STABILITY PROBLEM IN GRAVITY AND MAGNETIC MONOPOLES IN KALUZA-KLEIN THEORY*

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The stability of the Schwarzschild instanton in a Euclidean background metric is discussed as an introduction. The method is applied to the magnetic monopole solution in 5 dimensions and it is indicated that the classical solution is unstable against small perturbations of the metric.

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1. Introduction

Schwarzschild found the solution to Einstein's equation

$$R_{\mu\nu} = 0, \tag{1}$$

for a metric around a fixed spherically symmetrical center of mass

$$ds^2 = -V^{-1}dt^2 + Vdr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) = g_{\mu\nu}dx^\mu dx^\nu, \tag{2}$$

with $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi, V^{-1} = 1-(2M/r), M(\text{cm}) = 1.3\times 10^{-52}\,\frac{\text{cm}}{\text{GeV}}\,M(\text{GeV})$.

Is the solution stable with respect to small first order departures from the Schwarzschild metric? To examine this question, the background metric is indicated by $g_{\mu\nu}$ and the small perturbation to it by $h_{\mu\nu}$. The contracted Ricci tensor is called $R_{\mu\nu}$ if calculated from $g_{\mu\nu}$ and $R_{\mu\nu}+\delta R_{\mu\nu}$ if calculated from $g_{\mu\nu}+h_{\mu\nu}$, where $\delta R_{\mu\nu}$ is

$$\delta R_{\mu\nu} = -\delta\Gamma^\beta_{\mu\nu;\beta} + \delta\Gamma^\beta_{\mu\beta;\nu}, \tag{3}$$

$$\delta\Gamma^\mu_{\beta\gamma} = \tfrac{1}{2}\, g^{\mu\nu}(h_{\beta\nu;\gamma} + h_{\gamma\nu;\beta} - h_{\beta\gamma;\nu}),$$

$$\Gamma^\varrho_{\nu\sigma} = g^{\varrho\alpha}\Gamma_{\alpha\nu\sigma} = g^{\varrho\alpha}\,\tfrac{1}{2}\,(g_{\alpha\nu,\sigma} + g_{\alpha\sigma,\nu} - g_{\nu\sigma,\alpha}). \tag{4}$$

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In order to check the stability of the solution, the metric tensor $h_{\mu\nu}$ is separated into four factors each a function of a single variable or coordinate. The separation is obtained by Regge and Wheeler [1] by generalizing the spherical harmonics (known for vectors, scalars, and spinors) to tensors. The stability of the Schwarzschild solution can be studied by taking [1–4] $\delta R_{\mu\nu} = 0$ and looking for diverging solution with a factor of the form $F_0(t) = e^{-ikt}$. They found for various angular momentum states that $\mathrm{Im}\, k = 0$ and concluded that the solutions are stable.

2. *Schwarzschild instanton*

Perry studied the stability of the Schwarzschild instanton in a Euclidean background metric [5]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \quad x^\mu = (t, r, \theta, \phi), \tag{5}$$

where

$$g_{\mu\nu} = \begin{pmatrix} V^{-1} & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2s^2 \end{pmatrix},$$

$s = \sin\theta$, $V^{-1} = 1-(2M/r)$, and found a negative mode with $l = 0$, $k = 0$.

The equations can be separated in (t, r, θ, ϕ) coordinates. We follow Ref. [1] and take $m = 0$ in which case ϕ will disappear from the equations. The even parity metric tensor is written as [1]

$$h_{ab} = \begin{pmatrix} V^{-1}H_0 & H_2 & K_0\partial_\theta & 0 \\ & VH_1 & K_1\partial_\theta & 0 \\ & & r^2K & 0 \\ & & & r^2s^2K \end{pmatrix} e^{i\omega t}Y_{l0}(\theta), \tag{6}$$

$h_{ab} = h_{ba}$.

For the case $l = 0$, the θ also disappears and this mode is described by the following simple form

$$h_{ab} = \begin{pmatrix} V^{-1}H_0 & 0 & 0 & 0 \\ & VH_1 & 0 & 0 \\ & & r^2K^2 & 0 \\ & & & r^2s^2K \end{pmatrix}.$$

The h_{ab} is chosen to satisfy the transversality and tracelessness conditions,

$$h^{AB}{}_{;B} = 0, \quad g^{AB}h_{AB} = 0, \quad A, B = 0, 1, 2, 3. \tag{7}$$

The relations $h^{1B}{}_{;B} = 0$ and $g^{AB}h_{AB} = 0$ lead to

$$\left(1-\frac{2M}{r}\right)\frac{dH_1}{dr} + \left(\frac{2}{r} - \frac{3M}{r^2}\right)H_1 - \frac{M}{r^2}H_0 + \frac{1}{r^2}(r-2M)(H_0+H_1) = 0$$

or

$$H_0 = \frac{-r(r-2M)}{r-3M}\frac{dH_1}{dr} - \frac{(3r-5M)}{r-3M}H_1 \tag{8}$$

and

$$K = -\tfrac{1}{2}(H_0+H_1). \tag{9}$$

Equation (3) can be written as an eigenvalue equation [6]

$$-2\delta R_{AB} = -\Box h_{AB} - 2R_{ACBD}h^{CD} = \lambda h_{AB}, \tag{10}$$

where

$$\Box h_{AB} = g^{CD}h_{AB;CD}.$$

Positive (negative) values of λ will correspond to stable (unstable) fluctuations about the monopole solution.

For the component h_{11}, we obtain

$$\Box h_{11} + 2R_{1C1D}h^{CD} = -\lambda h_{11},$$

where

$$\Box h_{11} = g^{ik}h_{11;ik} = \frac{d^2H_1}{dr^2} + V\left(\frac{2M}{r^2} + \frac{2}{Vr}\right)\frac{dH_1}{dr} + \frac{2m^2V^2}{r^2}(H_0-H_1) - \frac{4}{r^2}H_1 + \frac{4}{r^2}K,$$

$$2R_{1C1D}h^{CD} = 2\left[\frac{2MV}{r^3}VH_0 - \frac{MV}{r}\frac{K}{r^2} - \frac{MV}{r^3}K\right] = \frac{4MV}{r^3}[H_0-K],$$

so the sum after dividing by V is

$$V^{-1}\frac{d^2H_1}{dr^2} + \left[\frac{2M}{r^2} + \frac{2}{Vr}\right]\frac{dH_1}{dr} + \frac{2M^2V}{r^4}(H_0-H_1)$$
$$-\frac{4}{r^2V}(H_1-K) + \frac{4M}{r^3}(H_0-K) = -\lambda H_1. \tag{11}$$

We use Eqs (8) and (9) to eliminate K and H_0 from (11) and obtain

$$\frac{1}{V}\frac{d^2}{dr^2}H_1 + \frac{4r^2-22Mr+24M^2}{r^2(r-3M)}\frac{d}{dr}H_1 - \frac{8M}{r^2(r-3M)}H_1 = -\lambda H_1. \tag{12}$$

The eigenvalue $\lambda = -0.19\,M^{-2}$ was found numerically [5] for this $l = 0$ case.

For the $l = 1$ case, we notice from Eq. (6) that the components of h_{ab} are multiplied by $\cos\theta$ or $\sin\theta$. The equations corresponding to (8) and (11) for the $l = 1$ case and $\omega = 0$ are obtained from (6), (7), (9) and (10);

$$\frac{1}{V}\frac{dH_1}{dr} + \frac{H_1}{r}\left(\frac{2}{r} - \frac{3M}{r^2}\right) - \frac{M}{r^2}H_0 + \frac{1}{rV}(H_1+H_0) - \frac{2K}{Vr^2} = 0,$$

$$\frac{1}{V}\frac{dK}{dr} + K\left(\frac{2}{r} - \frac{2M}{r^2}\right) - \tfrac{1}{2}(H_1+H_0) = 0, \tag{13}$$

$$\frac{1}{V^2}\frac{d^2H_1}{dr^2}+\frac{2(r-M)}{Vr^2}\frac{dH_1}{dr}+H_1\left(-\frac{6}{r^2}+\frac{26M}{r^3}-\frac{30M^2}{r^4}\right)$$
$$+H_0\left(-\frac{2}{r^2}+\frac{14M}{r^3}-\frac{18M^2}{r^4}\right)-\frac{2H_1}{Vr^2}+\frac{8M}{r^4V^2}=-\lambda\frac{H_1}{V}. \tag{14}$$

The solution of Eqs (13) and (14) for the zero mode $\lambda = 0$ is given by [5, 6] $h_{ab} = \varphi_{;ab}$ where $\varphi = (r-M)\cos\theta$, or $H_1 = H_0 = \frac{M}{r^2}$, $K = \frac{M}{r}$. The solution represents one component of the translations in the x, y, z directions of the origin of the Schwarzschild instanton. On the basis of this zero mode for $l = 1$, the existence of the negative mode for $l = 0$ was aptly explained by Witten [7]. His argument follows.

The most general traceless metric perturbation that preserves the rotational and time symmetry of Eq. (5) for the $l = 0$ case is

$$h_{\mu\nu}dx^\mu dx^\nu = BV^{-1}dt^2+AVdr^2-\tfrac{1}{2}(A+B)r^2(d\theta^2+s^2d\phi^2),$$

where A and B are two arbitrary functions of r. The transversality condition relates B to A as in Eq. (8). The resulting eigenvalue equation for A is a second order differential equation similar to Eq. (12). In order to look for metric perturbations for the $l = 1$ case the appropriate substitution is

$$A \to A\cos\theta \qquad B \to B\cos\theta.$$

The eigenvalue equation for $l = 1$ differs from that of $l = 0$ case by the presence of the positive angular momentum contribution corresponding to $l(l+1)/r^2$. The existence of a zero eigenvalue for $l = 1$ discussed above implies that there is a negative eigenvalue for $l = 0$ as shown by Perry [5].

3. *Magnetic monopole in five dimensions*

We extend the analysis of the stability problem given in the previous Sections to the magnetic monopole solution.

The magnetic monopole solution [8, 9] is a generalization of the Euclidean Taub–NUT solution described by the metric g_{AB} in 5 dimensions $A, B = 0, 1, 2, 3, 5$

$$ds^2 = dt^2+V(dx^5+Ad\phi)^2+V^{-1}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2) = g_{AB}dx^Adx^B, \tag{15}$$

where $x^A = (t, r, \theta, \phi, x^5)$,

$$g_{AB} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & V^{-1} & 0 & 0 & 0 \\ 0 & 0 & r^2V^{-1} & 0 & 0 \\ 0 & 0 & 0 & (r^2s^2+A^2V^2)/V & AV \\ 0 & 0 & 0 & AV & V \end{bmatrix} \equiv \begin{bmatrix} g_{\mu\nu}+AAV & AV \\ & \\ & \\ AV & V \end{bmatrix}. \tag{16}$$

Here $V^{-1} = 1+(4M/r)$, $s = \sin\theta$, and $A_a = 4M(1-\cos\theta)$ in the northern hemisphere and $A_b = -4M(1+\cos\theta)$ in the southern hemisphere. When $dt = 0$, the resulting subspace is the Taub–NUT instanton.

We first discuss the general angular dependence of the perturbative metric h_{AB}. Take the monopole harmonic [10] in R_a (northern hemisphere) and operate with the parity operator P which has the effect of taking $\theta \to \pi-\theta$, $\phi \to \pi+\phi$:

$$(Y_{qlm})_a = \Theta_{qlm}(x)e^{i(m+q)\phi}, \qquad x = \cos\theta, \tag{17}$$

and

$$P\Theta_{qlm}(x) = \Theta_{qlm}^{l+m}(-x) = (-1)^{l+m}\Theta_{-qlm}(x). \tag{18}$$

Now note that

$$P(Y_{qlm})_a = (-1)^{l+2m+q}(Y_{-qlm})_b, \tag{19}$$

and also similarly $P(Y_{-qlm})_b = (-1)^{l+2m+q}(Y_{qlm})_a$, where subscript b indicates southern hemisphere. Hence we can form states of definite parity $Y_{qlm}^{(1)}$ and $Y_{qlm}^{(2)}$ which have parities $(-1)^{l+2m+q}$ and $(-1)^{l+2m+q+1}$, respectively:

$$Y_{qlm}^{(1)} = \frac{1}{\sqrt{2}}[(Y_{qlm})_a+(Y_{-qlm})_b] = (\Theta_{qlm}(\theta)+\Theta_{-qlm}(\theta))e^{i(q+m)\phi}, \tag{20}$$

$$Y_{qlm}^{(2)} = \frac{1}{\sqrt{2}}[(Y_{qlm})_a-(Y_{-qlm})_b] = (\Theta_{qlm}(\theta)-\Theta_{-qlm}(\theta))e^{i(q+m)\phi}. \tag{21}$$

Thus here we have two scalar monopole harmonics of parity $(-1)^{l+2m+q}$ and $(-1)^{l+2m+q+1}$, respectively, each of which can be used to generate vector and tensor monopole harmonics. For example, for parity $(-1)^{l+2m+q}$ we have with $Y_{qlm}^{(1)} = Y^{(1)}$, $Y_{qlm}^{(2)} = Y^{(2)}$.

Scalar: $Y^{(1)}$

Vector: $\dfrac{\partial}{\partial x^\mu} Y^{(1)}, \quad \varepsilon_\mu^\nu \dfrac{\partial}{\partial x^\nu} Y^2$

Tensor: $\psi_{\mu\nu} = Y_{;\mu\nu}\begin{pmatrix}\text{covariant}\\ \text{derivative}\end{pmatrix}, \quad \gamma_{\mu\nu}Y^{(1)} = (g_{\mu\nu}/r^2)Y^{(1)}$

$$X_{\mu\nu} = \tfrac{1}{2}[\varepsilon_\mu^\lambda Y_{;\lambda\nu}^{(2)}+\varepsilon_\nu^\lambda Y_{;\lambda\mu}^{(2)}] \tag{22}$$

where $\varepsilon_2^2 = \varepsilon_3^3 = 0$, $\varepsilon_3^2 = \sin\theta$, $\varepsilon_2^3 = -\dfrac{1}{\sin\theta}$, $x^2 = \theta$, and $x^3 = \phi$. Any of the terms in (22) can be multiplied by an arbitrary function of r and t, without changing its transformation property under a rotation.

We now obtain an extension of Regge-Wheeler [1] $h_{\mu\nu}$ to five dimensions. Factoring out the time dependence and Kaluza-Klein coordinate dependence: $F_1(x^0)F_2(x^5)$, with

$F_1(x^0) = e^{-ikt}$, $F_2(x^5) = e^{inx^5/R}$ we have for the case $q = l = -m = 1$ and even parity,

$$Y^{(1)}_{1,1,-1} = \text{const. (independent of } \theta \text{ and } \phi). \tag{23}$$

With the aid of (23), we obtain for h_{AB}, $(h_{AB} = h_{BA})$

$$h_{AB} = \begin{bmatrix} H_0 & H_1 & 0 & -H_3 s^2 & H_{op} \\ & \dfrac{H_2}{V} & 0 & -H_4 s^2 & H_{1p} \\ & & r^2 K\gamma_{22} & 0 & 0 \\ & & & r^2 K\gamma_{33} & -H_{3p}s^2 \\ & & & & r^2 H_{2p}\gamma_{55} \end{bmatrix}. \tag{24}$$

The tracelessness condition becomes

$$H_0 + H_2 + 2K + H_{2p} + \frac{V^2 A^2}{r^2 s^2}(K + H_{2p}) + \frac{2VH_{3p}}{r^2} A = 0. \tag{25}$$

For terms independent of A we project out the P_0 part by $2\pi \int\limits_0^\pi \sin\theta d\theta$, whereas for terms $F(A)$ that depend on A, we use

$$(F(A))_0 \equiv 2\pi \int\limits_0^{\pi/2} F(A_a) \sin\theta d\theta + 2\pi \int\limits_{\pi/2}^{\pi} F(A_b) \sin\theta d\theta. \tag{26}$$

We then note that,

$$\frac{1}{2\pi}(V^2 A^2/r^2 s^2)_0 = 2a \int\limits_0^1 dx \frac{(1-x)^2}{(1-x^2)} = 2a(2\ln 2 - 1),$$

where $a = 4M^2V^2/r^2 = (1-V)^2$.

To solve for the ten amplitudes in (24), we use Eq. (25), and equations from 10 non--diagonal $-2\delta R_{AB} = \lambda h_{AB}$ and 5 diagonal $-2\delta R_{AA} = \lambda h_{AA}$ terms given in Eq. (10). The A dependent terms of all the equations above are projected as in (26).

Of the ten amplitudes, H_0, H_1, H_2, H_{2p}, K, H_{op}, H_{1p}, H_3, H_{3p}, H_4, the amplitudes H_0, H_1, H_3, H_{op} satisfy equations which are decoupled from the other amplitudes (Ref. [11]). It can be shown that one can consistently set the amplitudes H_{1p}, H_{3p} and H_4 equal to zero. Finally, we displayed in Ref. [11] above, the tracelessness condition, the transversality condition, and the coupled equations satisfied by the amplitudes H_2, K and H_{2p}. These coupled equations are eigenvalue equations involving the eigenvalue λ. If the equations possess solutions which go to zero at infinity (or bounded at ∞), for only positive values of λ, then the classical solution will be stable, while if it possesses acceptable solutions at $r \to \infty$ for negative values of λ also, then the classical monopole solution would be unstable. Thus one is led to examine the asymptotic solutions of these equations in the region of large r.

For convenience, we reproduce here the equations of Ref. [11] with which we will be concerned in a form suitable for further work. They are with the notations $n = a(2\ln 2-1)$, $a = (1-V)^2$, $V^{-1} = 1+(4M/r)$, the tracelessness condition [12].

$$(2+n)K+(1+n)H_{2p}+H_2+H_0 = 0, \tag{27}$$

the transversality condition,

$$(1+V)(2+n)K+\{1+n-V(1-n)\}H_{2p}-(3+V)H_2-2r\frac{dH_2}{dr} = 0, \tag{28}$$

and the differential equations which can be written as

$$\frac{d^2H_0}{dr^2}+\frac{2}{r}\frac{dH_0}{dr} = -\frac{\lambda}{V}H_0, \tag{29}$$

$$\frac{d^2H_2}{dr^2}+\frac{2}{r}\frac{dH_2}{dr}-a_1H_2-a_2K-a_3H_{2p} = -\frac{\lambda}{V}H_2, \tag{30}$$

$$\frac{d^2K}{dr^2}+\frac{2}{r}\frac{dK}{dr}-b_1H_2-b_2K-b_3H_{2p} = -\frac{\lambda}{V}K, \tag{31}$$

$$\frac{d^2H_{2p}}{dr^2}+\frac{2}{r}\frac{dH_{2p}}{dr}-c_1H_2-c_2K-c_3H_{2p} = -\frac{\lambda}{V}H_{2p}. \tag{32}$$

The coefficients a_i, b_i and c_i in Eqs (30)–(32) are:

$$a_1 = \left(4V+\frac{3a}{2}\right)\Big/r^2, \quad a_2 = \left\{2-6V-3a+\left(1-3V-\frac{3a}{2}\right)n\right\}\Big/r^2,$$

$$a_3 = \left\{-2+2V+\frac{3a}{2}+n\left(1-3V-\frac{3a}{2}\right)\right\}\Big/r^2; \tag{33}$$

$$b_1 = \left(1-3V-\frac{3a}{2}\right)\Big/r^2, \quad b_2 = \{-2+4V+2a+n(3a+4V)/2\}/r^2,$$

$$b_3 = \left\{1-V-\frac{3a}{2}-n\left(1-2V-\frac{3a}{2}\right)\right\}\Big/r^2; \tag{34}$$

and

$$c_1 = \left(-2-2V+\frac{3a}{2}\right)\Big/r^2, \quad c_2 = \left\{2-2V-3a+n\left(1-V-\frac{3a}{2}\right)\right\}\Big/r^2,$$

$$c_3 = \left\{-\frac{a}{2}+n\left(1-V-\frac{3a}{2}\right)\right\}\Big/r^2. \tag{35}$$

Let us consider the dimensionless variables $x = r/4M$, $\lambda_1 = (4M)^2\lambda$, then Eq. (29) with the substitution $H_0 = h_0/x$ is

$$\frac{d^2h_0}{dx^2} + \lambda_1\left(1+\frac{1}{x}\right)h_0 = 0. \tag{36}$$

For $\lambda_1 > 0$, the general solution of (36) is given by [13]

$$h_0 = A\sqrt{\lambda_1}\, xe^{-i\sqrt{\lambda_1}x} M\left(1+\frac{i}{2}\sqrt{\lambda_1}, 2, 2i\sqrt{\lambda_1}\, x\right). \tag{37}$$

For $\lambda_1 < 0$, $\lambda_1 = -\xi^2$ say, the equations are satisfied formally but the boundary condition at $x \to \infty$ is not, therefore there is no solution of h_0 for $\lambda_1 < 0$.

To solve Eqs (30)–(32), we substitute $H_2 = h_2/x$, $K = k/x$ and $H_{2p} = h_{2p}/x$ and expand h_2, k, and h_{2p} in terms of the complete set of eigenfunctions provided by (36). The problem of finding the eigenvalues and eigenfunctions of the coupled differential equations (30)–(32) is reduced to finding the eigenvalues and eigenvectors of the expansion coefficients. The coefficients are obtained with a computer. We found negative eigenvalues. Thus, even though the uncoupled Eq. (36) has only positive eigenvalues, the coupling between the different equations leads to negative eigenvalues. We have shown that bounded normalizable continuous solutions exist at $r = 0$ and $r = \infty$ for both signs of the eigenvalue, thus indicating instability of the classical solution for the mode $q = l = -m = 1$.

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