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EQUIVALENCE OF THE L-DIMINISHING OPERATOR AND THE HILL-WHEELER PROJECTION TECHNIQUE AND ITS APPLICATION TO THE FIVE-DIMENSIONAL HARMONIC OSCILLATOR*

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In the paper, a relation between the Hill-Wheeler integral and the angular momentum lowering operator is derived and used to construct the orthonormal physical basis for the Bohr-type and IBM collective Hamiltonians.

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1. Introduction

The Hill-Wheeler projection technique has been exploited in nuclear physics since 1953 [1]. The method permits us to obtain all the states required for a physical system by proper choice of a small set of states called "intrinsic states". The projection method is especially useful for the construction of the physical representation bases of a Hamiltonian symmetry group. In this way Elliot has constructed the basis for the symmetry SU(3) [2]. Following these works, in two successive papers by Kemmer, Pursey and Williams the non-orthogonal basis, in the form of the Hill-Wheeler integral, for the symmetric "non-physical" representations of SO(5) group, has been constructed [3]. As is known, over twenty years ago Bohr and Mottelson [4, 5] discussed the quadrupole vibrations of the liquid drop in the quantum-mechanical picture. This problem provided the basis for the introduction of the collective degrees of freedom in the description of nuclear dynamics. But, the explicit form of all eigenstates of the Bohr collective Hammiltonian, i.e. the basis for the symmetric physical representation of SO(5) group, has not been recognized until 1976 when four rather extensive papers on this problem were published [6–9]. One of them,

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written by Corrigan, Margetan and Williams [6] gives an exact solution of the Bohr Hamiltonian using the Hill-Wheeler integral. This solution is strictly related to the construction of the basis for SO(5) symmetry described in the present paper.

Another method has been applied in the later papers [10, 11]. L-diminishing angular momentum operator has been defined and used to obtain an orthonormal basis for the five dimensional harmonic oscillator in both, boson and Bohr collective variable pictures. In the present paper a connection between the Hill-Wheeler projection operator and the angular momentum lowering operator is studied and applied to construct another, simplified version of the basis. The basis vectors, it is worthwhile to note, are the eigenvectors for the "vibrational limit" of the Interacting Boson Model Hamiltonian [13] and small "anharmonicities" due to breaking of SU(5) dynamical symmetry can be easily treated by means of the perturbation theory. On the other hand the basis can be directly applied to diagonalize the Bohr collective Hamiltonian with deformation dependent mass parameters and potential energy calculated either from the cranking approach [14] or the generator coordinate method [15].

In general, the derived relation between the L-lowering operator method and the Hill-Wheeler integral (the projector) can be used to obtain all states required (with definite angular momentum quantum numbers) for a large class of physical systems.

2. *Equivalence of L-lowering operator and the Hill-Wheeler integral*

The angular momentum lowering operator $\hat{O}(L'lLM)$ has been introduced in Appendix A of [10]. The operator is defined by the equation

$$\hat{O}(L'lLM)\,|L'M' = L'\rangle = |LM\rangle, \quad (L' \geqslant L), \tag{1}$$

here L, L' and M, M' are the total angular momentum and its third component quantum numbers, respectively; l is a nonnegative arbitrary integer. It has been proved [10] that the operator

$$\hat{O}(L'lLM) = \sum_{m=L-L'}^{l} \beta_m(L'lL)\,(L_-)^{m+L'-M}(L_+)^{m+l}T^{(l)}_{-l}, \tag{2}$$

where $T^{(l)}$ is an irreducible tensor operator of the rank l under rotational transformations and $\beta_m(L'lL)$ are given by

$$\beta_m(L'lL) = \frac{(-1)^m}{(m+L'-L)!(m+L'+L+1)!}, \tag{3}$$

is of that property. L_-, L_+ with the additional operator L_0, stand for generators of the angular momentum group SO(3). The generators satisfy the known commutation relations

$$[L_0, L_+] = L_+, \quad [L_0, L_-] = -L_-, \quad [L_+, L_-] = 2L_0. \tag{4}$$

Equation (1) with the operator $\hat{O}$ given by (2) can be written in the form

$$|LM\rangle = \sum_{m=L-L'}^{l} \beta_m(L'lL)\,(L_-)^{m+L'-M}(L_+)^{m+l}T^{(l)}_{-l}|L'L'\rangle$$

$$= \sum_{m,I} \beta_m(L'lL)\,(l-lL'L'|I,\,L'-l)\,(L_-)^{m+L'-M}(L_+)^{m+l}\cdot(T^{(l)}\otimes|L'\rangle)^{I}_{L'-l}, \tag{5}$$

where $(l_1m_1l_2m_2|lm)$ denotes the Clebsch-Gordan coefficients for the group SO(3). After straightforward calculations, by making use of the following expressions:

$$(L_+)^pT^{(l)}_m|0\rangle = \left\{\frac{(l+m+p)!(l-m)!}{(l-m-p)!(l+m)!}\right\}^{1/2} T^{(l)}_{m+p}|0\rangle, \tag{6a}$$

$$(L_-)^pT^{(l)}_m|0\rangle = \left\{\frac{(l-m+p)!(l+m)!}{(l+m-p)!(l-m)!}\right\}^{1/2} T^{(l)}_{m-p}|0\rangle \tag{6b}$$

and

$$\sum_m \beta_m(L'lL)\frac{(I+L'+m)!}{(I-L'-m)!} = \delta_{IL}\frac{(-1)^{L-L'}}{2L+1}, \tag{6c}$$

where the last equation is a direct consequence of the construction of L-diminishing operator $\hat{O}(L'lLM)$ [10], we get

$$|LM\rangle = \frac{(-1)^{L-L'}}{2L+1}\left\{\frac{(L-L'+l)!(L-M)!}{(L+L'-l)!(L+M)!}\right\}^{1/2}$$

$$\times \sum_{M_1M_2}(l-lL'L'|L,\,L'-l)\,(lM_1L'M_2|LM)T^{(l)}_{M_1}|L'M_2\rangle. \tag{7}$$

Using standard properties of the Wigner functions $D^{(L)}_{M_1M_2}(\Omega)$, product of two Clebsch--Gordan coefficients can be replaced by the integral [12]

$$(L_1M_1L_2M_2|L'Q)\,(L_1N_1L_2N_2|L'P)$$

$$= (2L'+1)\int_{SO(3)} D^{(L_1)}_{M_1N_1}(\Omega)D^{(L_2)}_{M_2N_2}(\Omega)D^{(L')*}_{QP}(\Omega), \tag{8}$$

where

$$\int_{SO(3)} \equiv \frac{1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta\,\sin\beta \int_0^{2\pi} d\gamma.$$

Then, the combination of equations (7) and (8) yields

$$|LM\rangle = (-1)^{L-L'}\left\{\frac{(L-L'+l)!(L-M)!}{(L+L'-l)!(L+M)!}\right\}^{1/2}$$

$$\times \int_{SO(3)} D^{(L)*}_{M,L'-l}(\Omega)\hat{R}(\Omega)T^{(l)}_{-l}|L'L'\rangle, \tag{9}$$

where $\hat{R}(\Omega)$ is the rotation operator by the Euler angles (α, β, γ) [12]. The last formula is of the Hill-Wheeler type integral. Comparison of equations (1) and (9) proves that, in this case, the Hill-Wheeler projected technique is equivalent to L-diminishing operator method. The vectors $T^{(l)}_{-l}|L'L'\rangle$ play a role of the "intrinsic states" for a physical system to be considered.

3. *The orthonormal physical basis for the five-dimensional harmonic oscillator*

In the papers [10, 11] an orthonormal basis for the five-dimensional harmonic oscillator has been constructed in a boson picture by means of group theory methods. The basis of the form

$$|vxLM\rangle = \sum_{n_{-2},\ldots,n_2} \langle n_{-2}n_{-1}n_0n_1n_2|vxLM\rangle_0|n_{-2}\ldots n_2\rangle, \tag{10}$$

where v is the boson seniority number and x is the additional quantum number interpreted as a maximal number of boson triplets coupled to $L = 0$ contained in the state. For v, x and L the following conditions must be fulfilled:

$$0 \leqslant x \leqslant \tfrac{1}{3}v, \quad v-3x \leqslant L \leqslant 2(v-3x), \quad L \neq 2(v-3x)-1.$$

The "ket"

$$|n_{-2}n_{-1}n_0n_1n_2\rangle \equiv (n_{-2}!\ldots n_2!)^{-1/2}(d^+_{-2})^{n_{-2}}(d^+_{-1})^{n_{-1}}\ldots(d^+_2)^{n_2}|0\rangle$$

denotes the decoupled harmonic oscillator basis and d^+_m is the quadrupole boson creation operator. C-number coefficients $\langle n_{-2}\ldots n_2|vxLM\rangle_0$ are given by the recurrence formulae [10]. Following this work, we shall propose another version of the orthonormalized basis constructed by the Gramm orthonormalizing process. First, making use of the method described in [10] we get the nonorthogonal basis

$$|vxLM\rangle = \sum_{n_{-2},\ldots,n_2} \langle n_{-2}\ldots n_2|vxLM\rangle\, |n_{-2}\ldots n_2\rangle, \tag{11}$$

where the transformation coefficients are simple extensions of those obtained in [10] on arbitrary M, and are given by the equation

$$\langle n_{-2}\ldots n_2|vxLM\rangle = (2L+1)(n_{-2}!\ldots n_2!)^{1/2}(\sqrt{6})^{n_0}[(L-v+3x)!(L+v-3x)!$$
$$(L-M)!\,(L+M)!]^{1/2}\sum_k \sum_{\substack{p_1,\ldots,p_8\\ q_1,\ldots,q_5}} \binom{v-x}{p_1p_2\ldots p_8}\binom{x}{q_1\ldots q_5}$$
$$\times 2^{p_1+2p_3+2p_6+p_8+q_2+q_4}\frac{(-1)^{k+p_1+p_3+p_5+p_7+q_2+q_4}}{k!(L-v+3x-k)!(L-M-k)!(M+v-3x+k)!}$$
$$\times\frac{(k+p_2+p_3+2p_5+2p_6+2p_7+3p_8)!\,(L-M-k+p_2+p_6+p_8+q_2+2q_3+3q_4+4q_5)!}{(L+2v-p_4-p_5+1)!} \tag{12}$$

The following restrictions are imposed on the summations in (11) and (12):

$$\sum_{i=-2}^{2} n_i = v, \quad p_1+q_1 = n_{-2}, \quad p_8+q_5 = n_2.$$

$$\sum_{i=-2}^{2} in_i = M, \quad p_3+p_4+q_2 = n_{-1},$$

$$\sum_{i=1}^{8} p_i = v-x, \quad p_2+p_7+q_3 = n_0,$$

$$\sum_{i=1}^{5} q_i = x, \quad p_5+p_6+q_4 = n_1,$$

We are now in a position to find an equivalent form of the basis (11) by making use of equation (9). From straightforward but tedious calculations we get

$$\sum_{m=L}^{2v} \beta_m(L)\,(L_-)^{m-M}(L_+)^{m+v-3x}[(d^+ \otimes \tilde{d})^3_{-3}]^{v-x}(d_2^+)^v\,|0\rangle$$
$$= \frac{(-1)^L}{2L+1}\,\frac{v!}{2^{(v-x)/2}x!}\cdot\left\{\frac{(L-M)!\,(L+v-3x)!}{(L+M)!\,(L-v+3x)!}\right\}^{1/2}$$
$$\times \sum_{n_{-2},\ldots,n_2} \langle n_{-2}\ldots n_2|vxLM\rangle\,|n_{-2}\ldots n_2\rangle, \tag{13}$$

where

$$\beta_m(L) = \frac{(-1)^m}{(m-L)!\,(m+L+1)!}.$$

Then, taking equation (9) for $L' = 2v$ and $T_{-l}^{(l=3v-3x)} = [(d^+ \otimes \tilde{d})^3_{-3}]^{v-x}$ we have

$$\sum_{m=L}^{2v} \beta_m(L)\,(L_-)^{m-M}(L_+)^{m+v-3x}[(d^+ \otimes \tilde{d})^3_{-3}]^{v-x}(d_2^+)^v\,|0\rangle$$
$$= (-1)^L\left\{\frac{(L+v-3x)!\,(L-M)!}{(L-v+3x)!\,(L+M)!}\right\}^{1/2}\frac{v!}{2^{(v-x)/2}x!}$$
$$\times \int_{SO(3)} D^{(L)*}_{M,-v+3x}(\Omega)\hat{R}(\Omega)\,(d^+_{-1})^{v-x}(d_2^+)^x\,|0\rangle. \tag{14}$$

Comparison of equations (13) and (14) permits us to express the states (11) by the Hill-Wheeler type integral

$$|vxLM\rangle = \sum_{n_{-2},\ldots,n_2} \langle n_{-2}\ldots n_2|vxLM\rangle\,|n_{-2}\ldots n_2\rangle$$
$$= (2L+1)\int_{SO(3)} D^{(L)*}_{M,3x-v}(\Omega)\hat{R}(\Omega)\,(d^+_{-1})^{v-x}(d_2^+)^x\,|0\rangle. \tag{15}$$

Now, we shall proceed to obtain the overlap coefficients needed for orthonormalization of the basis (11) by means of the Gramm determinants. The overlaps are defined as the scalar product

$$c^{vl}_{x'x} = \langle vx'LM|vxLM\rangle. \tag{16}$$

The coefficients $c^{vL}_{x'x}$ can be effectively calculated by application of equation (15) to the scalar product (16)

$$\begin{aligned} c^{vL}_{x'x} &= (2L+1) \int_{SO(3)} D^{(L)*}_{M,3x-v}(\Omega) \langle vx'LM|\hat{R}(\Omega)\,|vx\rangle \\ &= (2L+1)\sum_K \int_{SO(3)} D^{(L)*}_{M,3x-v}(\Omega) D^{(L)}_{MK}(\Omega)\, \langle vx'LK|vx\rangle \\ &= \langle vxL,\, 3x-v|vx\rangle, \end{aligned} \tag{17}$$

where

$$|vx\rangle = (d^{+}_{-1})^{v-x}(d^{+}_{2})^{x}\,|0\rangle.$$

Insertion of the complete basis $|n_{-2} \ldots n_2\rangle$ between "bra" $\langle vx'L, 3x-v|$ and "ket" $|vx\rangle$ gives the simple result

$$c^{vL}_{x'x} = \sqrt{(v-x)!x!}\, \langle 0, v-x, 0, 0, x|vx'L, 3x-v\rangle. \tag{18}$$

The last coefficient is a special type of analytical formula (12). In Appendix A we find another expression for the coefficient .

$$\begin{aligned} \langle 0, v-x', 0, 0, x'|vxL, 3x'-v\rangle &= (2L+1)\,[(L-v+3x)!\,(L+v-3x)! \\ &\times (L-v+3x')!\,(L+v-3x')!]^{1/2} \cdot \sqrt{(v-x')!x'!} \\ &\times \sum_{s,k} (-1)^{x'-x+s} 2^{x'-x+2s} \binom{v-x}{x'-x+s}\binom{x}{s} \\ \times \frac{(-1)^k (3s+k)!\,(v+L+x'-2s-k)!}{k!\,(L+v-3x-k)!\,(L-v+3x'-k)!\,(3x-3x'+k)!\,(v+L+x'+s+1)!} \\ &\times {}_2F_1(x'-v+s,\, 3s+k+1;\, v+L+x'+s+2;\, 4). \end{aligned} \tag{19}$$

This equation is useful for the small quantum number x or x' (the following symmetry relation holds $c^{vL}_{x'x} = c^{vL}_{xx'}$). In practice, for the seniority number $v < 50$ the additional quantum number $x < 16$. An orthonormalizing process must be carried out only for the states with the same quantum numbers v, L, M but different x and x'. For the seniority number $v < 50$ maximal number of states with the same v, L and M is less than 9 i.e. the Gramm determinants are rather small and can be calculated effectively.

To sum up we give the explicit form of the orthonormal basis with arbitrary v, x, L

and M (in $c^{vL}_{xx'}$, we omit the indices v and L):

$$|vxLM\rangle_0 = (G_{x-1}\cdot G_x)^{-1/2}\begin{vmatrix} |\phi_0\rangle & |\phi_1\rangle & \dots & |\phi_x\rangle \\ c_{00} & c_{01} & \dots & c_{0x} \\ \cdot & \cdot & \cdot & \cdot \\ c_{x-1,0} & c_{x-1,1} & \dots & c_{x-1,x} \end{vmatrix},$$

where

$$G_x = \begin{vmatrix} c_{00} & c_{01} & \dots & c_{0x} \\ c_{10} & c_{11} & \dots & c_{1x} \\ \cdot & \cdot & \cdot & \cdot \\ c_{x0} & c_{x1} & \dots & c_{xx} \end{vmatrix} \qquad (20)$$

and

$$|\phi_{x'}\rangle = |vx'LM\rangle.$$

In addition, from equation (20) we get the transformation coefficients for the orthonormal basis written in the form (10)

$$\langle n_{-2}\dots n_2|vxLM\rangle_0 = (G_{x-1}G_x)^{-1/2}\begin{vmatrix} \langle n|\phi_0\rangle & \langle n|\phi_1\rangle & \dots & \langle n|\phi_x\rangle \\ c_{00} & c_{01} & \dots & c_{0x} \\ \cdot & \cdot & \cdot & \cdot \\ c_{x-1,0} & c_{x-1,1} & \dots & c_{x-1,x} \end{vmatrix}, \qquad (21)$$

where

$$\langle n|\phi_{x'}\rangle = \langle n_{-2}\dots n_2|vx'LM\rangle.$$

In the references [10, 11] the orthogonal basis for the five dimensional harmonic oscillator has been obtained using a time consuming recurrence procedure. The coefficients (21) allow for more efficient numerical calculations. The new version of the orthogonal basis for the Bohr collective Hamiltonian can be obtained directly from [11] replacing the coefficient $\langle n_{-2}, 0, n_0, 0, n_2|vvxLK\rangle_0$, (10), by the corresponding coefficients (21) and can be written as

$$\Phi_{NvxLM}(\Omega,\beta,\gamma) = F_{Nv}(\beta)\sum_K g_{vxLK}(\gamma)D^{(L)*}_{MK}(\Omega), \qquad (22)$$

where

$$F_{Nv}(\beta) = [2(\tfrac{1}{2}(N-v))!]^{1/2}\,[\Gamma(\tfrac{1}{2}(N+v+5))]^{-1/2}$$
$$\times\beta^v L^{v+3/2}_{(N-v)/2}(\beta^2)\exp(-\tfrac{1}{2}\beta^2), \qquad (23)$$

$$g_{vxLK}(\gamma) = \left[\frac{\Gamma(v+\frac{5}{2})}{\Gamma(\frac{5}{2})}\right]^{1/2}(1+\delta_{K0})^{-1}\sum_{n_0} 2^{n_0/2}$$
$$\times(n_{-2}!n_0!n_2!)^{-1/2}\langle n_{-2}0n_00n_2|vxLK\rangle_0(\sin\gamma)^{v-n_0}(\cos\gamma)^{n_0} \qquad (24)$$

and $D^{(L)}_{MK}(\Omega)$ are the Wigner rotational functions. The summation is taken only with respect to n_0 because

$$v = n_{-2}+n_0+n_2, \quad K = 2(n_2-n_{-2}),$$
$$n_{-2} = \tfrac{1}{4}(2v-K-2n_0), \quad n_2 = \tfrac{1}{4}(2v+K-2n_0). \tag{25}$$

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APPENDIX A

In order to derive the transformation coefficients $\langle 0, v-x', 0, 0, x'|vxL, 3x'-v\rangle$ we make use of the formula (B6) [10]

$$\langle n_{-2} \dots n_2|vxLM\rangle = \tfrac{1}{2}(2L+1)(n_{-2}! \dots n_2!)^{1/2} \sum_{\substack{r_i,s_i\\ r_i+s_i=n_i}} \binom{v-x}{r_{-2} \dots r_2}$$
$$\times \binom{x}{s_{-2} \dots s_2} \int_0^\pi d\beta \sin\beta (d^{(2)}_{-1-2}(\beta))^{r_{-2}} \dots (d^{(2)}_{-12}(\beta))^{r_2}$$
$$\times (d^{(2)}_{2-2}(\beta))^{s_{-2}} \dots (d^{(2)}_{22}(\beta))^{s_2} d^{(L)}_{3x-v,M}(\beta) \tag{A1}$$

Taking explicitly d-functions we get

$$\langle n_{-2} \dots n_2|vxLM\rangle = \tfrac{1}{2}(2L+1)(n_{-2}! \dots n_2!)^{1/2} \sum_{\substack{r_i,s_i\\ r_i+s_i=n_i}} \binom{v-x}{r_{-2} \dots r_2}\binom{x}{s_{-2} \dots s_2}$$
$$(-1)^{r_{-2}+s_{-1}+s_1+v-3x+M}(\sqrt{6})^{n_0}2^{r_{-2}+r_2+s_{-1}+s_1}.$$
$$\times [(L+v-3x)!\,(L-v+3x)!\,(L+M)!\,(L-M)!)]^{1/2}$$
$$\times \sum_k \frac{(-1)^k}{k!\,(L+v-3x-k)!\,(L+M-k)!\,(3x-v-M+k)!}$$
$$\times [\cos\tfrac{1}{2}\beta]^{2a}[\sin\tfrac{1}{2}\beta]^{2b}(2\cos\beta-1)^{r_{-1}}(\cos\beta)^{r_0}(2\cos\beta+1)^{r_1} \tag{A2}$$

(The coefficients a and b are defined below). Application of equation:

$$v-3x+M = -r_{-2}+r_0+2r_1+3r_2-4s_{-2}-3s_{-1}-2s_0-s_1 \tag{A3}$$

to equation (A.2) gives

$$\langle n_{-2} \dots n_2|vxLM\rangle = (2L+1)[(L-v+3x)!\,(L+v-3x)!\,(L+M)!\,(L-M)!]^{1/2}$$
$$\times (\sqrt{6})^{n_0}(n_{-2}! \dots n_2!)^{1/2} \sum_{s_{-2}\dots s_2} (-1)^{n_2-s_2+n_0-s_0} \binom{v-x}{n_{-2}-s_{-2}, \dots, n_2-s_2}$$

$$\times \begin{pmatrix} x \\ s_{-2} \dots s_2 \end{pmatrix} 2^{n_{-2}+n_2-s_{-2}+s_{-1}+s_1-s_2}$$

$$\times \sum_k \frac{(-1)^k}{k!\,(L+v-3x-k)!\,(L+M-k)!\,(3x-v-M+k)!}$$

$$\times \int_0^1 t^b(1-t)^a(1-4t)^{n_{-1}-s_{-1}}(1-2t)^{n_0-s_0}(3-4t)^{n_1-s_1}dt, \quad \text{(A4)}$$

where

$$a = v+L-x+n_2-2s_{-2}-s_{-1}+s_1+s_2-k,$$

$$b = n_{-2}+3s_{-2}+3s_{-1}+2s_0+s_1+k$$

and in the integral the variable is changed to $t = \sin^2 \frac{1}{2}\beta$. For the special case of the coefficients (A4), needed for the overlaps, we can simplify the formula (A4) by recognizing the hypergeometric function

$$\int_0^1 t^b(1-t)^a(1-4t)^{v-x'-s_{-1}}dt = \frac{\Gamma(b+1)\Gamma(a+1)}{\Gamma(a+b+2)}\, {}_2F_1(x'-v+s_{-1}, b+1, a+b+2; 4).$$

Then, the resulting expression is the desired relationship (19).

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