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ON THE FORM OF THE SUPERSYMMETRY ALGEBRA FOR MASSIVE FREE FIELDS

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It is shown that the supersymmetry algebra for massive free fields has to contain the translations — the absence of translations implies vanishing of supersymmetry altogether.

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1. *Introduction*

Some time ago P. K. Townsend wrote the paper [1] entitled "Local supersymmetry without gravity". He was able to give an example of lagrangian realizing local supersymmetry transformations, but with a peculiar feature of these transformations; the point is that the anticommutator of supersymmetry transformations Q_α did not produce the translations P_μ, and consequently this model does not contain gravity. At this point appears the discrepancy with the Haag-Łopuszański-Sohnius theorem [2] which states that in the massive field theory the anticommutator of supersymmetry generators Q_α and $Q^+_{\dot\alpha}$ has to have the form

$$\{Q_\alpha, Q^+_{\dot\beta}\} = \sigma^\mu_{\alpha\dot\beta} P_\mu \equiv P_{\alpha\dot\beta}. \tag{1}$$

This theorem is valid only for interacting fields and the model of Townsend is effectively free, as he showed in [1].

In this note we are going to look for the form of supersymmetry algebra for massive free fields. Especially, we are interested in the possibility of ruling out the translation operators at r.h.s. of (1). We will show that the form of supersymmetry transformations for the free massive fields is so restricted that it is not possible to get rid of P_μ on the r.h.s. of (1). Following sections contain the proof for the case of scalar–spinor and scalar–spinor–vector fields. The method of investigation is very similar to the one invented by Łupuszański [3].

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2. The case of scalar and spinor fields

Let us consider the model consisting of s scalar real and f spinor Majorana fields subjected to the free equations of motion and translationally covariant:

$$(\Box+m^2)\phi^{(i)}(x) = 0, \tag{2}$$

$$i[P_\mu, \phi^{(i)}(x)] = \partial_\mu\phi^{(i)}(x), \quad i = 1, 2, \ldots, s, \tag{3}$$

$$\partial^{\alpha\dot\beta}\psi_\alpha^{(a)}(x) = m\psi^{(a)\dot\beta+}(x), \tag{4}$$

$$i[P_\mu, \psi_\alpha^{(a)}(x)] = \partial_\mu\psi_\alpha^{(a)}(x), \quad a = 1, 2, \ldots, f. \tag{5}$$

The most general supersymmetry transformations preserving the equations of motion have the form

$$i[Q_\alpha, \phi^{(i)}(x)] = m\sum_{a=1}^{f} c^{ai}\psi_\alpha^{(a)}(x), \tag{6}$$

$$\{Q_\alpha, \psi_\beta^{(a)}(x)\} = -\varepsilon_{\alpha\beta}m\sum_{i=1}^{s} c^{ai}\phi^{(i)}(x). \tag{7}$$

Here c_{ai} are arbitrary c-numbers. From (4) and (7) we can compute the action of $Q_{\dot\alpha}^+$ on the ψ_β:

$$\{Q_{\dot\alpha}^+, \psi_\beta^{(a)}(x)\} = \sum_{i=1}^{s} \bar c^{ai}\partial_{\beta\dot\alpha}\phi^{(i)}(x). \tag{8}$$

Now we can compute the action of $\{Q_\alpha, Q_{\dot\beta}^+\}$ on scalar and spinor fields

$$i[\{Q_\alpha, Q_{\dot\beta}^+\}, \phi^{(i)}(x)] = m\sum_{k=1}^{s}\{\sum_{a=1}^{f}(\bar c^{ai}c^{ak}+c^{ai}\bar c^{ak})\}\partial_{\alpha\dot\beta}\phi^{(k)}(x), \tag{9}$$

$$i[\{Q_\alpha, Q_{\dot\beta}^+\}, \psi_\gamma^{(a)}(x)] = m\sum_{b=1}^{f}\{(\sum_{i=1}^{s} c^{ai}\bar c^{bi})\partial_{\alpha\dot\beta}\psi_\gamma^{(b)}(x)$$

$$+\sum_{i=1}^{s}(\bar c^{ai}c^{bi}-c^{ai}\bar c^{bi})\partial_{\gamma\dot\beta}\psi_\alpha^{(b)}(x)\}. \tag{10}$$

The r.h.s. of (9) and the first term on the r.h.s. of (10) express the action of P_μ (see (3) and (5)). Let us denote

$$A^{ik} \equiv \sum_{a=1}^{f}(\bar c^{ai}c^{ak}+c^{ai}\bar c^{ak}), \tag{11}$$

$$B^{ab} \equiv \sum_{i=1}^{s} c^{ai}\bar c^{bi}. \tag{12}$$

In order to get rid of the action of P_μ we have to put $A^{ik} = 0$ and $B^{ab} = 0$. In particular the diagonal elements of these matrices have to vanish, what leads to the conclusion that

$c^{ai} = 0, a = 1, \ldots, f, i = 1, \ldots, s$. But it means that the action of supersymmetry transformations also dissapears. In this way we obtain a rather surprising result that the supersymmetry algebra for free fields *has to contain the translation*. It is known that free fields have infinite number of ordinary (i.e. nonsupersymmetric) symmetries [4].

Let us remark that the anticommutator of Q's rotates the fields ϕ and ψ — so at first sight it seems to be the mixture of P_μ and internal symmetries. However, from (11) and (12) it follows that the matrices A and B are symmetric and hermitean, respectively. (The reality of A^{ik} is connected with the reality of fields $\phi^{(i)}$). So there exist matrices S and F which diagonalizes A and B, respectively, and we can write

$$i[\{Q_\alpha, Q_{\dot\beta}^+\}, \Phi^{(i)}(x)] = m\lambda^{(i)}\partial_{\alpha\dot\beta}\Phi^{(i)}(x),$$

$$i[\{Q_\alpha, Q_{\dot\beta}^+\}, \psi_\gamma^{(a)}(x)] = m\mu^{(a)}\partial_{\alpha\dot\beta}\Psi_\gamma^{(a)} + \sum_{b=1}^{f} \beta^{ab}\partial_{\gamma\dot\beta}\Psi_\alpha^{(b)},$$

where

$$\Phi^{(i)} = \sum_{k=1}^{s} (S^{-1})^{ik}\phi^{(k)}, \qquad \Psi_\alpha^{(a)} = \sum_{b=1}^{f} (F^{-1})^{ab}\Psi_\alpha^{(b)}.$$

and $\lambda^{(i)}$, $\mu^{(a)}$ are eigenvalues of A and B. So it turns out that the mixture of P_μ with internal symmetry is spurious. The above rotations do not violate the fact that fields are free, but the fields $\Psi_\alpha^{(a)}$ does not longer fulfil the Majorana condition, as $\partial^{\alpha\dot\beta}\Psi_\alpha^{(a)} \neq m\Psi^{\dot\beta +}$. Let us also remark that the transformations (6) and (7) does not require the equality between the numbers of degrees of freedom of scalar and spinor fields because some of the eigenvalues $\lambda^{(i)}$ and $\mu^{(a)}$ can be zeros and the arguments of [3] are not applicable here.

3. *The case of scalar, spinor and vector fields*

The anticommutator $\{Q_\alpha, Q_{\dot\beta}^+\}$ is a vector and the r.h.s. of it should be a combination of vectorial generators. In order to achieve that the translations in the supersymmetry algebra are absent we should introduce generators of vectorial character different from P_μ. It is known [5] that the free scalar and vector fields are invariant under the following action of a vectorial generator B_μ

$$i[B_\mu, \phi] = V_\mu, \tag{13}$$

$$i[B_\mu, V_\nu] = \left(g_{\mu\nu} + \frac{\partial_\mu\partial_\nu}{m^2}\right)\phi. \tag{14}$$

So we hope that the introducing of vector fields should give the richer structure of the supersymmetry algebra. The operator B_μ is generated by the following conserved current

$$j_{\mu\nu} = V_\nu\partial_\mu\phi - \phi\partial_\mu V_\nu.$$

Because this current is translationally covariant

$$i[P_\lambda, j_{\mu\nu}(x)] = \partial_\lambda j_{\mu\nu}.$$

the generator $B_\mu = \int d^3x j_{0\mu}$ commutes with P_μ.

Let us assume that beside scalar and spinor fields we have also v vector fields $V_{\alpha\dot{\beta}} \equiv (\sigma^\mu)_{\alpha\dot{\beta}} V_\mu$ fulfiling the following equations

$$(\Box + m^2) V^{(t)}_{\alpha\dot{\beta}}(x) = 0, \tag{15}$$

$$\partial^{\alpha\dot{\beta}} V^{(t)}_{\alpha\dot{\beta}}(x) = 0, \tag{16}$$

$$i[P_\mu, V^{(t)}_{\alpha\dot{\beta}}(x)] = \partial_\mu V^{(t)}_{\alpha\dot{\beta}}(x). \tag{17}$$

The most general supersymmetry transformations of scalar, spinor and vector fields being the symmetries of equations (2), (4), (15) and (16) have the following form

$$i[Q_\alpha, \phi^{(i)}(x)] = m \sum_{a=1}^{f} c^{ai} \psi^{(a)}_\alpha(x), \tag{18}$$

$$i[Q_\alpha, V^{(t)}_{\beta\dot{\delta}}(x)] = -\sum_{a=1}^{f} a^{at} \partial_{\beta\dot{\delta}} \psi^{(a)}_\alpha(x) + 2 \sum_{a=1}^{f} a^{at} \partial_{\alpha\dot{\delta}} \psi^{(a)}_\beta(x), \tag{19}$$

$$\{Q_\alpha, \psi^{(a)}_\beta(x)\} = -\varepsilon_{\alpha\beta} m \sum_{i=1}^{s} c^{ai} \phi^{(i)}(x) + \sum_{t=1}^{v} \partial^{\dot{\delta}}_\alpha V^{(t)}_{\beta\dot{\delta}}(x), \tag{20}$$

where c^{ai} and a^{at} are arbitrary coefficients. Now we can derive the action of $\{Q_\alpha, \overset{+}{Q}_{\dot{\beta}}\}$ on the fields.

After a tedious computations we obtain:

$$i[\{Q_\alpha, \overset{+}{Q}_{\dot{\beta}}\}, \phi^{(i)}(x)] = m \sum_{k=1}^{s} \{\sum_{a=1}^{f} (\bar{c}^{ai} c^{ak} + c^{ai} \bar{c}^{ak}) \partial_{\alpha\dot{\beta}} \phi^{(k)}(x)\}$$
$$+ m^2 \sum_{t=1}^{v} \{\sum_{a=1}^{f} (c^{ai} \bar{a}^{at} + \bar{c}^{ai} a^{at}) V^{(t)}_{\alpha\dot{\beta}}(x)\}, \tag{21}$$

$$i[\{Q_\alpha, \overset{+}{Q}_{\dot{\beta}}\}, \psi^{(a)}_\gamma(x)] = m \sum_{b=1}^{f} \{(\sum_{i=1}^{s} (c^{ai} \bar{c}^{bi}) + \sum_{t=1}^{v} (a^{at} \bar{a}^{bt}$$
$$+ 2\bar{a}^{at} a^{bt})) \partial_{\alpha\dot{\beta}} \psi^{(b)}_\gamma(x)\} + m \sum_{b=1}^{f} \{(\sum_{i=1}^{s} (c^{ai} \bar{c}^{bi} - \bar{c}^{ai} c^{bi})$$
$$+ \sum_{t=1}^{v} (a^{at} \bar{a}^{bt} - \bar{a}^{at} a^{bt})) \partial_{\gamma\dot{\beta}} \psi^{(b)}_\alpha(x)\}, \tag{22}$$

$$i[\{Q_\alpha, \overset{+}{Q}_{\dot{\beta}}\}, V^{(t)}_{\gamma\dot{\delta}}(x)] = m \sum_{t=1}^{v} \{\sum_{a=1}^{f} (\bar{a}^{ar} a^{at} + a^{ar} \bar{a}^{at}) \partial_{\alpha\dot{\beta}} V^{(t)}_{\gamma\dot{\delta}}(x)\}$$

$$+(\partial_{\alpha\dot\beta}\partial_{\gamma\dot\delta}+2m^2\varepsilon_{\alpha\gamma}\varepsilon_{\dot\beta\dot\delta})\sum_{i=1}^{s}\{\sum_{a=1}^{f}(\bar a^{ar}c^{ai}+a^{ar}\bar c^{ai})\phi^{(i)}\}$$

$$+m\sum_{t=1}^{v}\{\sum_{a=1}^{f}(\bar a^{ar}a^{at}-a^{ar}\bar a^{at})(\partial_{\gamma\dot\beta}V^{(t)}_{\alpha\dot\delta}-\partial_{\alpha\dot\delta}V^{(t)}_{\gamma\dot\beta})\}. \quad (23)$$

First terms in the above formulas describe the action of translations. In (21) and (23) we can also recover terms expressing the action of B_μ defined by (13) and (14)[1]. If we demand that the translations should be absent in the supersymmetry algebra we are confronted with the fact that c^{ai} and a^{at} have to vanish, what in turn leads to the vanishing of supersymmetry at all.

Let us also remark that, as in the previous section, the matrices rotating the fields and acting together with translations can be diagonalized.

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[1] Let us note that $\sigma^{\mu}_{\alpha\dot\beta}(\sigma_\mu)_{\gamma\dot\delta} = 2\varepsilon_{\alpha\gamma}\varepsilon_{\dot\beta\dot\delta}$.