ conformal Killing vectors corresponding to these regular solution then are

$$\hat{C}(\xi) = \sum_{n=-\infty}^{+1} (\varepsilon_n L_n + \bar{\varepsilon}_n \bar{L}_n), \tag{2.22}$$

where

$$\varepsilon_n = a_{1-n}, \qquad \bar{\varepsilon}_n = \bar{a}_{1-n};$$

$$L_n = z^{1-n}\partial_z, \qquad \bar{L}_n = \bar{z}^{1-n}\partial_{\bar{z}}, \qquad (n = 1, 0, -1, \ldots). \tag{2.23}$$

The operators L_n, $\bar{L}_n$ satisfy the commutator algebra

$$[L_n, L_m] = (n-m)L_{n+m}, \qquad (\text{id. } \bar{L}_n); \qquad (n \leqslant 1), \tag{2.24}$$

the Virasoro algebra. The somewhat strange numbering of the L_n and ε_n in (2.23) is for purely historical reasons, the form (2.24) of the Virasoro algebra being standard in the literature.

Two remarks concerning these results are in order.

— As noted before, the Virasoro algebra contains the finite conformal algebra (2.15) as a closed subalgebra. The precise correspondence is:

$$\begin{aligned} P_1 &= i(L_1-\bar{L}_1), \qquad & K_1 &= -i(L_{-1}-\bar{L}_{-1}), \qquad & M &= L_0-\bar{L}_0 \\ P_2 &= L_1+\bar{L}_1, \qquad & K_2 &= L_{-1}+\bar{L}_{-1}, \qquad & D &= L_0+\bar{L}_0. \end{aligned} \tag{2.25}$$

— The Virasoro algebra can be extended to include the conformal Killing vectors which are singular at the origin: $L_n = z^{1-n}\partial_z$, with $n > 1$. The form of the algebra then remains the same. However, the singularities lead to anomalies when we try to implement the symmetry algebra in the Hilbert space of a conformal quantum theory. This requires the quantum algebra to be extended with a central charge c:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta(m+n). \tag{2.26}$$

Note, that owing to the form of the last term on the r.h.s. it does not appear if we restrict ourselves to the regular Killing vectors with $-\infty < n \leqslant 1$; in other words, the algebra (2.24) is a closed subalgebra of (2.26). In the following this restriction is made, unless it is explicitly stated otherwise.

3. *A continuous basis for the Virasoro algebra*

The conventional form of the Virasoro algebra is the one given in (2.24), (2.26). However we find it convenient to change to a different basis, which is continuous rather than discrete: $L_n \to T_<(\zeta)$, where ζ is a continuous parameter labeling the generators. The transition is made by defining

$$T_<(\zeta) = \sum_{n=-\infty}^{+1} \zeta^{n-2} L_n. \tag{3.1}$$

In principle ζ is real, but analytic continuation to complex values of ζ is useful and is tacitly assumed in the following. Using the representation (2.23) of the generators L_n, we can easily prove that

$$T_{<}(\zeta) = \frac{1}{\zeta - z}\,\partial_z \quad \text{in the domain } |z| < |\zeta|. \tag{3.2}$$

We note in passing, that in the domain $|z| > |\zeta|$, the same operator has the expansion

$$\frac{1}{\zeta - z}\,\partial_z = -T_{>}(\zeta) = -\sum_{n=2}^{\infty} \zeta^{n-2} L_n \tag{3.3}$$

in terms of the singular conformal Killing vectors L_n, $n > 1$. On the boundary, with the exception of the point $\zeta = z$, $T_{<}(\zeta)$ and $-T_{>}(\zeta)$ are identical. Hence they are analytic continuations of each other, together covering the whole complex plane excluding the pole at $\zeta = z$. Thus we may actually drop the label $<$ and work with the operator

$$T(z, \zeta) = \frac{1}{\zeta - z}\,\partial_z = \theta(|\zeta| - |z|)T_{<}(\zeta) - \theta(|z| - |\zeta|)T_{>}(\zeta), \tag{3.4}$$

defined everywhere except at $\zeta = z$.

Similarly we have

$$\bar{T}(\bar{z}, \bar{\zeta}) = \frac{1}{\bar{\zeta} - \bar{z}}\,\partial_{\bar{z}}, \tag{3.5}$$

where $\bar{\zeta}$ is a second independent parameter labeling the $\bar{T}(\bar{z}, \bar{\zeta})$.

For $|\bar{z}| < |\bar{\zeta}|$ this has again an expansion in terms of regular conformal Killing vectors L_n, $n \leqslant 1$:

$$\bar{T}(\bar{z}, \bar{\zeta}) = \bar{T}_{<}(\bar{\zeta}) = \sum_{n=-\infty}^{+1} \bar{\zeta}^{n-2} \bar{L}_n, \tag{3.6}$$

whilst for $|\bar{z}| > |\bar{\zeta}|$ there is an expansion in terms of the $\bar{L}_n$ $(n > 1)$ which are singular at the origin:

$$\bar{T}(\bar{z}, \bar{\zeta}) = -\bar{T}_{<}(\bar{\zeta}) = -\sum_{n=2}^{\infty} \bar{\zeta}^{n-2} \bar{L}_n. \tag{3.7}$$

The equivalence of the discrete and continuous basis follows from the inversion of (3.4):

$$L_n = \oint_{\Gamma_z} \frac{d\zeta}{2\pi i}\,\zeta^{1-n} T(z, \zeta), \tag{3.8}$$

where the contour passes around the point $\zeta_0 = z$ (for $n \leqslant 1$ in the usual counterclockwise fashion, but for $n > 1$ in the clockwise direction). A similar inversion formula holds for the $\bar{L}_n$, $\bar{T}(\bar{z}, \bar{\zeta})$.

From the Virasoro algebra (2.24) we can derive the commutation relation for the $T(z, \zeta)$:

$$[T(z,\zeta), T(z,\zeta')] = \frac{1}{(\zeta'-z)^2}\, T(z,\zeta) + \frac{1}{(\zeta'-z)}\, T'(z,\zeta). \qquad (3.9)$$

Under integration over ζ' with an arbitrary (regular) function of ζ', and with $|z| < |\zeta'| < |\zeta|$ this is equivalent to

$$[T_<(\zeta), T_<(\zeta')] = -2\,\frac{T_<(\zeta')}{(\zeta-\zeta')^2} - \frac{T'_<(\zeta')}{\zeta-\zeta'}, \quad |\zeta| > |\zeta'|. \qquad (3.9')$$

Now consider an arbitrary element of the (restricted) Virasoro algebra: $X = \sum_{n\leqslant 1} X_n L_n$. In the new basis it can be written:

$$X = \oint_{\Gamma_z} \frac{d\zeta}{2\pi i}\, X(\zeta) T_<(\zeta), \qquad (3.10)$$

where

$$X(\zeta) = \sum_{n\leqslant 1} X_n \zeta^{1-n}. \qquad (3.11)$$

The contour is to be taken around $\zeta_0 = z$, as before. Thus, the restriction to the regular part of the Virasoro algebra makes $X(\zeta)$ regular around $\zeta = 0$ as well. The commutation relations (2.24) and (3.9) then give the commutator of two arbitrary elements X, Y of the algebra:

$$[X, Y] = \sum_{n=-\infty}^{+1} \Big(\sum_{m=n-1}^{+1} (2m-n) X_m Y_{n-m} \Big) L_n$$

$$= \oint_{\Gamma_z} \frac{d\zeta}{2\pi i} \big(X(\zeta)Y'(\zeta) - X'(\zeta)Y(\zeta)\big) T_<(\zeta). \qquad (3.12)$$

The last form is the basis for the formulation of the conformal gauge theory to be developed in the following sections.

4. *Conformal gauge theory*

The conformal transformations discussed above are rigid co-ordinate transformations in the 2-dimensional manifold which we take to be the world sheet of the string. With rigid I mean, that the parameters ε_n in (2.22) are co-ordinate independent. In the introduction I have argued, that it is desirable to have a theory in which these transformations are realized locally in the two-dimensional manifold. The first step in this program is to construct a gauge theory of the Virasoro algebra along the lines of standard Yang-Mills theory. Thus the Virasoro algebra is treated initially as an ordinary internal symmetry algebra. After this is accomplished it will become clear, that these "internal" conformal transformations can be re-interpreted as general co-ordinate transformations by imposing a suitable constraint on the theory. The resulting theory is $d = 2$ conformal gauge theory, which reduces by a suitable gauge fixing procedure to conformal gravity.

The Yang-Mills theory of the Virasoro algebra is constructed in terms of a set of gauge fields $(h_z, h_{\bar z})$ for the generators $T(z, \zeta)$ and $(\bar h_z, \bar h_{\bar z})$ for the generators $\bar T(\bar z, \bar\zeta)$. Since the algebra is a direct sum of the two isomorphic algebra's of $T(z, \zeta)$ and $\bar T(\bar z, \bar\zeta)$, we can phrase the discussion in terms of one of them only, for example $h_{z,\bar z}$ and $T(z, \zeta)$. The gauge fields are taken to be Lie-algebra valued and can therefore be expanded as in (3.10), (3.11). The gauge fields transform under infinitesimal local Virasoro transformations with parameter $\varepsilon(z, \bar z)$ as

$$\delta h_z = D_z\varepsilon = \partial_z\varepsilon - [h_z, \varepsilon], \; \delta h_{\bar z} = D_{\bar z}\varepsilon = \partial_{\bar z}\varepsilon - [h_{\bar z}, \varepsilon], \tag{4.1}$$

with

$$h_{z,\bar z} = \oint_{\Gamma_z} \frac{d\zeta}{2\pi i}\, h_{z,\bar z}(\zeta) T_<(\zeta), \qquad \varepsilon = \oint_{\Gamma_z} \frac{d\zeta}{2\pi i}\, \varepsilon(\zeta) T_<(\zeta). \tag{4.2}$$

Using (3.12) the components $h_z(z, \bar z; \zeta)$, $h_{\bar z}(z, \bar z; \zeta)$ transform as

$$\delta h_z(z, \bar z; \zeta) = D_z\varepsilon(z, \bar z; \zeta) = \partial_z\varepsilon - h_z\partial_\zeta\varepsilon + \partial_\zeta h_z\varepsilon, \qquad (z \to \bar z). \tag{4.3}$$

From the gauge fields we can construct a covariant tensor, the field strength or curvature:

$$R_{z\bar z} = \oint_{\Gamma_z} \frac{d\zeta}{2\pi i}\, R_{z\bar z}(\zeta) T_<(\zeta) = \partial_{[z}h_{\bar z]} - [h_z, h_{\bar z}] \tag{4.4}$$

with

$$R_{z\bar z}(\zeta) = \partial_{[z}h_{\bar z]}(\zeta) - h_{[z}\partial_\zeta h_{\bar z]}. \tag{4.5}$$

Note, that because of the antisymmetry in the indices the curvature tensor has only one independent component. Under the Virasoro transformations (4.1) $R_{z\bar z}$ transforms in the adjoint representation:

$$\delta R_{z\bar z} = [\varepsilon, R_{z\bar z}]. \tag{4.6}$$

Eqs. (4.1)–(4.6) follow the usual formalism of Yang-Mills theory with the Virasoro algebra being treated as an ordinary internal-symmetry algebra. In order to make contact with conformal space-time symmetries, we now consider the behaviour of $d = 2$ space-time vectors and tensors like $(h_z, h_{\bar z})$ and $R_{z\bar z}$ under general co-ordinate transformations (GCT), as in (2.2). In light-cone co-ordinates the infinitesimal GCTs are given by

$$\delta(\xi)h_z = -(\xi^z\partial_z + \xi^{\bar z}\partial_{\bar z})h_z - \partial_z\xi^z h_z - \partial_z\xi^{\bar z} h_{\bar z},$$

$$\delta R_{z\bar z} = -(\xi^z\partial_z + \xi^{\bar z}\partial_{\bar z})R_{z\bar z} - \partial_z\xi^z R_{z\bar z} - \partial_{\bar z}\xi^{\bar z} R_{z\bar z}. \tag{4.7}$$

These can be rewritten in a covariant way as follows:

$$\delta(\xi)h_z = \xi^{\bar z}R_{z\bar z} - D_z(\xi^z h_z + \xi^{\bar z}h_{\bar z})$$
$$\delta(\xi)R_{z\bar z} = -D_z(\xi^z R_{z\bar z}) - D_{\bar z}(\xi^{\bar z}R_{z\bar z}) - [\xi^z h_z + \xi^{\bar z}h_{\bar z}, R_{z\bar z}]. \tag{4.8}$$

Thus we see, that a GCT can be decomposed into a covariant GCT with parameters $(\xi^{\bar z}R_{z\bar z}, \xi^z R_{z\bar z})$ and a field dependent (local) Virasoro gauge transformation with parameter $\varepsilon(h) = -\xi \cdot h$. It is now obvious how the Virasoro-Yang-Mills theory can be converted into conformal gauge theory: one imposes the constraint $R_{z\bar z} = 0$. In components:

$$\partial_{[z}h_{\bar z]} - h_{[z}\partial_\zeta h_{\bar z]} = 0, \qquad \partial_{[z}\bar h_{\bar z]} - \bar h_{[z}\partial_{\bar\zeta}\bar h_{\bar z]} = 0. \tag{4.9}$$

Then the parameter of the covariant GCT vanishes, and an ordinary GCT is simply a local gauge transformation with parameter $\varepsilon(h)$. For complete equivalence the converse should be true as well, that is any gauge transformation must be interpretable as a (possibly field--dependent) GCT. Consider the complete set of gauge transformations with parameters $(\varepsilon, \bar\varepsilon)$, and the GCT with parameters $(\xi^z, \xi^{\bar z})$. With the constraint (4.9) they are transformed into each other by

$$\begin{pmatrix}\varepsilon\\ \bar\varepsilon\end{pmatrix} = -\begin{pmatrix}h_z & h_{\bar z}\\ \bar h_z & \bar h_{\bar z}\end{pmatrix}\begin{pmatrix}\xi^z\\ \xi^{\bar z}\end{pmatrix} \equiv -\vec{\vec{h}} \cdot \vec\xi. \tag{4.10}$$

The inverse exists if the matrix $\vec{\vec{h}}$ is non-singular, implying

$$h = -\det \vec{\vec{h}} = (\bar h_z h_{\bar z} - h_z \bar h_{\bar z}) \neq 0. \tag{4.11}$$

Then $\vec\xi = -\vec{\vec{h}}^{-1} \cdot \vec\varepsilon$, with

$$\vec{\vec{h}}^{-1} = \begin{pmatrix}h^{-1z} & \bar h^{-1z}\\ h^{-1\bar z} & \bar h^{-1\bar z}\end{pmatrix} = \frac{1}{h}\begin{pmatrix}-\bar h_{\bar z} & h_{\bar z}\\ \bar h_z & -h_z\end{pmatrix}. \tag{4.12}$$

From (4.11) we find that h_z and $h_{\bar z}$ cannot vanish simultaneously, and similarly for $\bar h_z$ and $\bar h_{\bar z}$.

We conclude, that the conformal gauge theory is obtained from the Virasoro-Yang--Mills theory by imposing the constraints $R_{z\bar z} = 0$ and $h \neq 0$.

5. Some properties of h

The gauge fields $(h, \bar h)$ contain infinitely many components h_n, $\bar h_n$ which are obtained from their Taylor expansion in ζ and $\bar\zeta$ respectively. The first components ($n = 1$) correspond to the zweibein e^a_μ, the gauge field of local translations (the exact correspondence follows from (2.25)); similarly the $n = 0$ components correspond to the dilatation gauge field b_μ and the $d = 2$ spin connection ω_μ; the $n = -1$ components to the gauge fields of special conformal transformations f^a_μ, etc. [9, 10]. Therefore $h = \det(-\vec{\vec{h}})$ is a direct generalization of $e = \det e^\mu_a$, which is in fact its lowest component, obtained by taking $\zeta = 0$.

Under the local conformal transformations (4.3) h transforms as

$$\delta h = (D\varepsilon + \bar{D}\bar{\varepsilon})h \quad \text{or} \quad \delta \log h = D\varepsilon + \bar{D}\bar{\varepsilon}, \tag{5.1}$$

where

$$D\varepsilon = \partial\varepsilon - \partial_\zeta \varepsilon + \frac{1}{h}\, \bar{h}_{[z}\partial_\zeta h_{\bar{z}]}\varepsilon,$$

$$\bar{D}\bar{\varepsilon} = \bar{\partial}\bar{\varepsilon} - \partial_\zeta \bar{\varepsilon} - \frac{1}{h}\, h_{[z}\partial_{\bar{\zeta}} \bar{h}_{\bar{z}]}\bar{\varepsilon} \tag{5.2}$$

and

$$\partial = h^{-1z}\partial_z + h^{-1\bar{z}}\partial_{\bar{z}}, \quad \bar{\partial} = \bar{h}^{-1z}\partial_z + \bar{h}^{-1\bar{z}}\partial_{\bar{z}}. \tag{5.3}$$

Because of its transformation law (5.1) we call h a conformal density with weights $(-1, -1)$. We return to the subject of conformal densities after we have discussed representations of the local algebra, since they are important for the construction of invariant actions.

Another somewhat technical point is the possibility of using the conformal gauge fields $(h, \bar{h})$ to convert world vectors $(V_z, V_{\bar{z}})$ (or any tensor for that matter) to world scalars $(V, \bar{V})$ by contraction with $(h^{-1}, \bar{h}^{-1})$. This is similar to the construction of a Lorentz vector out of a world vector and vice-versa. The precise relations are:

$$\begin{pmatrix} V \\ \bar{V} \end{pmatrix} = \begin{pmatrix} h^{-1z} & h^{-1\bar{z}} \\ \bar{h}^{-1z} & \bar{h}^{-1\bar{z}} \end{pmatrix} \begin{pmatrix} V_z \\ V_{\bar{z}} \end{pmatrix} \tag{5.4}$$

and

$$\begin{pmatrix} V_z \\ V_{\bar{z}} \end{pmatrix} = \begin{pmatrix} h_z & \bar{h}_z \\ h_{\bar{z}} & \bar{h}_{\bar{z}} \end{pmatrix} \begin{pmatrix} V \\ \bar{V} \end{pmatrix}. \tag{5.5}$$

These relations also hold for the derivatives $(\partial_z, \partial_{\bar{z}})$ and $(\partial, \bar{\partial})$, as in (5.3).

6. *A special gauge*

The gravitational field in 2 dimensions has no physical degrees of freedom, because there are no transverse directions. As a result all the components of the gravitational fields are gauge degrees of freedom. We can expect the same to be true for the conformal gauge theory, which includes the gravitational field. This result will now be established. A preliminary remark about the choice of gauge is however in order. We have already found at the end of Section 4, that the two components $(h_z, h_{\bar{z}})$ cannot vanish simultaneously. Actually, we know that a 2-dimensional manifold can always be parametrized locally in such a way that it is manifestly conformally flat [1]. Since we have local Weyl invariance as well, we can choose the conformal factor to be unity: $e^a_\mu = \delta^a_\mu$. For the conformal gauge fields this corresponds to the gauge $h_z = 0$, $h_{\bar{z}} = i\sqrt{2}$ (this is a consequence of the relation (2.25)). Therefore I want to demonstrate that this gauge can be obtained in our theory.

First the fact that $\varepsilon(z, \bar{z}; \zeta)$ is an arbitrary function of its arguments (although non--singular at $\zeta = 0$) allows us to take $h_z = 0$. Now the constraint (4.9) implies immediately, that

$$h_{\bar{z}}(z, \bar{z}; \zeta) = h_{\bar{z}}(0, \bar{z}; \zeta). \tag{6.1}$$

Furthermore, this choice of gauge still allows residual gauge transformations with the parameter ε $(z, \bar{z}; \zeta)$ satisfying:

$$\partial_z \varepsilon = 0 \to \varepsilon(z, \bar{z}; \zeta) = \varepsilon(0, \bar{z}; \zeta). \tag{6.2}$$

Since the dependence of the residual gauge parameter on $(\bar{z}, \zeta)$ is still arbitrary, we can now choose a gauge for $h_{\bar{z}}$ as well. The only restriction is, that we have to respect the condition $h \neq 0$.

Thus we may indeed choose the gauge

$$h_z = 0, \quad h_{\bar{z}} = i\sqrt{2} \tag{6.3}$$

and similarly

$$\bar{h}_z = -i\sqrt{2}, \quad \bar{h}_{\bar{z}} = 0. \tag{6.4}$$

Clearly this gauge choice satisfies $h = 2 \neq 0$ as well as the constraint $R_{z\bar{z}} = 0$.

It may come as a surprise, that (6.3–4) does not yet fix all the residual gauge freedom. By requiring any residual gauge transformations to satisfy

$$D_z \varepsilon = \partial_z \varepsilon = 0,$$
$$D_{\bar{z}} \varepsilon = \partial_{\bar{z}} \varepsilon - i\sqrt{2}\, \partial_\zeta \varepsilon = 0, \tag{6.5}$$

we find that we are still free to make transformations with parameter

$$\varepsilon(0, 0; \zeta + i\sqrt{2}\,\bar{z}) = \varepsilon(0, \bar{z} - i(\zeta/\sqrt{2}); 0) = \varepsilon^{+1}(0, \bar{z} - i(\zeta/\sqrt{2})). \tag{6.6}$$

These can either be interpreted as rigid conformal transformations or as "semi-local" translations, i.e. a translation depending on only one of the light-cone co-ordinates (in both cases we have to shift variables). In the gauge (6.3–4) we thus obtain theories with rigid conformal invariance in (locally) flat 2-dimensional space-time.

We close this section by noting, that in the special gauge the world-scalar derivatives become:

$$\partial = \frac{-i}{\sqrt{2}}\partial_{\bar{z}}, \quad \bar{\partial} = \frac{i}{\sqrt{2}}\partial_z. \tag{6.7}$$

This is a useful result for calculations in the special gauge.

7. *Representations: conformal fields*

Representations of the conformal algebra in $d = 2$ have been constructed and analysed in detail in [4]. Here I present a generalization of the concept of primary field for the case of local conformal transformations. From (3.12), or alternatively the transformation law of the gauge fields, we know that the commutator of two local conformal transformations with parameters ε_1, ε_2 is a conformal transformation with parameter $\varepsilon_3 = \varepsilon_{[1}\partial_\zeta \varepsilon_{2]}$. This algebra can be realized on fields $A(z, \bar{z}; \zeta, \bar{\zeta})$ by defining

$$\delta A = \varepsilon DA + \bar{\varepsilon}\bar{D}A + (\Delta \partial_\zeta \varepsilon + \bar{\Delta}\partial_{\bar{\zeta}}\bar{\varepsilon})A. \tag{7.1}$$

Here Δ, $\bar{\Delta}$ are two real parameters labeling the representation and called the conformal weights. The (world scalar) covariant derivatives DA and $\bar{D}A$ are defined by:

$$DA = \partial A - \frac{\Delta}{h}\bar{h}_{[z}\partial_\zeta h_{\bar{z}]}A - \frac{\bar{\Delta}}{h}\bar{h}_{[z}\partial_{\bar{\zeta}}\bar{h}_{\bar{z}]}A,$$

$$\bar{D}A = \bar{\partial}A + \frac{\Delta}{h}h_{[z}\partial_\zeta h_{\bar{z}]}A + \frac{\bar{\Delta}}{h}h_{[z}\partial_{\bar{\zeta}}\bar{h}_{\bar{z}]}A. \tag{7.2}$$

Checking the closure of the algebra is somewhat tedious but straightforward, using the transformation rule of $(h_z, h_{\bar{z}})$ and h as given in (4.3) and (5.1). In the course of the computation, one also needs the transformation rule of DA:

$$\delta DA = \varepsilon DDA + \bar{\varepsilon}\bar{D}DA + [(\Delta+1)\partial_\zeta\varepsilon + \bar{\Delta}\partial_{\bar{\zeta}}\bar{\varepsilon}]DA + \Delta\partial_\zeta^2 \varepsilon A. \tag{7.3}$$

Here the second derivatives are

$$DDA = \partial DA - \frac{(\Delta+1)}{h}\bar{h}_{[z}\partial_\zeta h_{\bar{z}]}DA - \frac{\bar{\Delta}}{h}\bar{h}_{[z}\partial_{\bar{\zeta}}\bar{h}_{\bar{z}]}DA - \frac{\Delta}{h}\bar{h}_{[z}\partial_\zeta^2 h_{\bar{z}]}A,$$

$$\bar{D}DA = \bar{\partial}DA + \frac{(\Delta+1)}{h}h_{[z}\partial_\zeta h_{\bar{z}]}DA + \frac{\bar{\Delta}}{h}\bar{h}_{[z}\partial_{\bar{\zeta}}\bar{h}_{\bar{z}]}DA$$

$$+ \frac{\Delta}{h}h_{[z}\partial_\zeta^2 h_{\bar{z}]}A. \tag{7.4}$$

A direct computation shows that $[D, \bar{D}]A = 0$, as expected from the Ricci identity and the constraint (4.9).

We note, that the $(\zeta, \bar{\zeta})$ dependence of A implies, that the representation actually has infinitely many components $A_{n,m}$, obtained by a power series expansion of A in $(\zeta, \bar{\zeta})$. However, it is possible to reduce this to a finite dimensional (actually: one-component) representation by imposing a constraint on the $(\zeta, \bar{\zeta})$-dependence:

$$\partial_\zeta A = DA, \quad \partial_{\bar{\zeta}}A = \bar{D}A. \tag{7.5}$$

Then in the special gauge (6.3–4) and using (6.7), the transformation law of the field A reduces to that of a primary field of Ref. [4]. Thus all results of Ref. [4] apply here in the special gauge upon imposing the reducibility constraint (7.5). For this reason we call $A(z, \bar{z}; \zeta, \bar{\zeta})$ a primary field of the local conformal gauge theory.

8. *Invariant actions*

The last topic I would like to discuss here is the construction of invariant actions for the conformal fields A introduced above. I begin by observing, that if F is a conformal field with conformal weights (1, 1), then $\mathscr{L} = hF$ transforms into a total derivative: $\delta_\varepsilon \mathscr{L} = -\partial_{[z}(\varepsilon h_{\bar{z}]}F)$. $\mathscr{L}$ is now a conformal density of weight (0, 0), which is what we require for the construction of an invariant action: by construction

$$S = \int d^2 z \mathscr{L} = \int d^2 z h F \tag{8.1}$$

is invariant (modulo surface terms) under local conformal gauge transformations. Note, that the densities are still allowed to have arbitrary $(\zeta, \bar{\zeta})$ dependence. This means, that by expanding in these parameters we obtain an infinite set of actions for all the components of the conformal field $F(z, \bar{z}; \zeta, \bar{\zeta})$. However, by imposing the constraint (7.5) on F, we get an ordinary space-time action $eF(z, \bar{z})$.

If we now want to write down an action for fields A^i which is quadratic in derivatives, we should clearly start with $\Delta_i = \bar{\Delta}_i = 0$ for all A^i. Then

$$F = Z_{ij}[A]\partial A^i \bar{\partial} A^j \tag{8.2}$$

is a conformal field of weight (1, 1) and hF is an invariant (modulo total derivatives). For real fields A^i hermiticity requires $Z_{ij}(A)$ to have the form

$$Z_{ij}(A) = g_{ij}(A) + i t_{ij}(A) \tag{8.3}$$

where $g_{ij}(A)$ is symmetric and $t_{ij}(A)$ anti-symmetric. If we impose the reducibility condition (7.5) on the fields A^i, we obtain the standard $d = 2$ σ-model with Wess-Zumino term. In the special gauge thus action reduces to the standard form

$$\mathscr{L} = Z_{ij}\partial A^i \bar{\partial} A^j = g_{ij}\eta^{\mu\nu}\partial_\mu A^i \partial_\nu A^j + t_{ij}\varepsilon^{\mu\nu}\partial_\mu A^i \partial_\nu A^j. \tag{8.4}$$

The actions presented here form the starting point for string theory. In the form (8.4) we have fixed a gauge, and the theory actually requires extra ghost contributions to restore the local symmetry in the form of BRST-invariance [2]. These still have to be added to (8.4).

It is an interesting problem, what kind of theories we obtain if we do not impose the reducibility condition (7.5). In fact, the dependence of $(\zeta, \bar{\zeta})$ is reminiscent of a dimensionally reduced 4-dimensional theory. Fixing the dependence on $(\zeta, \bar{\zeta})$ would then correspond to choosing a particular compactification. However, it may be that other choices are possible.

The treatment of conformal invariance and the construction of a conformal gauge theory presented here can be generalized to the case of the superconformal algebra. It is obviously important in the context of the spinning string. Full results have been obtained in collaboration with R. Gastmans, A. Sevrin, W. Troost and A. Van Proeyen, and will appear in a joint publication [11].

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