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WEAK COUPLING EXPANSION IN LATTICE GAUGE THEORIES

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We discuss problems of the lattice regularization of the quantum chromodynamics. We derive on the two-loop level a finite relation between the coupling constants of two versions of the lattice action: Wilson and mixed fundamental-adjoint. This relation maps onto each other the weak coupling predictions of theories built with these actions. We discuss also finite size effects, and in particular the role played by the zero-momentum sector of the lattice gauge theory. We derive a form of the leading-order contribution to the averages of Wilson loops coming from this sector valid for $d = 4$ and the SU(N) group with $N \geqslant 3$.

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1. Introduction

In the last years Monte Carlo simulations of systems described by the lattice quantum chromodynamics [1, 2] became one of the important tools in studying the physics of strong interactions [3, 4]. The lattice was introduced originally to regularize the standard continuum theory [1] and proved very useful in understanding such physical phenomena as confinement, which escape if one uses only the perturbative tools of the theory.

Lattice gauge theories became a subject in itself and a lot of theoretical effort was put in studying their properties. A novelty of this approach was existence of a new regime of the theory: *strong coupling regime* where physically interesting quantities (Green's functions) can be calculated as power series in the inverse of coupling constant using non-perturbative technique of cluster expansions (cf. e.g. [5–9]). In the strong coupling regime the phenomenon of confinement is satisfied, in a sense, by definition. Strong-coupling expansions can easily be generated to very high orders and there was a hope that one can reach by some extrapolation technique the opposite limit in the coupling constant, namely that of the *weak coupling*, which is the one related to the original continuous theory [10–13].

Unfortunately this turned out to be impossible. Transition between the two regimes

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of the theory, at least for physically interesting gauge groups SU(N) with $N \geqslant 2$, does not seem to be connected with a phase transition, however one cannot exclude the infinite-order phase transition of the surface roughening type. The weak coupling version of the theory is again confining, but the nature of confinement changes. The strong coupling series seem to have a finite radius of convergence, which corresponds to a position of the first singularity in the applied extrapolated formulas (e.g. Padé approximants). This singularity marks the place where the transition between the two regimes takes place [14–19].

Monte Carlo simulations proved to be a method to fill the gap between the two regimes. Physically interesting quantities can be numerically computed in the Euclidean version of the theory, where the measure of Feynman's path integral becomes real and positive and can be interpreted as probability [4]. For obvious reasons lattice systems which can be handled this way have finite sizes, say 16^4 sites of a hypercubic lattice, but special purpose computers are being constructed to deal with bigger systems. The size of a system is not very important if one wants to get information about local quantities, like an average action. Physically interesting however are phenomena which are related to the large distance objects and their averages. Such objects are e.g. Wilson loops which can give information about the lattice string tension, and consequently about the lattice scale Λ_{lattice}. Large distance behaviour of correlations of certain objects can be used to measure the mass spectrum of the theory. The conflict between the limited possibilities of computer experiments and unlimited demand for hidden information can be resolved only if the theory is understood sufficiently well to ask the computer questions it can answer.

An example of such understanding is provided by the renormalization group equations [20–22], which relate the lattice spacing a between the lattice sites with the value of the bare coupling constant. The renormalization group tells us that in the weak coupling limit this spacing goes to zero exponentially with the squared inverse of the bare coupling. This explains what physically happens when we work with a system which is finite in lattice units. If in the Monte Carlo experiment we decrease the value of the coupling constant it means that also the physical size of a system is decreased. At a certain moment our system inevitably will become too small to see phenomena which have fixed *physical* scale. On the other hand when this value is increased we increase at the same time the lattice spacing a and at certain moment our lattice approximation becomes too crude to describe the physics of continuum. This apparently happens when the transition between the strong and weak coupling regime occurs. It is therefore crucial, if the Monte Carlo method is to be used to find a region in the coupling constant which is at all suitable to measure physics. The fact that physically interesting results were already obtained in Monte Carlo experiments must be considered to be a lucky coincidence!

An important factor, which can possibly increase the predictive power of lattice gauge theories is the non-uniqueness of the lattice action used to regularize the continuum theory [23–28]. There are in fact infinitely many possible choices. All of them should give the same physics in the continuum, but their behaviour for finite coupling constants is different, although related [24, 29–31]. The first part of this paper is devoted to the study of this particular problem. In Section 2 we introduce our basic notations recalling the main properties of the continuous QCD. The lattice version of QCD is discussed in Section 3 on an

example of the Wilson theory [1]. Another possible choice of the lattice action: the mixed action is also introduced [24]. In Section 4 we discuss in more detail the concept of universality and its implications. We shall derive an approximate (in the sense of perturbation expansion) relation which maps the weak coupling regime of the mixed theory onto that of Wilson theory. This derivation is based in the big part on our earlier work [32]. These results were generalized in Ref. [33] to other forms of the lattice action. The method we shall use is the background field method [30, 34–39], which is introduced in Section 5. In Section 6 we derive the one-loop form of the relation for the physical coupling constant (in terms of the bare one) for the Wilson theory [30]. The diagrammatic structure of this relation will be used in Section 7 to derive the required two-loop formula relating the coupling constants in the mixed theory to that of the equivalent Wilson theory. In Section 8 we present a comparison of our prediction with numerical experiments and draw conclusions concerning their results. A more detailed discussion can be found in Ref. [40]. The main conclusion is not very optimistic and can be summarized as follows: values of the lattice scale obtained in numerical experiments have probably systematic errors of the order of 10%.

The second part of this paper is devoted to study implications of the finiteness of the lattice system and its periodicity on the perturbative expansion. The more complete study of this problem was published as Ref. [41]. The first implication of the finiteness is that the momentum spectrum of a theory becomes discrete. Modes with non-zero momentum become Gaussian modes [42, 43], but the zero-momentum modes appear in the action only in the quartic terms. Our analysis is devoted to study the contribution from these particular terms to the partition function. Zero-momentum modes have an important physical meaning describing the possible structure of the finite box vacuum. Similar problems were discussed in our earlier publication [44], where we introduced the concept of a *toron* and of a *toron manifold*. The present paper and the Ref. [41] can be considered as a sequel to it. The structure remains important also when the size of the system goes to infinity. In Section 9 we discuss in more detail the structure of the standard weak coupling expansion on a finite lattice and discuss the possible saddle points of the action. In Section 10 we discuss the non-zero momentum sector of the theory, gauge fixing in this sector and the ghost determinant. In all these calculations we keep the structure of the saddle point possibly general. Integrating out the non-constant modes gives us an effective action for the zero-momentum sector. The one-loop form of this action is discussed in Section 11. The derivation of the general form of this action is given in Appendix A. Section 12 deals with the zero-momentum sector of the theory. We show that the structure of the vacuum (or the structure of the dominating saddle point) depends on the gauge group and the dimensionality of the system. We show that for $d = 4$ and group SU(N) with $N \geqslant 3$ the structure of the vacuum corresponds to a naive vacuum, with the quartic scaling law for the zero-momentum fluctuations. This result is based on the inequalities derived in Appendix B. For this case the leading order behaviour of the partition function is derived in Section 13. We evaluate also the leading order contribution coming from the zero-momentum sector for the avarages of Wilson loops and show that for large loops this contribution is non-negligible. In Section 14 we present a discussion of this result and its possible implications.

2. Continuum gauge theory

In the following we shall always discuss the Euclidean versions of all considered theories, i.e. we shall assume that the space-time is Euclidean. The analytic continuation to Minkowski space is a standard problem and will not be treated here.

We shall start our discussion by recalling the standard, *continuous* version of the quantum chromodynamics (QCD). The basic physical concept is a vector potential $A_\mu(x)$ which is a $N\times N$ hermitean matrix field which can be decomposed in the colour space as

$$A_\mu(x) = \tfrac{1}{2} A^a_\mu(x)\lambda_a, \tag{2.1}$$

with λ_a — the SU(N) group generators which satisfy the commutation relations

$$[\lambda_a, \lambda_b] = if^{abc}\lambda_c, \tag{2.2}$$

with SU(N) structure constants f^{abc}. For λ_a we assume in the $N\times N$ representation

$$\mathrm{Tr}\,\lambda_a\lambda_b = 2\delta_{ab}. \tag{2.3}$$

From $A_\mu(x)$ we construct the field strength tensor $G_{\mu\nu}(x)$:

$$G_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + i[A_\mu(x), A_\nu(x)], \tag{2.4}$$

where $\partial_\mu \equiv \dfrac{\partial}{\partial x_\mu}$. In the colour coordinates

$$G^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) - f^{abc}A^b_\mu(x)A^c_\nu(x). \tag{2.5}$$

In all these expressions the summation over the repeated indices is understood. The position of both colour and space-time indices is not important.

The fundamental concept in QCD is that of the gauge transformation. Assume the existence of the SU(N) field $\Omega(x)$, under which the vector potentials transform as

$$A'_\mu(x) = \Omega^\dagger(x)A_\mu(x)\Omega(x) - i\Omega^\dagger(x)\partial_\mu\Omega(x). \tag{2.6}$$

As can easily be checked under transformation (2.6) the field strength tensor $G_{\mu\nu}(x)$ transforms locally:

$$G'_{\mu\nu}(x) = \Omega^\dagger(x)G_{\mu\nu}(x)\Omega(x), \tag{2.7}$$

ad the so called action density $\mathscr{S}(A)$

$$\mathscr{S}(A) = \tfrac{1}{2}\,\mathrm{Tr}\,G_{\mu\nu}(x)G_{\mu\nu}(x) = \tfrac{1}{4}\sum_a G^a_{\mu\nu}(x)G^a_{\mu\nu}(x) \tag{2.8}$$

is an invariant. Consequently the global action of the system $S(A)$

$$S(A) = \int d^dx\,\mathscr{S}(A) \tag{2.9}$$

is invariant under local gauge transformations (2.6). The physical properties of the system are defined by specifying the measure in the Feynman's path integral. This measure is taken to be

$$d\mu(A) = \prod_{x,\mu,a} dA^a_\mu(x) \exp\left[-\frac{1}{g^2} S(A)\right] \tag{2.10}$$

and is again invariant under local gauge transformations (2.6). With the measure defined by (2.10) the Green's functions of the system can be computed. The gauge invariance implies that only the gauge-invariant part of the averaged quantities has a physical sense. In other words only gauge invariant quantities should be considered. Averages of such quantities computed with the help of the measure (2.10) are however ill-defined, since they are always proportional to the *(infinite)* volume of the local gauge transformation (SU(N) group volume at each space-time point x). This means that gauge-fixing is mandatory if finite results are to be obtained. Problems of this type are typical for field theories with local gauge symmetry (i.e. quantum electrodynamics) and will be discussed later in the context of the lattice version of the model.

To give meaning to the product over space-time points, which appears in (2.10), one usually is forced to interprete the functional integrals as a limit of the ordinary multi-dimensional integrals, in which the space-time is discretized in some sense and the distance a between the points goes to zero. This approach means that, in general, the expression for the action $S(A)$ has to be redefined: derivatives appearing there have to be approximated by differences in such way that in the limit $a \to 0$ one obtains the standard expression (so called naive continuum limit). Obviously there are infinitely many possible expressions which have a common limit when $a \to 0$. All these will correspond to different, but hopefully equivalent ways to regularize the functional integrals.

In order that the regularisations be really equivalent it is necessary however to preserve certain important symmetries (like the local gauge symmetry) of the system. This particular one can easily be broken and as one can easily convince oneself is in fact broken in all naive attempts to discretize the gauge theory. The reason is the essentially non-local character of transformation (2.6) (derivative term becomes local only in the $a \to 0$ limit) and a proper definition of the discrete version of gauge theory has to take it into account [1].

3. *Lattice gauge theory*

In this chapter we shall discuss the basic concepts connected with the lattice formulation of the gauge theory [1]. The first step will be to discretize the space-time. In all which follows we shall take the space-time to be a set of points on a hypercubic lattice in d dimensions, where d is the dimensionality of the space-time. Assuming lattice spacing a in all directions to be the same the possible positions of the points in the space-time are restricted to take values

$$x = a\{n_1, n_2, \ldots n_d\}, \tag{3.1}$$

with integer n_μ, $\mu = 1, \ldots, d$.

The important geometrical concepts will be that of a *link* and *plaquette*. *Link* is an oriented line connecting two nearest neighbour points. It is characterized by a set $\{n, \mu\}$, where n is a point from which it emanates and μ a unit vector in direction μ. Two points joined by this link are thus n and $n+\mu$. Notice that links are oriented, i.e. every link has its partner with opposite orientation $\{n+\mu, -\mu\}$.

Plaquette is a set of four oriented links forming a closed square. Plaquette is characterized by two orientations μ and ν and the vertices of the square are n, $n+\mu$, $n+\mu+\nu$ and $n+\nu$. The same as for links, every plaquette has its partner with the links having opposite orientations.

The vector potential $A_\mu(x)$ from the continuous theory will be replaced by the link variable

$$U_\mu(n) = \exp\left[\tfrac{1}{2} iaA^a_\mu(n)\lambda_a\right] \tag{3.2}$$

defined on the link (n, μ). As is clear from the definition (3.2) variables $U_\mu(n)$ are SU(N) matrices. Variable defined on the link with opposite orientation $(n+\mu, -\mu)$ is an inverse:

$$U_{-\mu}(n+\mu) = U_\mu^{-1}(n) = U_\mu^\dagger(n). \tag{3.3}$$

For every plaquette one can define a matrix $P_{\mu\nu}(n)$ by

$$P_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\mu)U_\mu^\dagger(n+\nu)U_\nu^\dagger(n), \tag{3.4}$$

which is a product of four link variables along the plaquette. Similarly one can construct a matrix corresponding to the plaquette with opposite orientation: $P_{\nu\mu}(n) = P^\dagger_{\mu\nu}(n)$.

As in the continuous theory we shall consider the local gauge symmetry. In each point n of the lattice we shall define the SU(N) matrix $\Omega(n)$ — an analog of the gauge matrix $\Omega(x)$ from the preceding paragraph. Matrices $\Omega(n)$ will transform link variables $U_\mu(n)$ as

$$U'_\mu(n) = \Omega^\dagger(n)U_\mu(n)\Omega(n+\mu). \tag{3.5}$$

As can be easily checked, the $P_{\mu\nu}(n)$ matrix transforms locally:

$$P'_{\mu\nu}(n) = \Omega^\dagger(n)P_{\mu\nu}(n)\Omega(n). \tag{3.6}$$

In the analogy with the continuous case we can easily construct gauge-invariant quantities, i.e. quantities invariant under transformation (3.4). The simplest example is Tr $P_{\mu\nu}(n)$ or the character of $P_{\mu\nu}(n)$ in the fundamental SU(N) representation. More complicated examples can involve traces of SU(N) matrices obtained as products of link variables along any closed curve on a lattice, so called Wilson loops, products of such traces, or finally more complicated functions of such traces (these last can be understood as e.g. characters in other than fundamental representations of the SU(N) group).

We are now in a position to define the lattice action of the system. We shall require that it satisfies two basic requirements: a) local gauge invariance and b) correct naive continuum limit (when $a \to 0$ the lattice action should approach the continuous one). The first requirement means that we shall have to construct our action from the gauge invariant

objects as described above. The second gives certain limitations on possible objects we can use. There are infinitely many possible constructions satisfying both conditions [23–28], all of which should in principle be equivalent. The exact meaning of this equivalence will be made clear in the next Section. Here we shall explain only some possible constructions, adding as a third requirement c) quasi locality. This last requirement means that the action will be constructed only as a functional of the plaquette variables $P_{\mu\nu}(n)$.

Making use of the definition (3.2) of the gauge potential $A_\mu(n)$ in terms of the link variable $U_\mu(n)$, and assuming regularity of potentials when the limit $a \to 0$ is approached, i.e.

$$\Delta_\mu \to a\partial_\mu + O(a^2) \tag{3.7}$$

(where Δ_μ is a difference operator), which is supposed to be satisfied wherever differences of potentials appear, we can easily show that if

$$P_{\mu\nu}(n) = \exp\left[ia^2 G_{\mu\nu}(n)\right] \tag{3.8}$$

then

$$G_{\mu\nu}(n) = \frac{1}{a}(\Delta_\mu A_\nu - \Delta_\nu A_\mu) + i[A_\mu, A_\nu] + O(a). \tag{3.9}$$

The simplest choice for the lattice action satisfying requirements listed above is

$$S_W(A) = -\tfrac{1}{2}\sum_{\substack{\mu,\nu \\ n}} \left[(\mathrm{Tr}\, P_{\mu\nu}(n) - N) + \mathrm{c.c.}\right], \tag{3.10}$$

so called Wilson action. The sum in (3.10) is over all plaquettes with positive orientation. Plaquettes with negative orientation are in the complex conjugate part. As can easily be checked

$$S_W(A) = a^4 \sum_{\substack{\mu,\nu \\ n}} \mathrm{Tr}\, G_{\mu\nu}(n) G_{\mu\nu}(n) + O(a^5) \xrightarrow[a\to 0]{} S_{\mathrm{cont}}(A). \tag{3.11}$$

As for the continuous theory we shall define the Feynman's integral for the Wilson theory with the help of the measure

$$d\mu(U) = \prod_{\mu,n} DU_\mu(n) \exp\left[-\beta S_W(U)\right]. \tag{3.12}$$

DU_μ is the invariant (Haar) measure on the group SU(N) normalized to

$$\int DU = 1. \tag{3.13}$$

Parameter $\beta \sim 1/g_0^2$ plays the role of the bare coupling constant (cf. (2.10)). The fact that the field manifold becomes compact is typical for the lattice regularization of the gauge theory, independent on the particular form of the lattice action. The measure (3.12) shows the main properties of the lattice regularization of the gauge theory. As in the case of the continuum the measure (3.12) has a local gauge invariance. This means that physical quantities should also have this property, otherwise when averaged with (3.12) only their gauge-

-invariant part will contribute. Unlike in the continuum resulting integrals are not by themselves divergent — the volume of the gauge symmetry group is because of (3.13) finite and, at least in principle, the gauge fixing is not necessary. As will be made clear in the next Sections, we shall nevertheless need the gauge fixing to be able to study the weak coupling limit ($\beta \to \infty$) of the theory, where the measure (3.12) is dominated by field configurations which are fluctuations around some saddle point(s).

As we have already mentioned the Wilson action [1] is one among infinitely many actions with the correct naïve continuum limit [23–28]. Another simple example is the action, where the plaquette variable appears in the adjoint representation. The mixed action S_m which contains both fundamental and adjoint representations [23, 24] has two coupling constants β_F and β_A and has a form:

$$S_m(A) = -\tfrac{1}{2} \sum_{\substack{\mu<\nu \\ n}} (\mathrm{Tr}\, P_{\mu\nu}(n) + \mathrm{Tr}\, P^{\dagger}_{\mu\nu}(n)) - \beta_A \sum_{\substack{\mu<\nu \\ n}} |\mathrm{Tr}\, P_{\mu\nu}(n)|^2, \tag{3.14}$$

where we have included the coupling constants into the action (this should not lead to any complications) and where we have omitted the unimportant constant terms. Using (3.8) we can find the naive continuum limit of (3.14) which, as expected, is the continuum theory but with the coupling constant

$$\frac{1}{g_m^2} = \frac{\beta_F}{2} + N\beta_A. \tag{3.15}$$

Of course higher order (in a) terms of the pure Wilson ($\beta_A = 0$) and mixed actions are different and we may ask the question what is the relation between these two theories.

4. Problem of universality

As was mentioned in the last Section the problem of defining a lattice action with the proper naive continuum limit has in general infinitely many possible solutions. In this chapter we shall try to analyze the possible consequences of different choices and try to describe the relations between them. For the lattice gauge theory we would like physically interesting quantities to be gauge invariant, i.e. expressible in terms of traces of *loops*. A concept of a *loop* was introduced in the last Section as a product of link variables along a closed curve on the lattice. Using the transformation properties of link variables (3.5) and denoting by $U(C)$ the loop variable for the loop C, starting and finishing at point n we see that under a gauge transformation (3.5) $U(C)$ transforms locally:

$$U'(C) = \Omega^{\dagger}(n) U(C) \Omega(n). \tag{4.1}$$

Quantity $U(C)$ is called the Wilson loop. Average of its trace, where the average of quantity $X(U)$ is defined as

$$\langle X(U) \rangle = \frac{\int d\mu(U) X(U)}{\int d\mu(U)}, \tag{4.2}$$

has a very important physical meaning. Consider the loop C to be a rectangle of size $R \times T$, $T \gg R$. R can be interpreted as the spacial distance between the external *heavy* quark and antiquark sources. T is the Euclidean time during which the pair interacts. For loop C

$$W(C) = \left\langle \frac{1}{N} \operatorname{Tr} U(C) \right\rangle \underset{T\to\infty}{\sim} \frac{1}{N} \exp\left[-(V(R)-E_0)T\right], \tag{4.3}$$

where $V(R)$ is the pair energy and E_0 the ground-state energy. For large loops $W(C)$ can be parametrized

$$W(C) = \exp\left[-\varrho_e(\beta)RT - \varepsilon(\beta)(R+T) - \gamma(\beta)\ldots\right]. \tag{4.4}$$

$W(C)$ plays a role of an order parameter of the theory. Phases of the system can be distinguished by the value of parameter $\varrho_e(\beta)$ (zero or not). This parameter is an electric string tension, since from (4.3) and (4.4) we have

$$\varrho_e(\beta)R = E(R) - E_0. \tag{4.5}$$

A lattice theory will be confining if this parameter is non-zero. Notice that in calculating $W(C)$ the physical length a has disappeared. ϱ_e is in this case a function of β or the *bare* coupling constant g^2 since by comparing (2.10) and (3.13) we have

$$\beta = \frac{2}{g^2}. \tag{4.6}$$

We should notice that in (4.3) and (4.4) the size of the Wilson loop is measured in the lattice units (R and T are integers). The real size of the loop is $Ra \times Ta$ and

$$\varrho_e(\beta) = \sigma(\beta)a^2, \tag{4.7}$$

where $\sigma(\beta)$ is a physical string tension. In the continuous theory σ can be used to determine the scale (renormalization of the theory can be done in such a way that $\sigma(\beta)$ or rather $\sigma(g^2)$ is kept fixed). Eq. (4.7) can then be regarded as a relation between the physical scale a^2 and the bare coupling constant g^2 [20]

$$\sigma_0 a^2 = \varrho_e(g^2). \tag{4.8}$$

This relation can be perturbatively calculated in the continuum when a is treated as an inverse of the cutoff. The clue is to study the renormalization group equation satisfied by the physical coupling constant $g^2(a)$ [20–22]:

$$a\frac{d}{da}g^2(a) = 2\beta_0 g^4(a) + 2\beta_1 g^6(a) + O(g^6(a)), \tag{4.9}$$

where coefficients β_0 and β_1 are independent on the regularization scheme (it starts with g^6 terms [45]) and their values are [46–48]

$$\beta_0 = (11/3)\frac{N}{(4\pi)^2}, \qquad \beta_1 = (34/3)\left(\frac{N}{(4\pi)^2}\right)^2. \tag{4.10}$$

The solution of Eq. (4.9) can be written in the form

$$\sigma a^2 = \Lambda^2 \exp\left[-\frac{1}{\beta_0 g^2}\right](\beta_0 g^2)^{-\beta_1/\beta_0^2}(1+O(g^2)), \tag{4.11}$$

where Λ^2/σ is an integration constant, which cannot be computed perturbatively. σa^2 can be interpreted as a lattice string tension. Relation (4.11) means that a continuum limit of the theory ($a \to 0$) corresponds to a transition $g^2 \to 0$, where all physical quantities should scale according to (4.11). The scale Λ in general depends on the employed regularization scheme [29–31].

The renormalization group relation (4.11) was derived in the continuum theory. The coupling constant g^2, which appears in (4.11) should carry an index r determining which regularization scheme has been used. Different prescriptions result in a *finite* redefinition of the coupling constant

$$g_{r'}^2 = g_r^2(1+O(g^2)). \tag{4.12}$$

In (4.12) $g_{r'}^2$ and g_r^2 are coupling constants corresponding to two different regularization schemes. The *finite* coefficients which appear in (4.12) are perturbatively calculable. One can check that terms of order g^4 in (4.12) (marked $O(g^2)$ there) have no influence on the form of Eq. (4.11) except on the value of the scale parameter Λ^2, which is changed by a finite factor. Higher order terms will in general be affected, but these are not specified in (4.11).

At this point we are ready to specify what we shall mean by universality. We shall call two lattice theories equivalent if in the limit $a \to 0$ they can be mapped one onto another with the help of a *finite* relation of the type (4.12) between their bare coupling constants. This means that both theories will predict the same physics in the continuum limit.

If the Wilson lattice theory in the limit $a \to 0$ is equivalent to a continuum theory (regularized in some other way) there must exist a finite relation between the bare coupling constant g_r of the continuum theory and that of the Wilson theory

$$g_r^2 = \frac{2}{\beta}\left(1+O\left(\frac{1}{\beta}\right)\right). \tag{4.13}$$

This gives a very definite prediction for the large-β behaviour of the lattice string tension $\varrho_e(\beta)$, which can be obtained by introducing (4.13) to (4.11). This prediction has been confirmed in numerous numerical experiments [4].

The problem of universality will be discussed in the following Sections *not* as a relation between the lattice and continuum theory but as a relation between various forms of lattice theories. We shall concentrate here on one particular relation, namely that between the Wilson (3.10) and mixed (3.14) theory. These two theories can be considered as two different regularizations of the same continuum theory. In the weak coupling limit we expect the existence of a finite relation

$$\beta_W = \beta_W(\beta_F, \beta_A) \tag{4.14}$$

which can map predictions of the mixed theory onto that of a Wilson theory. We shall derive the form of this relation perturbatively to the two-loop order.

5. *The background field technique*

In this Section we shall discuss the background field method which will be used in the next Sections to compute the effective coupling constants. This method, originally proposed for the continuum theory [34–37] was later generalized for lattice gauge theories [38, 39, 30].

The background field method introduces the background gauge field $\boldsymbol{B}_\mu(n) \equiv \frac{1}{2} B^a_\mu \lambda_a$ (λ_a are as before the SU(N) generators, $a = 1, \ldots, N^2-1$, satisfying $\mathrm{Tr}\, \lambda_a \lambda_b = 2\delta_{ab}$) by splitting each link variable into a product

$$\begin{aligned} \boldsymbol{U}_\mu(n) &= \exp\left[ig_0 \boldsymbol{A}_\mu(n)\right] \exp\left[ia\boldsymbol{B}_\mu(n)\right] \\ &= \boldsymbol{U}^A_\mu(n)\boldsymbol{U}^B_\mu(n). \end{aligned} \tag{5.1}$$

The background field $\boldsymbol{B}_\mu(n)$ can be considered as some nontrivial solution of the classical equation of motion (a saddle point of the integrand in the Feynman's path integral) and $\boldsymbol{A}_\mu(n)$ as a quantum fluctuation around such a solution. The background field method simplifies considerably the computational effort necessary to evaluate perturbatively the behaviour of the effective coupling constant. The background field in (5.1) sets the physical scale: we assume that variations of $\boldsymbol{B}_\mu(n)$ become appreciable only over distances $L \gg a$.

Using the parametrization (5.1) we shall formally proceed to evaluate the functional $Z(\boldsymbol{U}^B)$

$$Z(\boldsymbol{U}^B) = \int\!\!\prod_{\mu,n} D\boldsymbol{U}^A_\mu(n) \exp\left[-S_{\mathrm{inv}}(\boldsymbol{U}^B_\mu \boldsymbol{U}^A_\mu)\right], \tag{5.2}$$

where S_{inv} is an invariant action (for example the Wilson's action). This calculation will make sense only in the weak coupling limit, where as we have mentioned before we shall assume that the background field $\boldsymbol{U}^B_\mu(n)$ is some nontrivial saddle point of the integrand in (5.2). Matrices $\boldsymbol{U}^A_\mu$ will be expanded in terms of vector potentials $\boldsymbol{A}_\mu(n)$ and the integral (5.2) will be computed with the saddle point method (after introducing the gauge fixing necessary to make such expansion sensible) to give an effective action of the background field

$$Z(\boldsymbol{U}^B) = \exp\left[-\Gamma(\boldsymbol{B})\right]. \tag{5.3}$$

The essential feature of the background field method is to perform the computation of (5.3) in such a way that everywhere the invariance of the action under the gauge transformations of the background field is preserved. This guarantees that also $\Gamma(\boldsymbol{B})$ will be a locally gauge-invariant functional of $\boldsymbol{B}_\mu(n)$. $\Gamma(\boldsymbol{B})$ is in general a complicated functional of the background field. Expanding $\Gamma(\boldsymbol{B})$ in terms of $\boldsymbol{B}_\mu(n)$ we shall concentrate on the lowest nontrivial second order term (constant terms in $\Gamma(\boldsymbol{B})$ have no physical meaning and linear terms are absent if the background field is indeed a saddle point of the integral)

$$\Gamma(\boldsymbol{B}) = \Gamma_2(\boldsymbol{B}) + \Gamma_3(\boldsymbol{B}) + \ldots \tag{5.4}$$

In the limit $a \to 0$, $\Gamma_2(\boldsymbol{B})$ will have an expansion

$$\Gamma_2(\boldsymbol{B}) = \frac{1}{G^2} S_{\mathrm{cont}}(\boldsymbol{B}) + O(a), \tag{5.5}$$

where

$$S_{\text{cont}}(B) = \tfrac{1}{4}\int d^dx \sum_{\substack{\mu,\nu \\ a}} (G^a_{\mu\nu}(x))^2 \tag{5.6}$$

and $G^a_{\mu\nu}(x)$ is the field strength tensor of the background field (cf. Eq. (2.5)), or rather its part linear in B_μ. G^2 in (5.5) is the renormalized coupling constant

$$\frac{1}{G^2} = \frac{1}{g_0^2}(1+O(g_0^2))$$

with g_0 — the bare coupling constant of the considered theory. Subsequent terms in the expansion of $1/G^2$ as a function (in general divergent) of g_0 can be computed perturbatively.

Before we start computing let us specify how the gauge transformations act on link variables parametrized by (5.1). Using (3.5) we can write

$$U^A_\mu(n)U^B_\mu(n) = \Omega^\dagger(n)U^A_\mu(n)\Omega(n)\Omega^\dagger(n)U^B_\mu(n)\Omega(n+\mu). \tag{5.7}$$

Adopting this transformation law we have indeed a parametrisation in which the quantum fields transform as local matter fields and the background part behaves as a real gauge field. This symmetry should obviously be present in the $\Gamma(B)$ functional — that is how we can predict the form of its two-point local part.

We define two lattice covariant derivatives

$$D^+_\mu(B)F_\nu(n) = U^B_\mu(n)F_\nu(n+\mu)U^{B\dagger}_\mu(n) - F_\nu(n), \tag{5.8a}$$

$$D^-_\mu(B)F_\nu(n) = U^{B\dagger}_\mu(n-\mu)F_\nu(n-\mu)U^B_\mu(n-\mu) - F_\nu(n) \tag{5.8b}$$

by their action on any matter field $F_\nu(n)$. In contrast with the ordinary lattice derivative $\Delta^+_\mu F_\nu$ the left-hand sides of (5.8) transform again as matter fields. Covariant derivatives will be used to rewrite the plaquette matrix $P_{\mu\nu}(n)$ (cf. (3.4)) as

$$P_{\mu\nu}(n) = \exp\left[ia^2 G_{\mu\nu}(n)\right]\exp\left[ig_0 F_{\mu\nu}(n)\right], \tag{5.9}$$

where using the Baker-Campbell-Hausdorff formula

$$\exp(gA)\exp(gB) = \exp\Big(g(A+B)+\frac{g^2}{2}[A,B]+\frac{g^3}{12}([A,[A,B]] + [[A,B],B])+O(g^4)\Big),$$

we find

$$G_{\mu\nu}(n) = \frac{1}{a}(\Delta^+_\mu B_\nu(n) - \Delta^+_\nu B_\mu(n)) + i[B_\mu(n), B_\nu(n)] + O(a), \tag{5.10}$$

(Δ^+_μ is the lattice derivative $D^+_\mu(B=0)$ and

$$F_{\mu\nu}(n) = F^{(1)}_{\mu\nu}(n) + g_0 F^{(2)}_{\mu\nu}(n) + g_0^2 F^{(3)}_{\mu\nu}(n) + \ldots \tag{5.11}$$

In (5.11)

$$F^{(1)}_{\mu\nu}(n) = D^+_\mu A_\nu(n) - D^+_\nu A_\mu(n),$$

$$F^{(2)}_{\mu\nu}(n) = i([A_\mu, A_\nu] - \tfrac{1}{2}[D^+_\nu A_\mu, D^+_\mu A_\nu] - \tfrac{1}{2}[D^+_\nu A_\mu, A_\mu] + \tfrac{1}{2}[D^+_\mu A_\nu, A_\nu]) \tag{5.12}$$

etc.

$F^{(3)}_{\mu\nu}$ can be written in terms of double commutators of fields A_μ and their covariant derivatives.

Let us now proceed to express the lattice action in terms of quantities introduced above. Our first example will be the Wilson action (3.10) [30]

$$\beta S_W = -\frac{\beta}{2} \sum_{\substack{\mu<\nu \\ n}} ([\operatorname{Tr} P_{\mu\nu}(n) - N] + \text{c.c.}). \tag{5.13}$$

To terms at most quadratic in both B_μ and A_μ it equals

$$\beta S_W = \beta \sum_{\substack{\mu<\nu \\ n}} \left(a^2 g_0 \operatorname{Tr}(G_{\mu\nu}(n) F_{\mu\nu}(n)) + \frac{a^4}{2} \operatorname{Tr} G^2_{\mu\nu} \right.$$

$$\left. + \frac{g_0^2}{2} \operatorname{Tr} F^2_{\mu\nu} - \frac{a^4 g_0^2}{4} \operatorname{Tr}(G^2_{\mu\nu} F^2_{\mu\nu}) + \ldots \right). \tag{5.14}$$

The first term on the right-hand side of (5.14) contains part linear in the quantum field A_μ:

$$\beta a^2 g_0 \sum_{\substack{\mu<\nu \\ n}} \operatorname{Tr}(G_{\mu\nu}(n) F^{(1)}_{\mu\nu}(n))$$

$$= \beta a^2 g_0 \sum_{\substack{\mu<\nu \\ n}} \operatorname{Tr}(G_{\mu\nu}(n)(D^+_\mu A_\nu - D^+_\nu A_\mu)). \tag{5.15}$$

This term can be made zero if we use the identity

$$\sum_n \operatorname{Tr}(f(n) D^+_\mu g(n)) = \sum_n \operatorname{Tr}[(D^-_\mu f(n)) g(n)] \tag{5.16}$$

and the classical field equations, which we assume to be satisfied by the background field B_μ:

$$D^-_\mu G_{\mu\nu}(n) = 0. \tag{5.17}$$

In the following we shall introduce the gauge fixing procedure, which will eliminate the gauge zero-modes of the A_μ field and which will guarantee that $A_\mu = 0$ is a saddle point of the action (5.14). Assuming now that this gauge fixing has been performed we get on the tree level:

$$\beta S_W = \frac{\beta}{2} a^{4\frac{1}{4}} \sum_{\substack{\mu,\nu \\ a,n}} (G^a_{\mu\nu}(n))^2 \tag{5.18}$$

and we shall relate $\beta/2$ of the Wilson action to the bare coupling constant g_0, which we used to scale the quantum fields in (5.1):

$$\frac{\beta}{2} = \frac{1}{g_0^2}. \tag{5.19}$$

It is important to note here that we could use a different relation between β and g_0 of the form

$$\frac{\beta}{2} = \frac{1}{g_0^2}(1+\alpha_1 g_0^2+\alpha_2 g_0^4+ \ldots),$$

with arbitrary finite α_i. Such redefinition would eventually affect higher order terms in the perturbative expansion. We can consider (5.19) to be a definition of the bare coupling constant g_0 for the Wilson theory. Freedom to redefine g_0 will be used in Sections 6 and 7. The physical coupling constant G^2

$$\frac{1}{G^2} = \frac{1}{g_0^2}(1+O(g_0^2)), \tag{5.20}$$

where $O(g_0^2)$ (in general divergent) corrections will come from the one-loop terms.

We shall proceed now to construct the perturbative expansion based on the expression (5.14) for the invariant action. The Gaussian part of (5.14) is

$$\beta S_W^{(2)} = \frac{\beta g_0^2}{2} \sum_{\substack{\mu<\nu \\ n}} \mathrm{Tr}\,(D_\mu^+ A_\nu - D_\nu^+ A_\mu)^2 + \ldots \tag{5.21}$$

In (5.21) the factor $\beta g_0^2/2$ equals unity if we accept the relation (5.19). We shall rewrite (5.21) as

$$\begin{aligned}\beta S_W^{(2)} &= \sum_{\substack{\mu,\nu \\ n}} (\mathrm{Tr}\,(D_\mu^+ A_\nu)^2 - \mathrm{Tr}\,(D_\mu^+ A_\nu)(D_\nu^+ A_\mu) + \ldots) \\ &= \sum_{\nu,n} \mathrm{Tr}\,(A_\nu \sum_\mu (D_\mu^- D_\mu^+) A_\nu) \\ &\quad - \sum_n \mathrm{Tr}\,(\sum_\nu D_\nu^- A_\nu)^2 + \sum_{\substack{\mu,\nu \\ n}} \mathrm{Tr}\,(A_\mu [D_\mu^+, D_\nu^-] A_\nu) \\ &\quad + \ldots \end{aligned} \tag{5.22}$$

The last sum in (5.22) contains the commutator of covariant derivatives and is therefore at least linear in the background field. Form (5.21) is identical to the sum of N^2-1 abelian gauge fields (except that covariant derivatives in general do not commute) and requires gauge fixing to eliminate the gauge zero-modes

$$A_\mu \to A_\mu + D_\mu^+ \omega + \ldots \tag{5.23}$$

We chose the gauge fixing term to be

$$\beta S_W \to \beta(S_W + \frac{1}{\alpha} S_{g.f.}),$$

where

$$\beta S_{g.f.} = \sum_n \mathrm{Tr}\,(\sum_\nu D_\nu^- A_\nu)^2 \tag{5.24}$$

and the (renormalized) value of α will be taken one. The gauge fixing term generates the ghost term. Its form to the one-loop level is

$$S_{gh} = \sum_{n,a,b} \bar{\eta}^a(n)\,(\sum_\mu D_\mu^- D_\mu^+)^{ab} \omega^b(n). \tag{5.25}$$

The last information we need to start the perturbative calculation is the expression for an invariant measure $DU_\mu(n)$ in terms of A_μ. This expression is

$$\prod_{\mu,n} DU_\mu(n) = \exp(-S_m) \prod_{\substack{\mu,n \\ a}} dA_\mu^a(n), \tag{5.26}$$

where

$$S_m = g_0^2 \sum_{\mu,n} \tfrac{1}{24}\,\mathrm{Tr}\, A_\mu^2 + O(g_0^4) \tag{5.27}$$

results from the change of variables (5.26) and starts with the terms $O(g_0^2)$. The measure term (5.27) is a lattice artifact. It contributes in perturbative expansion terms which violate the standard loop counting. It will eventually contribute only on the two-loop level.

Another aspect of the change of variables (5.26), typical for perturbative expansions, which is not at first sight apparent is the change in the domain of integration over the A_μ fields in the path integral. This will now extend to infinity in contrast with the finite group integration of the original formulation. We can hope that with the properly chosen gauge fixing one can neglect contribution to the path integral coming from the region of very large fields.

6. Computation of the one-loop effective coupling

In this Section we shall outline the structure of diagrams necessary to calculate the effective coupling constant for the Wilson action on the one-loop level. Without actually performing this calculation we shall proceed to derive the two-loop formula relating the bare coupling constants of the Wilson and mixed (fundamental–adjoint) actions.

In the last Section we formulated the perturbative expansion of the Wilson theory. As we argued in this Section we can assume that the background field $B_\mu(n)$ is a saddle point of the integrand (after the gauge fixing term has been introduced). Since we are interested in computing the effective action for the background field only on one-loop level, we shall need terms at most quadratic in the background field and exactly quadratic in gluon fields. To the one-loop order the gluon and ghost parts of the action are independent.

We can also neglect the measure term. For the gluon sector we have

$$\beta(S_W+S_{g.f.}) = S_0+S_1+S_2+S_3, \tag{6.1}$$

where we have separated the free part of the action

$$S_0 = \sum_{\nu,n} \mathrm{Tr}\,[A_\nu \sum_\mu (\Delta_\mu^- \Delta_\mu^+) A^\nu] \tag{6.2}$$

which does not depend on the background field,

$$S_1 = \sum_{\nu,n} \mathrm{Tr}\,[A_\nu \sum_\mu (D_\mu^- D_\mu^+) A_\nu] - S_0 \tag{6.3}$$

contains terms both linear and quadratic in the background field.

$$S_2 = a^2 \sum_{\substack{\mu<\nu \\ n}} \mathrm{Tr}\, G_{\mu\nu} R_{\mu\nu}. \tag{6.4}$$

Here

$$R_{\mu\nu} = F_{\mu\nu}^{(2)} + i[A_\mu(n+\nu), A_\nu(n+\mu)], \tag{6.5}$$

with the last term coming from the commutator of covariant derivatives in (5.22). Finally

$$S_3 = -\frac{a^4}{2} \sum_{\substack{\nu<\nu \\ n}} \mathrm{Tr}\,(G_{\mu\nu}^2 F_{\mu\nu}^2). \tag{6.6}$$

A similar decomposition can be made for the ghost part

$$S_{gh} = S_{gh_0} + S_{gh_I}, \tag{6.7}$$

where

$$S_{gh_0} = \sum_{a,n} \bar{\eta}^a [\sum_\mu \Delta_\mu^- \Delta_\mu^+] \omega^a \tag{6.8}$$

is the free ghost part, which does not depend on the background. S_{gh_I} contains terms linear and quadratic in the background field. We can observe that the gluon field couples to the background field either by the nontrivial dependence in covariant derivatives (term S_1) or directly to the $G_{\mu\nu}$ tensor (S_2 and S_3). Mixed dependence is also possible (cf. Eq. (5.12) for $F_{\mu\nu}^{(2)}$). We observe also that with our gauge choice the gluon propagator generated by the term S_0 is diagonal both in space and colour indices. The ghost propagator, generated by S_{gh_0} is diagonal in colour indices. In Fig. 1 we list the diagrams which will contribute to the two-point function $\Gamma_2(B)$ on the one-loop level. Gluon propagators are represented by continuous lines, dashed lines are the ghost propagators. Wiggly lines are the background field external lines. Vertices are marked with numbers, corresponding to the decomposition (6.1) and (6.7). Diagrams with one interaction vertex (and two background field lines attached to it) are finite. Diagrams with two interaction vertices (one background

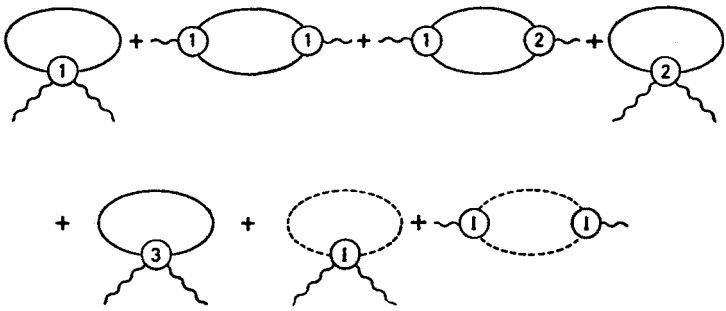

Fig. 1

line at each of them) are in general logarithmically divergent. We shall not perform here the one-loop calculation. This was presented extensively in the literature (cf. e.g. [30]). For our later discussion we shall need the form of interaction vertices and diagrams representing $\Gamma_2(\boldsymbol{B})$.

We shall now proceed to discuss similar expansion for the mixed fundamental–adjoint action. As was mentioned in Section 3 this action has two coupling constants, β_F and β_A

$$S_M = -\frac{\beta_F}{2}\sum_{\substack{\mu<\nu \\ n}}[\mathrm{Tr}\, \boldsymbol{P}_{\mu\nu}(n)+\mathrm{c.c.}]-\beta_A\sum_{\substack{\mu<\nu \\ n}}|\mathrm{Tr}\, \boldsymbol{P}_{\mu\nu}(n)|^2 \tag{6.9}$$

multiplying respectively characters of the plaquette variable $\boldsymbol{P}_{\mu\nu}(n)$ in its fundamental and adjoint representations.

It is obvious that if $\beta_A = 0$ lattice action (6.9) becomes the Wilson action. It is also known from numerical experiments [23, 24, 49] that (at least for some N) a lattice system described by the action (6.9) behaves differently than the Wilson's system, developing complicated phase structure in the (β_F, β_A) plane. This need not contradict the concept of universality (or equivalence of these two theories) since this applies only to the weak coupling limit of the theory. We shall in fact show that these two are equivalent at least up to two-loop level. We shall derive a finite relation, mapping all the diagrams derived from the action (6.9) onto those of the Wilson's action. By all diagrams we mean those necessary to compute the two-point function $\Gamma_2(\boldsymbol{B})$. Mapping the two theories on each other in the weak coupling limit will correspond to finding a suitable expression for g_0.

For the action (6.9) we shall again use parametrization (5.1) in terms of the quantum and background fields. Before we start to construct the perturbative expansion let us first rewrite (6.9) as

$$S_M = -\tfrac{1}{2}(\beta_F+2N\beta_A\omega)\sum_{\substack{\mu<\nu \\ n}}[\mathrm{Tr}\, \boldsymbol{P}_{\mu\nu}(n)+\mathrm{c.c.}]$$

$$-\beta_A\sum_{\substack{\mu<\nu \\ n}}|\mathrm{Tr}\, \boldsymbol{P}_{\mu\nu}-N\omega|^2+\beta_A N^2\omega^2. \tag{6.10}$$

In (6.10) $\omega = \omega(Ng_0^2)$ is chosen to be

$$\omega(Ng_0^2) = \left\langle \frac{\mathrm{Tr}\, \boldsymbol{P}_{\mu\nu}}{N} \right\rangle_{\mathrm{W}} \tag{6.11}$$

the average action per plaquette computed with the Wilson action with the bare coupling constant $\beta = 2/g_0^2$ (cf. Eq. (4.6)). The sense of such reparametrization of (6.9) will be made clear soon. The last term of the action in (6.10) does not depend on dynamical variables and can be dropped when averages will be computed. Notice that we still have not specified the relation between g_0^2 and β_{F} and β_{A}. We shall now proceed to construct the perturbation expansion as in the case of the Wilson action. Our starting point will be the form (6.10) of the mixed action where the second term will be treated as a perturbation over the first (Wilson) term. If we simply neglect this second term we see that the resulting Wilson's action will have the bare coupling constant g_0 satisfying nonlinear equation

$$\frac{2}{g_0^2} = \beta_{\mathrm{F}} + 2N\beta_{\mathrm{A}}\omega(Ng_0^2) \tag{6.12}$$

to be compared with (4.6) for the pure Wilson case. Eq. (6.12) can be considered as a generalization of the naive relation (3.16). Obviously for $\beta_{\mathrm{A}} = 0$ the two coincide. The new interaction vertex resulting from (6.10) can be represented diagrammatically as two semicircles, each corresponding to one of the factors $(\mathrm{Tr}\, \boldsymbol{P}_{\mu\nu} - N\omega)$ or its complex conjugate. Each such vertex carries a factor β_{A}. Using expression (5.9) for $\boldsymbol{P}_{\mu\nu}$ we can attach gluon and background field lines to each of these semicircles. Again we shall be interested in terms which are at most quadratic in the background field. A new interaction vertex will appear in the diagrams. We can now easily understand the effect of rewriting our action in the form (6.10). In order to do so it is important to realize what is the diagrammatic expansion of the average plaquette action in the weak coupling limit of Wilson theory. Using formalism introduced above we can expand $\mathrm{Tr}\, \boldsymbol{P}_{\mu\nu}$ in terms of gluon fields and compute its average. Obviously no vertices with background field lines attached to them would appear. Representing diagramatically $\mathrm{Tr}\, \boldsymbol{P}_{\mu\nu}$ as a semicircle we get for its average a set of connected diagrams built from the gluon and ghost lines (these would already appear on the two-loop level) with one semicircle vertex and arbitrary number of interaction vertices resulting from the Wilson action (cf. Fig. 2). It is now easy to see that the role of the redefinition

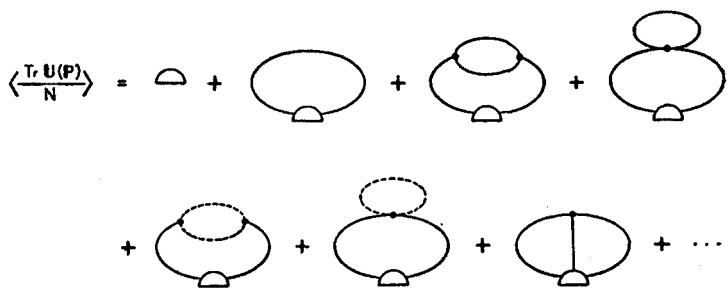

Fig. 2

(6.10) is to subtract from the perturbative expansion of any average computed with the mixed action diagrams with at least one β_A vertex where one of the semicircles has the structure described above. Examples of such diagrams are presented in Fig. 3a. A simple consequence is that now we need only to consider diagrams where to each semicircle appearing in a diagram at least one, and in practice two lines must be attached (gluon or background).

We shall use an even stronger rule to reduce the number of diagrams. We shall discard all the diagrams with an arbitrary number of β_A and Wilson vertices which become dis-

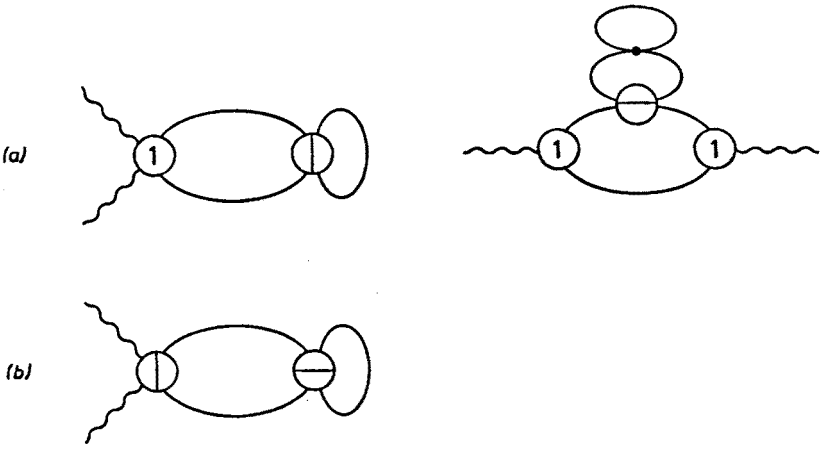

Fig. 3

connected when two semicircles forming any of β_A vertices are taken apart and when one of the resulting subdiagrams has no background field lines (Fig. 3b). Such diagrams will represent one of the terms of the expansion of $\omega(Ng_0^2)$ with g_0 satisfying the complete relation of which (6.12) is only the first approximation. In general we expect

$$\frac{2}{g_0^2} = \beta_F + 2N\beta_A\omega(Ng_0^2) + N\sum_{i=1}^{\infty}(\beta_A)^i f_i(Ng_0^2), \tag{6.13}$$

where $f_i(Ng_0^2)$ are finite functions of the coupling constant. These functions are themselves expressible as infinite series of (Ng_0^2) starting with terms proportional to $(Ng_0^2)^{2i-1}$ (this corresponds to the graph with i β_A vertices and a minimal number of gluon lines). We shall compute the form of equation (6.13) in the two-loop approximation, which means that only a finite number of terms of the infinite sum (two) will be computed. Also functions f_i, $i = 1, 2$, will be computed only to the two-loop order. The quality of our approximation will obviously depend on the values of (Ng_0^2) and β_A.

With the help of expansion (5.9) we can now evaluate one-loop corrections to the Wilson $\Gamma_2(B)$ which result from introducing the perturbation β_A term in (6.10). It is easy

to realize that only one diagram contributes (Fig. 4) which gives

$$\delta\Gamma_2(\boldsymbol{B}) = \beta_A a^4 \frac{g_0^2}{4} \sum_{\substack{\mu<\nu \\ a,b \\ n}} G^a_{\mu\nu}(n) G^b_{\mu\nu}(n) \langle F^{(1)a}_{\mu\nu}(n) F^{(1)b}_{\mu\nu}(n) \rangle_W, \tag{6.14}$$

where the average is calculated with the Wilson action with the coupling constant g_0. Average appearing in (6.14) can easily be computed. For $d = 4$ (dimensionality of the space-time)

$$\langle F^{(1)a}_{\mu\nu}(n) F^{(2)b}_{\mu\nu}(n) \rangle = \frac{\delta^{ab}}{2} \tag{6.15}$$

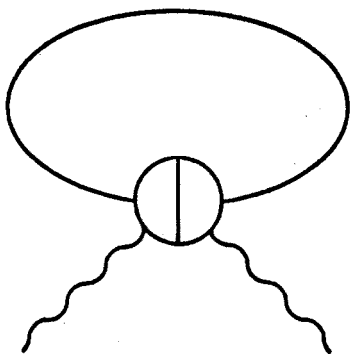

Fig. 4

which is independent on n. Dimensionality d affects only the coefficient in front of $\delta^{ab}(2/d)$, but not the other properties of (6.15). From (6.15) we can easily compute the one-loop corrected form of formula (6.12).

$$\frac{2}{g_0^2} = \beta_F + 2N\beta_A \omega(Ng_0^2) - N \frac{\beta_A}{2N^2} (Ng_0^2). \tag{6.16}$$

In (6.16) $\omega(Ng_0^2)$ should be taken to the one-loop (Ng_0^2) terms only to maintain the one-loop character of the formula. We must remember that all the averages computed here are the averages with the Wilson action and with the gauge fixing term chosen as in (5.24) with $\alpha = 1$. The two-loop corrections to formula (6.16) will be discussed in the next Section.

7. *Two-loop corrections*

We shall now proceed to compute the two-loop corrections to the formula (6.12). We shall first make a comment about the free part of the action used in our calculations. Expanding (6.10) in analogy with the Wilson case we see that the part of the action quadratic

in the quantum field is multiplied by the factor

$$\varphi = (\beta_F + 2N\beta_A \omega(Ng_0^2))\frac{g_0^2}{2}$$

$$= 1 - \frac{Ng_0^2}{2}\sum_{i=1}^{\infty}(\beta_A)^i f_i(Ng_0^2), \tag{7.1}$$

where we made use of (6.13). This factor was equal one if g_0^2 satisfied the zero-loop expression (6.12). The gauge fixing term used in analogy with the Wilson's case was

$$\frac{1}{\alpha}S_{g.f.} = \frac{1}{\alpha}\sum_{n}\mathrm{Tr}\left(\sum_{\nu} D_\nu^- A_\nu\right)^2, \tag{7.2}$$

with $\alpha = 1$. As we have shown in the last chapter the one-loop equation satisfied by g_0^2 is now (6.16). To this order (7.1) equals

$$\varphi = 1 + \frac{\beta_A}{4N^2}(Ng_0^2)^2 = 1 + \Delta. \tag{7.3}$$

In effect the whole quadratic part of the action (including terms interacting with the background field) will be increased by the same factor

$$S^{(2)} \to S^{(2)} + \Delta S^{(2)}. \tag{7.4}$$

The gauge fixing part of the action is not affected by this rescaling. To compute the two-loop form of the two-point function $\Gamma_2(B)$ we must take into account corrections to the one-loop diagrams resulting from (7.4). These will exactly cancel the two-loop diagrams in which one of the loops is connected with the β_A vertex and is obtained by joining with the gluon line two semicircles forming this vertex. As a simple example we can consider a diagram of the form represented in Fig. 5, where the β_A vertex appears as an insertion on the gluon propagator (no background field lines present). This subdiagram has exactly the same structure as the one-loop correction term discussed in the previous chapter, except

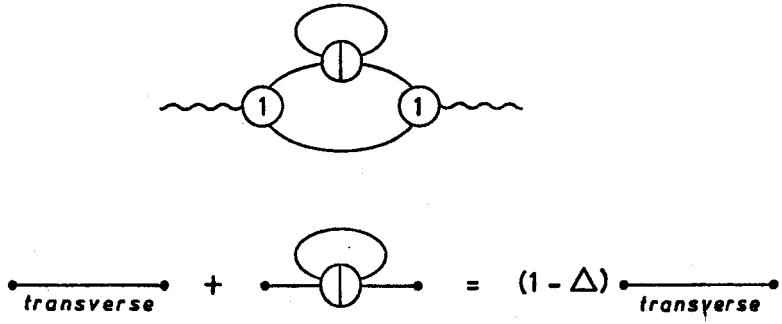

Fig. 5

that now the quantum lines replace the background field lines. As can be easily checked the two-loop diagrams containing a subdiagram of this form are exactly canceled by the corresponding one-loop diagrams, where the gluon propagator is rescaled according to (7.4). As can be seen from Fig. 5 the subdiagram of this form is equivalent with rescaling the transverse part of the gluon propagator by a factor Δ in the one-loop diagram. Rescaling (7.4) will produce a diagram of exactly the same form, but of opposite sign, the longitudinal part of the propagator being unaffected by this rescaling.

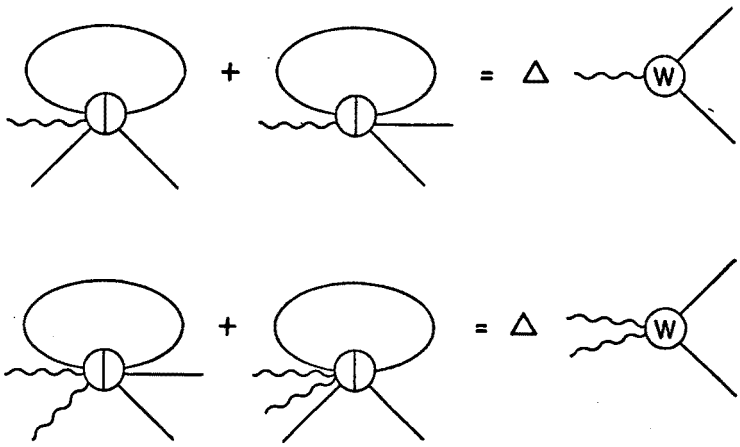

Fig. 6

Similarly we can show that subdiagrams similar to that in Fig. 5, but with one or two background field lines emanating, each have its partner resulting from the rescaling (7.4) of the quadratic part of the action. This is represented schematically in Fig. 6. To prove this relation we made use of the diagonal in colour and space indices form of the gluon propagator and of the antisymmetry of the SU(N) structure constants f^{abc}. In effect diagrams with one β_A vertex, where the two semicircles are connected with the gluon line are exactly canceled by the one-loop counterterms resulting from (7.4).

The remaining two-loop diagrams are either pure Wilson diagrams (with no β_A vertices) or *finite* two-loop diagrams with β_A vertices which do not contain subdiagrams described above. All these diagrams combine to the two-loop correction to the equation (6.13) satisfied by the bare coupling constant. As we can see, this way the diagrammatic expansion of $\Gamma_2(B)$ remains up to two loops exactly the same for the Wilson and mixed actions. We have only to compute the two-loop form of (6.13). Diagrams we have to calculate fall into two classes

(*i*) diagrams with one β_A vertex of the form similar to the one-loop diagram from Section 6 (Fig. 5), with one background field line $G^a_{\mu\nu}$ attached to each semicircle, but with two-loop structure on the gluon propagator, joining these semicircles (Fig. 7);

(*ii*) diagrams represented in Fig. 8 with two β_A vertices or one β_A and one Wilson vertex. Both vertices are linear in $G^a_{\mu\nu}$ and cubic in $F^{(1)b}_{\mu\nu}$.

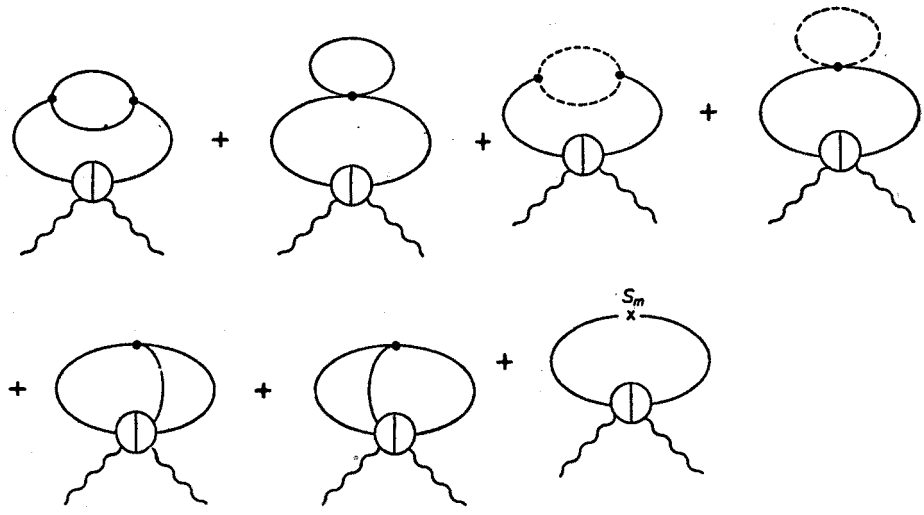

Fig. 7

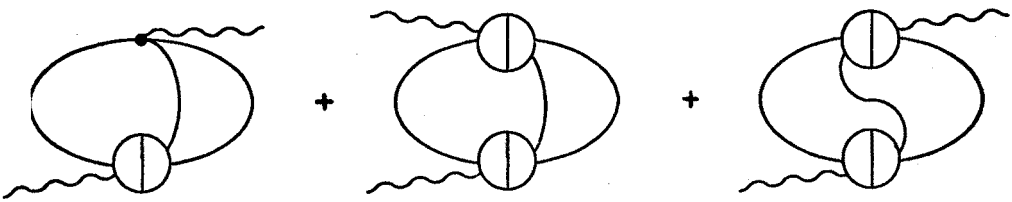

Fig. 8

We can easily obtain the closed form for all diagrams of the form described in (*i*). They have the general structure

$$\tfrac{1}{4}\beta_A a^4 \sum_{\substack{\mu<\nu \\ a,b \\ n}} G^a_{\mu\nu}(n) G^b_{\mu\nu}(n) \langle \mathrm{Tr}\, \lambda_a P_{\mu\nu}\, \mathrm{Tr}\, \lambda_b P^\dagger_{\mu\nu} \rangle, \tag{7.5}$$

where the average is computed with the Wilson action, with the coupling constant g_0. Notice that (7.5) contains also the one-loop result. From the colour symmetry we can argue that the average in (7.5) is of the form $\delta_{ab}X$. With the help of the formula

$$\sum_a (\lambda_a)_{ij}(\lambda_a)_{kl} = 2\left(\delta_{jk}\delta_{il} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) \tag{7.6}$$

X can be related to

$$X = \frac{2N}{N^2-1}\left(1 - \left\langle \frac{\mathrm{Tr}\, P_{\mu\nu}}{N} \frac{\mathrm{Tr}\, P^\dagger_{\mu\nu}}{N} \right\rangle\right)$$

$$= \frac{2N}{N^2-1}\left(1 - \omega^2(Ng_0^2) - \frac{1}{N^2}\langle \mathrm{Tr}\, P_{\mu\nu}\, \mathrm{Tr}\, P^\dagger_{\mu\nu} \rangle_c\right), \tag{7.7}$$

where $\omega(Ng_0^2)$ is the average action per plaquette in the Wilson's theory. The connected part in (7.7) is easily calculated on the two-loop level:

$$\frac{1}{N^2}\langle \mathrm{Tr}\, P_{\mu\nu}\, \mathrm{Tr}\, P_{\mu\nu}^{\dagger}\rangle_c = \frac{1}{32N^2}\left(1-\frac{1}{N^2}\right)(Ng_0^2)^2. \tag{7.8}$$

Function $\omega(Ng_0^2)$ was computed to two-loops and equals [50]

$$\omega(Ng_0^2) = 1-\frac{N^2-1}{8N^2}(Ng_0^2)-\frac{N^2-1}{4N^2}\left(0.0203-\frac{1}{32N^2}\right)(Ng_0^2)^2+\ldots \tag{7.9}$$

Diagrams (*ii*) forming the second subset of nonvanishing diagrams can be easily computed when we transform them to the momentum space. They all involve integral of the type

$$\mathscr{I} = \int \frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\, Y_{12}(p_1)Y_{12}(p_2)Y_{12}(p_1+p_2), \tag{7.10}$$

where

$$Y_{12}(p) = \frac{|\mathscr{P}_1(p)|^2+|\mathscr{P}_2(p)|^2}{\sum\limits_{\mu}|\mathscr{P}_\mu(p)|^2}, \tag{7.11}$$

with $\mathscr{P}_\mu(p) = e^{ip_\mu}-1$, and p_μ the lattice momentum. To evaluate $\mathscr{I}$ we notice that it does not depend on the pair of directions [1, 2]. We can write

$$Y_{12}(p) = 1-Y_{34}(p) \tag{7.12}$$

for $p = p_1$ and $p = p_2$ in Eq. (7.10) to get an equation

$$\mathscr{I} = 1-3\int \frac{d^4p_1}{(2\pi)^4}\, Y_{34}(p_1)+3\int \frac{d^4p_1 d^4p_2}{(2\pi)^4(2\pi)^4}\, Y_{34}(p_1)Y_{34}(p_2)-\mathscr{I}. \tag{7.13}$$

In obtaining (7.13) we changed variables where necessary, making use of the symmetry of (7.10) with respect to the momenta p_1, p_2 and p_1+p_2. Obviously from (7.13) and from the directional independence of integrals we find

$$\int \frac{d^4p_1}{(2\pi)^4}\, Y_{34}(p_1) = \tfrac{1}{2}, \tag{7.14}$$

and therefore $\mathscr{I} = \frac{1}{8}$. Diagrams considered above yield for $a \to 0$

$$N\left[\frac{\beta_A}{48N^2}\left(1-\frac{3}{2N^2}\right)(Ng_0^2)^2+\tfrac{1}{64}\frac{N^2+1}{N^4}\,\beta_A^2(Ng_0^2)^3\right]S_{\mathrm{cont}}(B), \tag{7.15}$$

where in derivation we made use of Eq. (7.6). Our final two-loop equation for the bare coupling constant is

$$\frac{2}{g_0^2} = \beta_F + 2N\beta_A\omega(Ng_0^2) + 2N\beta_A\left[-\frac{1}{N^2-1}(1-\omega^2(Ng_0^2))\right.$$
$$\left. - \frac{1}{48N^2}\left(1-\frac{3}{N^2}\right)(Ng_0^2)^2\right] + 2N\beta_A^2\left(-\tfrac{1}{64}\frac{N^2+1}{N^4}(Ng_0^2)^3\right). \tag{7.16}$$

Expressions for $\omega(Ng_0^2)$ and $1-\omega^2(Ng_0^2)$ should be used in (7.16) to the two-loop order $(Ng_0^2)^2$.

The method outlined above can in principle be continued to higher loops. The two-loop equation (7.16) will again introduce rescaling of the bare coupling constant, necessary to reduce all divergent three-loop diagrams with β_A vertices to one- and two-loop diagrams of the pure Wilson's theory.

The last remark in this Section considers the form of (7.16) when the N of the gauge group $N \to \infty$. In this limit $G_\infty^2 = (Ng_0^2)$ remains finite (and also $\beta_\infty = \beta_F/N$ rather than β_F) and we should rewrite (7.16) as equation for this quantity. When $N = \infty$ this equation simplifies considerably:

$$\frac{1}{G_\infty^2} = \frac{\beta_\infty}{2} + \beta_A\omega(G_\infty^2). \tag{7.17}$$

It is exactly the form predicted on the basis of general considerations making use of the factorization of the Wilson loops in this limit by Makeenko and Polikarpov [51]. All higher-loop corrections are suppressed as powers of $1/N^2$.

8. Comparison with experiment

In the last few years the numerical Monte Carlo simulations have given many interesting results concerning lattice gauge theories. Among them probably the most important was possibility to check experimentally relation (4.11) between the lattice string tension and the bare coupling constant [4]. In all numerical experiments the lattice was finite with periodic boundary conditions. Since, as we have shown in Section 4, the physical length of the lattice bonds is also related by the same relation to the bare coupling constant, the physical size of the lattice shrinks to zero exponentially when coupling gets smaller. In effect it is by no means obvious that the scaling regime, when Eq. (4.11) holds, can at all be reached in computer experiments and the fact that apparently it can, must be considered as a lucky coincidence. This means that there exists a region in the coupling constant where the physical size of the lattice is big enough for the Wilson loops which can be built in such a box to show the *area* law (4.4). This region in the coupling constant must be finite, since decreasing the coupling constant we finally reach a limit when the box is too small to see physically interesting effects. Relation (4.11) is thus tested for values of g_0^2 far from zero and one may expect that higher order effects may be very important if one wants to extract from the experiment the value of the lattice scale Λ_{lattice}.

With the help of universality relation derived in the last chapter we can try to check the importance of higher-order effects in the region of the coupling constant where experiments are performed. We shall do it by comparing results obtained for a Wilson system with those for the mixed system. Universality relation (7.16) (or in general (6.13)) predicts that on the $\{\beta_F, \beta_A\}$ plane one can draw lines corresponding to the same value of g_0^2, along which physics is the same. The same physics means in particular the same string tension. By measuring string tension in various points on the $\{\beta_F, \beta_A\}$ plane the shape of these lines can be obtained experimentally and compared with theoretical prediction.

The present comparison [40] is based on the numerical data of Bowler et al. [52] to get an overall view and on the high statistics data of Barkai et al. [53, 54] to settle the fine points. The numerical data in all these cases are for the SU(3) system with the mixed [54] and Wilson [53] action.

Our universality formula can be rewritten as an equation for the constant physics line

$$\beta_F = \beta_F(\beta_A, \beta_W), \tag{8.1}$$

where $\beta_W = \frac{2}{g_0^2}$ is the corresponding value of the pure Wilson theory. Identifying (Ng_0^2) with $2N/\beta_W$ ($N = 3$ in this case) we get

$$\beta_F = \beta_W - 6\beta_A\left(\omega - \tfrac{1}{8}(1-\omega^2) - \frac{1}{18\beta_W^2}\right) + \beta_A^2 \tfrac{5}{2}\frac{1}{\beta_W^3}, \tag{8.2}$$

where $\omega = \omega(6/\beta_W)$ is the average plaquette given to the two-loop order by formula (7.9). In Fig. 9 we compare the lines given by (8.2) for various β_W with equal-string-tension curves from the Monte Carlo experiment [52]. The continuous lines are the experimental curves and the dotted line is the prediction of formula (8.2). The dashed line and the crosses represent location of the Monte Carlo experiment [54]. In all figures β_F and β_A are scaled by an extra factor 9 ($= N^2$). This comes from the different definition of coupling constants used in these papers. We can see that the agreement is not very good, particularly at low β_W (larger g_0^2). For larger β_W the agreement improves. Experimental lines are practically straight which can be well understood by the smallness of the leading term proportional to β_A^2 in (8.2), at least in the range of β_A considered in this experiment. The main discrepancy seems to come from the error in the slope (part proportional to β_A in (8.2)). The range of β_W considered above is rather limited. Of course if it would be numerically possible to go beyond this range, agreement would get much better. As we explained before this would however require using much bigger lattices than those in the experiment. The β_W value, where according to the authors the scaling regime sets in is supposed to be $9\beta_W \sim 5.8$.

To trace the source of discrepancy it is instructive to compare the experimentally measured values of the average plaquette action in the considered range of the coupling constant β_W with its two-loop form (7.9). Experimental points have negligible errors and we can expect to compare this way the exact function with its two-loop approximation. The effect of this exercise is displayed in Fig. 10. The two-loop curve is represented by

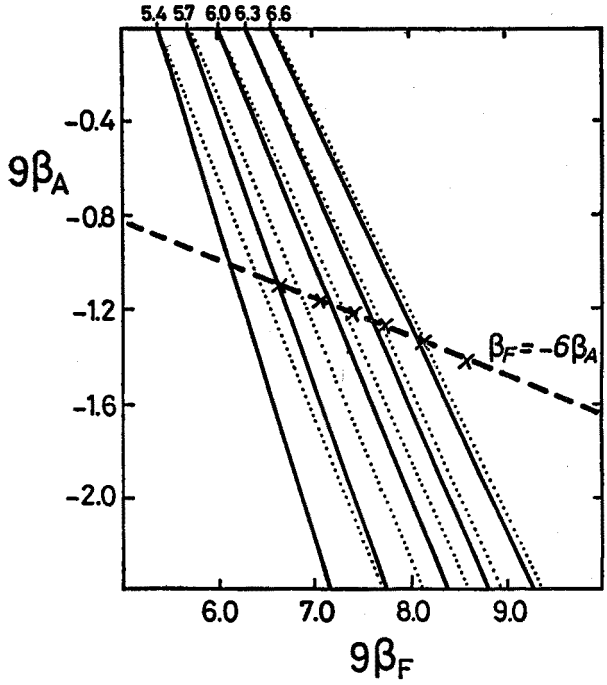
5.4 5.7 6.0 6.3 6.6
-0.4
-0.8
-1.2
-1.6
-2.0
9βA
6.0
7.0
8.0
9.0
9βF
βF = -6βA

Fig. 9

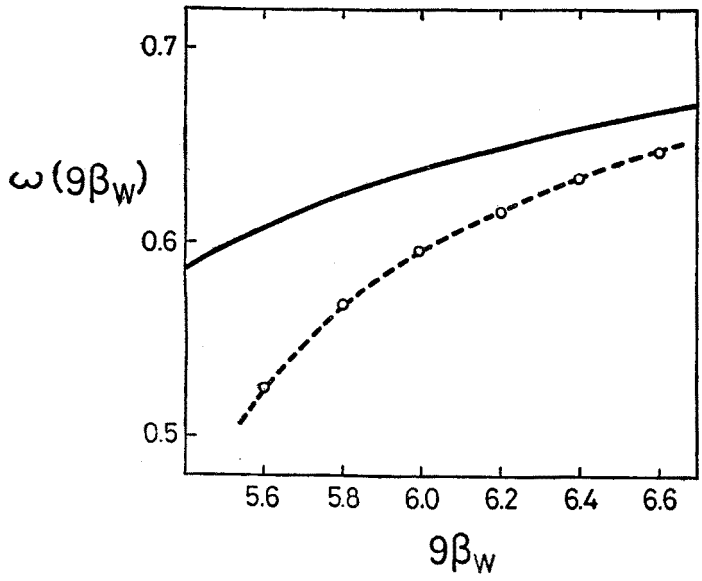
0.7
0.6
0.5
ω(9βW)
5.6
5.8
6.0
6.2
6.4
6.6
9βW

Fig. 10

a continuous line. Experimental values (open circles) are taken from Ref. [53] and the broken line extrapolates between these points. Note that the scale on the ω axis is very large to show better the difference between the two values. We can see that in the measured region the two-loop approximation becomes steadily worse and probably third- or even higher-order terms would be necessary to improve the agreement. In Fig. 11 we show the comparison of the experimental slope of the equal-string tension curves with that predicted

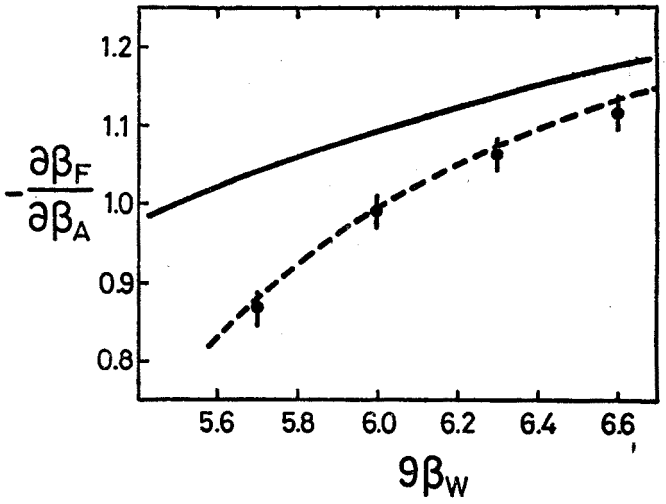

Fig. 11

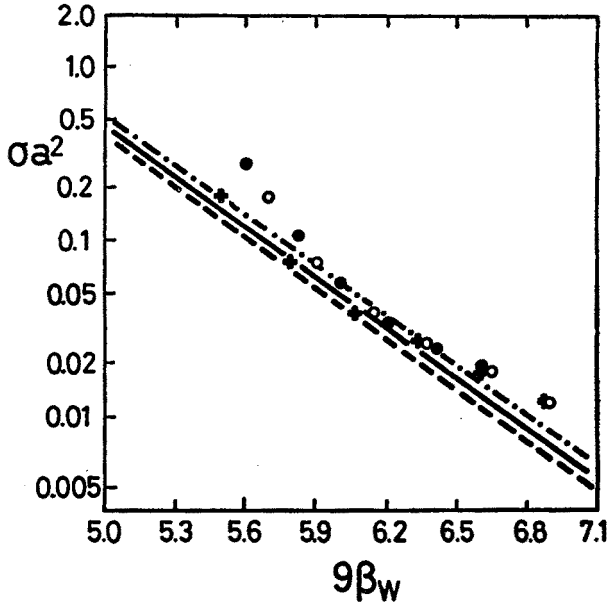

Fig. 12

at $\beta_A = 0$ by formula (8.2). Again the two-loop prediction is represented by a solid line. The curve, where experimental values of ω were taken instead of the computed two-loop values is also displayed (broken line). Agreement of the latter with experimental points is remarkable, showing that probably the universality relation is dominated by diagrams discussed as point (*i*) in Section 7.

In Fig. 12 we show the string tension for the pure Wilson theory (black dots) taken from Ref. [53] compared with that for the mixed action (Ref. [54]) extrapolated with the two-loop formula (crosses) and the semi-empirical one (ω taken from the experiment — open circles). Again the consistency of the latter with the Wilson experiment is remarkable.

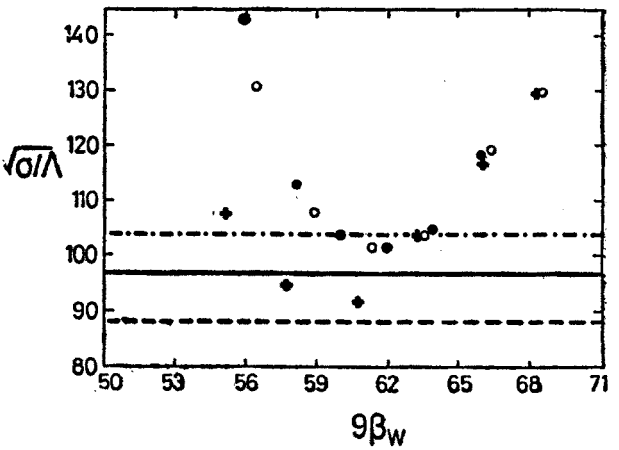

Fig. 13

In Fig. 13 we show the string-tension scale parameter ratio $\sqrt{\sigma}/\Lambda$ obtained by using formula (4.11) for various g_0^2. The two forms of extrapolation are again used for the mixed action results (crosses and open dots), the points for the Wilson action are also displayed (black dots). Figure shows that again points for the mixed action give results consistent with the Wilson theory only if the semiempirical extrapolation formula is used. If we would like to extract from this figure experimental value for $\sqrt{\sigma}/\Lambda$ using the two-loop extrapolation formula (8.2) and following the method proposed in Ref. [53] for the Wilson action we would get for the mixed action

$$\sqrt{\sigma}/\Lambda = 94.0$$

to be compared with $\sqrt{\sigma}/\Lambda = 103.8$ for the Wilson theory. These two numbers should be the same if the two-loop extrapolation formula was exact!

It is important to observe at this point that the method employed to determine the lattice scale uses itself the two-loop formula (4.11). Our exercise shows that the region of the coupling constant used in this experiment is still too distant from the "two-loop regime" to have a confidence in the consistency of results. In effect we expect that the "measured" values of the lattice scale may be subject to a systematic error of 5%–10%.

One way to improve the accuracy is to increase the size of the lattice system used in

Monte-Carlo simulations and push measurements to smaller values of the coupling constant g_0. This should be possible with the help of the new generation of special-purpose computers which now are being constructed.

9. *Weak coupling expansion on a finite lattice*

In the Monte Carlo experiments Wilson loops are measured on lattices which are always finite with periodic boundary conditions. In the following we shall discuss properties of the weak coupling expansion formulated on such lattices [40]. We shall be interested in particular in the role played by the zero momentum modes, which escape the standard perturbative treatment of the theory.

In the following we shall always consider the Wilson action for the lattice system. Results of this analysis can easily be generalized for other forms of the action. To understand better the structure of the perturbative expansion remember the form of the action we discussed in Section 5. Let us consider the case $B_\mu(n) = 0$ when all the covariant derivatives can be replaced by ordinary lattice derivatives. Let us parametrize the link variables $U_\mu(n) = \exp[iA_\mu(n)]$ (we did not include g_0 in the exponent) and use $\beta = \frac{2}{g_0^2}$. For the quadratic part of the action we get [42, 43]

$$S_W + S_{g.f.} = \frac{2}{g_0^2} \sum_{\nu,n} \mathrm{Tr}\left(A_\nu(n) \sum_\mu (\Delta_\mu^- \Delta_\mu^+) A_\nu(n) \right). \tag{9.1}$$

Let us assume that our system is a finite periodic box $L_1 \times L_2 \times \ldots L_d$. Introduce Fourier momentum components of the fields

$$A_\nu(n) = \frac{1}{V^{1/2}} \sum_{\{p\}} Q_\nu(p) e^{-ip\cdot n},$$

$$Q_\nu(p) = \frac{1}{V^{1/2}} \sum_{\{n\}} A_\nu(n) e^{ip\cdot n}, \tag{9.2}$$

where the momentum p_μ in direction $\mu = 1, \ldots, d$ takes discrete values

$$p_\mu = \frac{2\pi}{L_\mu} n_\mu, \qquad n_\mu = 0, 1, \ldots L_\mu - 1 \tag{9.3}$$

and where

$$Q_\nu(-p) = Q_\nu^\dagger(p). \tag{9.4}$$

We can rewrite (9.1) as

$$S_W + S_{g.f.} = \frac{2}{g_0^2} \sum_{\nu,p} \mathrm{Tr}\left(Q_\nu(-p) \sum_\mu |\mathscr{P}_\mu^\pm - 1|^2 Q_\nu(p) \right), \tag{9.5}$$

where

$$\mathscr{P}_\mu^{\pm} = e^{\pm ip_\mu} - 1. \tag{9.6}$$

From (9.5) we see that all the momentum-carrying modes are indeed Gaussian modes and scale for $g_0 \to 0$ as

$$Q_\mu(p) \to g_0 \tilde{Q}_\mu(p). \tag{9.7}$$

Interactions of these modes will produce a standard perturbative expansion [42].

The zero-momentum field is a zero mode of the quadratic action, but because of the non-abelian character of the theory, it will appear in the interaction terms. To understand better the nature of this field let us consider a configuration, where only this field is present. This configuration corresponds to a constant field

$$B_\mu = \frac{1}{V^{1/2}} Q_\mu(0) \tag{9.8}$$

or constant link variable $U_\mu = \exp(iB_\mu)$ on all links in direction μ. The plaquette variables $P_{\mu\nu}$ are also constant

$$P_{\mu\nu} = U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger \tag{9.9}$$

and when we expand around $B_\mu = 0$ terms like $((\mathrm{Tr}\, P_{\mu\nu} - N) + \mathrm{c.c.})$ we get for the corresponding Wilson action

$$S_W^{(0)} = -\frac{V}{g_0^2} \sum_{\mu<\nu} \mathrm{Tr}\,[B_\mu, B_\nu]^2 + O(B^5). \tag{9.10}$$

Equation (9.10) suggests that zero-momentum modes may have a different scaling than the momentum-carrying modes. This includes not only the g_0 but also volume dependence. In effect these modes may compete in perturbative expansion with other, more populous modes. In fact the form of the zero-momentum part of the action is the same as the Wilson action on a one-point periodic box. Properties of such a system were discussed in an earlier publication [41] and many results obtained there will find a direct application here. Eq. (9.9) shows us that one can also expect a more complicated structure for the vacuum (minimum of the action). All configurations with

$$[U_\mu, U_\nu] = 0 \tag{9.11}$$

are classically equivalent, but may have different quantum properties.

10. Non-constant modes in the finite box

In this Section we shall discuss the computation of the lattice partition function Z on a periodic box

$$Z = \int \prod_{\mu,n} DU_\mu(n) \exp\left(-\frac{1}{g_0^2} S(U)\right), \tag{10.1}$$

where $S(U)$ is the Wilson action (3.10). We shall use the parametrisation in which the constant commuting modes play a special role. Since all translationally invariant saddle points of the action, satisfying (9.11) can be brought to a set of constant diagonal fields we parametrize the link matrices as

$$U_\mu(n) = \exp(iA_\mu(n)) \exp(iB_\mu), \tag{10.2}$$

where B_μ are diagonal and $A_\mu(n)$ satisfy the constraint

$$Q_\mu^{\text{diag}}(0) = \frac{1}{V^{1/2}} \sum_n A_\mu^{\text{diag}}(n) = 0. \tag{10.3}$$

Equations (10.2) and (10.3) define B_μ and $A_\mu(n)$ in terms of $U_\mu(n)$. Reparametrisation (10.2) will modify the measure in (10.1). Let us compute the Jacobian which goes with it

$$\prod_n DU_\mu(n) = \prod_n dA_\mu(n) dB_\mu \delta\left(\frac{1}{V^{1/2}} \sum_n A_\mu^{\text{diag}}(n)\right) J_\mu(A, B). \tag{10.4}$$

Consider an arbitrary diagonal and constant unitary matrix C_μ. For a given $U_\mu(n)$ define the matrix $A_\mu^C(n)$ by

$$\exp(iA_\mu^C(n)) \equiv U_\mu(n) C_\mu^\dagger(n). \tag{10.5}$$

Consider the diagonal part of $A_\mu^C(n)$ and form the following functional of the configuration $\{U_\mu(n)\}$:

$$I_\mu(U) \equiv \int dC_\mu \delta\left(\frac{1}{V^{1/2}} \sum_n A_\mu^{C\text{diag}}(n)\right). \tag{10.6}$$

This functional is clearly an invariant on all the configurations of the form $U_\mu(n)V_\mu$ with arbitrary diagonal and constant V_μ since V_μ can be reabsorbed in the integration (10.6). In (10.6) the measure dC_μ is flat.

The Jacobian J_μ becomes

$$J_\mu(A, B) = \exp(-\sum_n S_m(n) - \log I_\mu(U)), \tag{10.7}$$

where S_m is the traditional local measure factor defined by

$$DU_\mu = dA_\mu \exp(-S_m(A_\mu)). \tag{10.8}$$

We can verify (10.7) by writing (10.4) as

$$\prod_n DU_\mu(n) = \prod_n dU_\mu(n) \int dC_\mu \delta\left(\frac{1}{V^{1/2}} \sum_n A_\mu^{C\text{diag}}(n)\right) I_\mu^{-1}$$

$$= \int dB_\mu \prod_n dA_\mu \exp\left(-\sum_n S_m(n) - \log I_\mu\right) \delta\left(\frac{1}{V^{1/2}} \sum_n A_\mu^{\text{diag}}(n)\right).$$

In the last equality we used the fact that the measure of the abelian subgroup is flat $dC_\mu = dB_\mu$. The importance of (10.7) lies in the fact that I_μ depends only on A_μ, furthermore both S_m and $\log I_\mu$ behave like A^2 for small A.

In this way we have checked that, when the gauge fixing will be introduced, measure terms in the action will not influence our parametrisation in which the diagonal B fields are minima of the action and A fields are fluctuations around them. This is in contrast to parametrisation where one studies fluctuations around *any* set of constant potentials. See e.g. Refs [55–58].

Before we proceed with the computation of the partition function Z let us observe an important symmetry of the parametrisation (10.2). Consider a gauge transformation $\Omega_\mu(n)$ which is periodic modulo the element of the center $Z(N)$ of the group SU(N) and has a form:

$$Q_\mu(n) = \exp\left(-i\frac{2\pi H_\mu}{NL_\mu}n_\mu\right). \tag{10.9}$$

Here H_μ is a diagonal, traceless integer matrix and there is *no* summation over μ. The eigenvalues h_μ^i of the matrix H_μ can be written as

$$h_\mu^i = k_\mu + Nl_\mu^i,$$

where

$$k_\mu = -\sum_i l_\mu^i.$$

Under such transformation the link variable becomes

$$U'_\mu(n) = \Omega_\mu^\dagger(n)U_\mu(n)\Omega_\mu(n+\mu). \tag{10.10}$$

This results in a local rotation of $A_\mu(n)$ and a shift on B_μ

$$A'_\mu(n) = \Omega_\mu^\dagger(n)A_\mu(n)\Omega_\mu(n),$$

$$B'_\mu = B_\mu - \frac{2\pi}{NL_\mu}H_\mu. \tag{10.11}$$

Variables $A_\nu(n)$ for $\nu \neq \mu$ are only rotated and corresponding B_ν are unchanged. Diagonal matrices B_μ can be parametrized by their diagonal elements B_μ^i with a constraint $\sum_i B_\mu^i = 0$.

The shift in Eq. (10.10) can be rewritten in terms of these elements as

$$B_\mu^{i'} = B_\mu^i - \frac{2\pi}{L_\mu}l_\mu^i - \frac{2\pi}{NL_\mu}k_\mu,$$

which shows that we can always find a gauge transformation of the form (10.9) which transforms a configuration with arbitrary values of B_μ^i into a configuration where all $N(N-1)$

variables B_μ^{ij}:

$$B_\mu^{ij} \equiv B_\mu^i - B_\mu^j,$$

$$-\frac{2\pi}{L_\mu} < B_\mu^{ij} < \frac{2\pi}{L_\mu}. \tag{10.12}$$

As we shall see variables B_μ^{ij} will play a special role.

Combining transformations of the form (10.9) we can satisfy relations (10.12) for all $\mu = 1, \ldots, d$. The importance of these constraints will become clear if we compute the effect of the rotation (10.11) on the fields $Q_\nu(p)$. We shall express the result as a relation for the (i,j) elements Q^{ij} and $Q^{ij'}$ of the $N \times N$ matrices Q. Note that this relation will be the same for all ν. Recalling definition (9.2) of the field Q_ν we see that

$$Q_\nu^{ij'}(p) = Q_\nu^{ij}(p'),$$

where

$$p'_\mu = p_\mu + \frac{2\pi}{L_\mu}(l_\mu^i - l_\mu^j), \quad p'_\varrho = p_\varrho, \varrho \neq \mu. \tag{10.13}$$

Eq. (10.13) means that the effect of the gauge transformation (10.9) on the fields $Q_\nu(p)$ is a translation in the momentum space by $(l_\mu^i - l_\mu^j)$ units (cf. (9.3)). The parametrisation which satisfies (10.12) is necessary in order to have a well defined concept of momentum for the Q fields. Configurations where (10.12) is not satisfied are gauge equivalent to configurations with shifted B and redefined momenta.

Let us now discuss in more detail the choice of the gauge fixing term. This term must be chosen in such a way that quantum variables $Q_\mu(p \neq 0)$ tend to zero when g_0 goes to zero. The gauge choice which seems the most promising for our objective is the background gauge (5.24), where we identify the background field with the constant, diagonal and commuting field B_μ. In the momentum representation the gauge fixing term becomes

$$S_{\text{g.f.}}(p \neq 0) = \tfrac{1}{2} \sum_{\substack{p \neq 0 \\ i,j}} |\sum_\mu \mathscr{P}_\mu^-(p + B^{ij}) Q_\mu^{ij}(p)|^2. \tag{10.14}$$

In (10.14) Q_μ^{ij} is the (i,j) element of the $N \times N$ matrix Q_μ and the lattice momentum $\mathscr{P}_\mu^\pm(q)$ is defined as before:

$$\mathscr{P}_\mu^\pm(q) \equiv e^{\pm i q_\mu} - 1. \tag{10.15}$$

The background field enters (10.14) only through the differences of eigenvalues (10.12). We can note here that the momenta $\mathscr{P}_\mu(p + B^{ij})$ are never vanishing as long as $p \neq 0$, provided B_μ^{ij} satisfy constraint (10.12). Again this constraint is essential, since otherwise we could get new zero modes of the quadratic part of the action. As can be seen from the discussion these new "would be zero-modes" are simply the gauge copies of the zero modes we already have.

Let us postpone the discussion of the gauge fixing in the zero-momentum sector to the next Section. It will turn out that the $Q_\mu^{ij}(p=0)$ variables are at least of order $g_0^{1/2}$ for any B field configuration if we choose the right gauge in this sector.

The ghost determinant one obtains from (10.15), admitting only non-constant ($p \neq 0$) gauge transformations, reads to leading order

$$S_{\mathrm{gh}}(p \neq 0) = \sum_{\substack{p \neq 0 \\ i,j}} \bar{\eta}^{ij}(p) \mathscr{P}^2(p + B^{ij}) \omega^{ij}(-p) + O(g_0^{1/2}), \tag{10.16}$$

where

$$\mathscr{P}^2(p) = \sum_\mu |\mathscr{P}_\mu^{\pm}(p)|^2. \tag{10.17}$$

Including all the terms discussed so far we get an effective action

$$S_{\mathrm{eff}} = S_{\mathrm{W}} + S_{\mathrm{g.f.}} + S_{\mathrm{gh}} + S_{\mathrm{ms}}, \tag{10.18}$$

where $S_{\mathrm{ms}} = g_0^2 \log J_\mu(Q)$. Rescaling the momentum-carrying modes by a factor g_0 (cf. (9.7)) we get explicitly:

$$\begin{aligned} \frac{1}{g_0^2} S_{\mathrm{eff}} = & \sum_{\substack{p \neq 0 \\ \mu,i,j}} (Q_\mu^{ij}(p) \mathscr{P}^2(p + B^{ij}) Q_\mu^{ij}(-p)) \\ & + \sum_{\substack{p \neq 0 \\ i,j}} (\bar{\eta}^{ij}(p) \mathscr{P}^2(p + B^{ij}) \omega^{ij}(-p)) \\ & + \frac{V}{g_0^2} \sum_{\mu,\nu} \mathrm{Tr}\,(\mathbf{1} - U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger) + O(g_0^{1/2}). \end{aligned} \tag{10.19}$$

The terms with order $g_0^{1/2}$ start with cubic terms with one zero-momentum Q field. As will be discussed in the next Section, it is more convenient to keep the zero-momentum action in its integrality, without scaling out the coupling. The variables U_μ are defined as

$$U_\mu = \exp(iQ_\mu(p=0)) \exp(iB_\mu) \tag{10.20}$$

and $Q_\mu(p=0)$ having no diagonal elements.

We can now perform in the partition function all the integrals over the non-zero-momentum degrees of freedom in the leading order. To the one loop terms the result is a determinant which can be computed immediately by observing that the corresponding part in (10.19) is diagonal. The result is

$$Z_\varphi(B) = \prod_{\substack{p \neq 0 \\ i,j}} [\mathscr{P}^2(p + B^{ij})]^{-\frac{d-2}{2}}.$$

The free energy $F_\varphi(B)$ is defined as

$$F_\varphi(B) = -\log Z_\varphi(B)$$
$$= \frac{d-2}{2} \sum_{\substack{p \neq 0 \\ i,j}} \log (\mathscr{P}^2(p+B^{ij})). \tag{10.21}$$

The term proportional to d comes from the d polarisations of the Q variables, whereas the term -2 comes from the ghost.

Let us normalize this free energy by that of a free gluon gas

$$F_{\mathrm{gl}} = \frac{d-2}{2}(N^2-1) \sum_{p \neq 0} \log \mathscr{P}^2(p) \tag{10.22}$$

and define

$$\Delta F_\varphi(B) = F_\varphi(B) - F_{\mathrm{gl}}. \tag{10.23}$$

This function contains all the information concerning nontrivial B dependence of the determinant (10.21). In view of (10.12) it is useful to consider ΔF_φ as a function of $\hat{B}_\mu$ where

$$\hat{B}_\mu = L_\mu B_\mu \tag{10.24}$$

and the remaining L_μ dependence is tacitly understood.

11. Behaviour of the determinant of the non-constant modes

In the last Section we have computed the determinant over the non-constant modes of the system. This determinant was given in terms of a finite, but in general large product. Here we shall discuss its behaviour for small B and some aspects of the global behaviour of this determinant. The derivation of the global form of this determinant will be given in the Appendix A.

The first remark to be made is that $\Delta F_\varphi(\hat{B})$ is symmetric under $\hat{B}^{ij}_\mu \to -\hat{B}^{ij}_\mu$. Therefore, if it is analytic at $\hat{B} = 0$, it admits the expansion

$$\Delta F_\varphi(\hat{B}) = \tfrac{1}{2} \sum_\mu m_\mu^2 \sum_{i,j} (\hat{B}^{ij}_\mu)^2 + O(\hat{B}^4). \tag{11.1}$$

The coefficients in such an expansion can be computed by directly performing the sum over one of the momentum components. If we use the identity

$$\prod_{p_\mu} (z - e^{ip_\mu}) = z^{L_\mu} - 1 \tag{11.2}$$

the following quantity comes in naturally when doing the summation:

$$\mathrm{ch}\, q(p_\perp) \equiv 1 + 2 \sum_{\varrho \neq \mu} \sin^2 \frac{p_\varrho}{2}. \tag{11.3}$$

For small p_ϱ $q(p_\perp)$ reduces to the modulus $(\sum_{\varrho \neq \mu} p_\varrho^2)^{1/2}$.

Finally we obtain for the *mass* in (11.1)

$$\frac{m_\mu^2}{d-2} = -\tfrac{1}{6}\left(1-\frac{1}{L_\mu^2}\right) + \tfrac{1}{2}\sum_{p_\perp \neq 0} \sinh^{-2}(\tfrac{1}{2} L_\mu q(p_\perp)). \tag{11.4}$$

In (11.4) the sum is over $d-1$ dimensional lattice of momenta with $\varrho \neq \mu$. Expression (11.4) coincides with the one obtained in Ref. [42] and [43]. Their results were derived, strictly speaking, by expanding around constant *non*-commuting background fields. However, to order $\hat{B}^2$ the resulting free energy is the same, since commutator terms can appear only in higher orders of expansion (11.1). One may be surprised to discover terms of the form (11.1) after a discussion concerning the effective action $\Gamma_2(B)$ in Section 5. Mass terms like (11.1) would break the gauge invariance of $\Gamma_2(B)$ strongly advocated there. The answer to this controversy is that (11.1) is a typical finite-size effect, which will not play a role in global quantities like $\Gamma_2(B)$. It can be considered as a modification in the measure of integration for the zero-modes.

We shall now look at various limits for these expressions in the case $d = 4$:

(*i*) Isotropic case ($L_\mu = L$, $\mu = 1, \ldots, 4$, $L \to \infty$). In this case $m_\mu^2 = m^2$ and m^2 is *negative* (it was erroneously assumed to be positive in Ref. [43]). Numerically we get

$$m^2 = -0.140\ldots$$

In the isotropic infinite-volume limit the quantity $L_\mu q(p_\perp)$ stays always finite and non-zero. Therefore, no infrared problems are met and this remains true for all higher orders in $\hat{B}$: function $\Delta F_\varphi(\hat{B})$ is analytic in this limit. Thus the $\hat{B}$-dependent part of the free energy is of order V^{-1} with respect to the free energy of a free gluon gas. This means that the one mode we did *not* take into account, namely the $p = 0$ mode, may well modify it (and actually does, as will be shown in Section 13).

(*ii*) Finite-temperature limit. We keep $L_4 = T^{-1}$ fixed and let $L_1 = L_2 = L_3 = L_s$ become large. Infrared effects start to play a role because $L_4 q(p_\perp) \to 0$ as $L_s \to \infty$. This implies for the *mass* in the fourth direction:

$$\tfrac{1}{2} m_4^2 = L_s^3 \int \frac{d^3p}{(2\pi)^3} \sinh^{-2}(\tfrac{1}{2} L_4 q(p_\perp)) = L_s^3 \frac{1}{3L_4^3}. \tag{11.5}$$

We neglected terms with less than volume divergence and took the continuum limit. This result, first obtained by Weiss, represents the electric mass the gluon acquires in the high-temperature phase [59–61]. This mass is positive. The other, spacelike, masses are all equal in this limit. For the computation of, say, m_3 one has to look at the behaviour of $L_3 q(p_\perp)$ (Eq. (11.3)). Since $L_3 \to \infty$, this quantity will stay finite only if $p_4 = 0$, and never becomes zero. The free energy per spacelike volume $f(\hat{B}) = \Delta F_\varphi(\hat{B})/L_s^3$ does to order $\hat{B}^4$ not depend any more on the spacelike background fields. The latter do still play a role in the terms of order L_s^{-3} as we shall show in the Appendix A.

The coefficient of $\hat{B}^4$ has a divergence of order L_s^4. This means that function $f(\hat{B})$ is non-analytic due to infrared divergencies.

The global behaviour of determinant (10.21) is discussed in Appendix A. For our discussion the most interesting is case (*i*). The result can be summarized as follows: in the region of integration over the $\hat{B}$ fields determinant is an analytic function of $\hat{B}$. It becomes singular only when the constraint (10.12) is violated. If the $\hat{B}$ field scales as $g_0^{1/2}$ this determinant will give a contribution $O(g_0)$ to the partition function Z. In order that a perturbation theory for these modes could be constructed we shall need a gauge fixing in the zero-momentum sector such that the non-diagonal zero-momentum fields $Q_\nu(0)$ are forced to zero for all values of $\hat{B}$ (also if $B \sim O(g_0^{1/2})$). This problem will be discussed in the next Section.

12. Contribution from the zero-momentum sector

As we mentioned in the last Section the determinant over non-constant modes of the action (10.19) is an analytic function of $\hat{B}_\mu^{ij} = L_\mu B_\mu^{ij}$ in the integration domain (10.12) when we take the isotropic limit $L_\mu = L \to \infty$. This result remains true also in the limit $L_\mu = a_\mu L, L \to \infty, a_\mu$ finite. This means that information about the weak coupling behaviour of the zero-momentum sector is contained in the zero-momentum part of S_{eff}, Eq. (10.19) and that the computed determinant will contribute only in higher orders of perturbation around the saddle point of the action $S_{\text{eff}}^{(0)}$

$$\frac{1}{g_0^2} S_{\text{eff}}^{(0)} = \frac{V}{g_0^2} \sum_{\mu,\nu} \operatorname{Tr} (\mathbf{1} - U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger), \tag{12.1}$$

where $\mathbf{1}$ is $N \times N$ unit matrix and

$$U_\mu = \exp (iQ_\mu(p = 0)) \exp (iB_\mu). \tag{12.2}$$

Action (12.1) is invariant under a global gauge transformation and under a multiplication of a link variable U_μ by an element of the center of the SU(N) group. We can rewrite (12.1) as

$$S_{\text{eff}}^{(0)} = \frac{V}{2} \sum_{\mu,\nu} \operatorname{Tr} G_{\mu\nu} G_{\mu\nu}^\dagger, \tag{12.3}$$

where

$$G_{\mu\nu} \equiv [U_\mu, U_\nu]. \tag{12.4}$$

(12.4) shows that saddle points of (12.1) correspond to $G_{\mu\nu} = 0$ and can be brought, by means of the global gauge transformation, to the diagonal form. Parametrisation (12.2) can be used to study fluctuations around the saddle points or, as we shall call them, *torons*. Let us introduce the "momentum" operator $\mathscr{P}_\mu^\pm(B)$

$$\mathscr{P}_\mu^\pm(B) Q_\nu = e^{\pm iB_\mu} Q_\nu e^{\mp iB_\mu} - Q_\nu. \tag{12.5}$$

Using this operator we can expand the action (12.1) in powers of Q_ν:

$$S_{\text{eff}}^{(0)} = V \sum_{\mu,\nu} \text{Tr}\,\{(\mathscr{P}_\mu^+(B)Q_\nu - \mathscr{P}_\nu^+(B)Q_\mu)^2$$
$$+(\mathscr{P}_\mu^+(B)Q_\nu - \mathscr{P}_\nu^+(B)Q_\mu)\,(2[Q_\mu, Q_\nu] + [Q_\mu, \mathscr{P}_\mu^+(B)Q_\nu]$$
$$+[Q_\nu, \mathscr{P}_\nu^+(B)Q_\mu] - [\mathscr{P}_\nu^+(B)Q_\mu, \mathscr{P}_\mu^+(B)Q_\nu])$$
$$-[Q_\mu, Q_\nu]^2 + \ldots\}. \tag{12.6}$$

It is useful to note that if B becomes small and of order Q, then (12.6) becomes

$$S_{\text{eff}}^{(0)} = -V \sum_{\mu,\nu} \text{Tr}\,[A_\mu, A_\nu]^2, \tag{12.7}$$

where $A_\mu \equiv B_\mu + Q_\mu$.

The action of the momentum operator (12.5) on the (i,j) element of the matrix Q_ν is

$$\mathscr{P}_\mu^+(B)Q_\nu^{ij} = (e^{i(B_\mu{}^i - B_\mu{}^j)} - 1)Q_\nu^{ij} = \mathscr{P}_\mu^{ij}(B)Q_\nu^{ij}. \tag{12.8}$$

If the matrix B_μ is degenerate, i.e. eigenvalues B_μ^i and B_μ^j are the same, then the corresponding $\mathscr{P}_\mu^{ij}(B) = 0$. For one μ we can use the global gauge symmetry to order eigenvalues of B_μ in such a way that equal eigenvalues follow each other. If we write $\mathscr{P}_\mu^{ij}(B)$ as a $N \times N$ matrix then we shall get a block of zeros for indices (i,j) where eigenvalues B_μ^i are degenerate. Assume now that we have the same structure of blocks of zeros for *all* $\mathscr{P}_\mu^\pm$, $\mu = 1, \ldots, d$. This means that *all* matrices have the form

$$\mathscr{P}_\mu^\pm(B) = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \quad \mu = 1, \ldots, d, \tag{12.9}$$

where the first block of zeros has dimension V_1, second V_2, up to V_P. If there is *no* degeneracy of eigenvalues $P = N$, $V_r = 1, \ldots P$. If the degeneracy is *complete* $P = 1$, $V_1 = N$. Of course always

$$\sum_{r=1}^{P} V_r = N. \tag{12.10}$$

Partition $\{V_r, r = 1, \ldots, P\}$ classifies the toron. Writing effective action (12.6) in terms of Q_ν^{ij} we see that for (i,j) inside the block of zeros there will be *no* corresponding *quadratic* term in the action. The number of quadratic modes is

$$d(N^2 - \sum_{r=1}^{P} V_r^2), \tag{12.11}$$

but among them we have the gauge modes

$$Q_\mu^{ij} \sim \mathscr{P}_\mu^{ij}(B). \tag{12.12}$$

These have to be eliminated with the help of the gauge fixing procedure leaving $E_2(V)$ *Gaussian* modes:

$$E_2(V) = (d-1)(N^2 - \sum_{r=1}^{P} V_r^2). \tag{12.13}$$

The modes Q_μ^{ij} which are not Gaussian appear in the action (12.6) in the cubic terms at most linearly and only in *quartic* terms they appear alone. If we recall that effective action is multiplied by a factor $1/g_0^2$, we can expect that Gaussian modes will scale as g_0, and the quartic modes as $g_0^{1/2}$. Each block $V_r \times V_r$ has $V_r^2 - 1$ quartic modes. The remaining mode is the "toron" mode, which only changes value of the degenerate eigenvalue B_μ^i in this block. The number of quartic modes is thus

$$E_4(V) = d \sum_{r=1}^{P} (V_r^2 - 1). \tag{12.14}$$

If we introduce the scaling factors g_0 and $g_0^{1/2}$ we obtain the jacobian proportional to

$$(g_0^2)^{\frac{1}{4}E_2(V) + \frac{1}{2}E_4(V)} \tag{12.15}$$

and the partition function Z will be dominated by the partition $\{V\}$ for which the power M in (12.15) is the smallest. In Appendix B we show that for every group SU(N) the structure of partition depends on the dimension d of the system and that there is a critical dimension $d_c = 2N/(N-1)$ such that:

— if $d \leqslant d_c$ $\{V\}$ is the purely quadratic one, $P = N$, $V_r = 1$, $r = 1, \ldots, d$. This corresponds to the *regular toron.*

— if $d \geqslant d_c$ $\{V\}$ is the purely quartic one, (the *singular toron*) $P = 1$, $V_1 = N$. The power law in these two cases is given by

$$\begin{aligned} M &= \tfrac{1}{2}(d-1)N(N-1), \qquad d \leqslant d_C, \\ M &= \tfrac{1}{4}d(N^2-1), \qquad\quad d \geqslant d_C. \end{aligned} \tag{12.16}$$

$d = 4$ is a critical dimension for SU(2). These regions on the $\{d, N\}$ plane are drawn in Fig. 14 together with the critical line $N = d/(d-2)$ which separates the two regimes. As was shown in Ref. [44] this means that naive power counting presented above may not be sufficient, and that there will be an extra logarithmic factor in (12.15). For any SU(N), $N \geqslant 3$, $d = 4$ is a dimension where the purely quartic behaviour dominates. This means that all matrices B_μ, $\mu = 1, \ldots, 4$ are completely degenerate i.e. they are proportional to the unit matrix. The link variables $U_\mu^0 = e^{iB_\mu}$ must belong to the center $Z(N)$ of the group SU(N), which in view of the structure of (12.1) permits us to consider only fluctuations around $B_\mu = 0$.

To be able to propose a systematic procedure to compute the partition function Z we have to define the gauge fixing procedure for the zero-momentum sector. If we simply add a term analogous to the one for $p \neq 0$, namely

$$S_{\text{g.f.}}^{(0)} = \tfrac{1}{2} \sum_{i,j} | \sum_\mu \mathscr{P}_\mu^- (B^{ij}) Q_\mu^{ij}(p=0)|^2$$

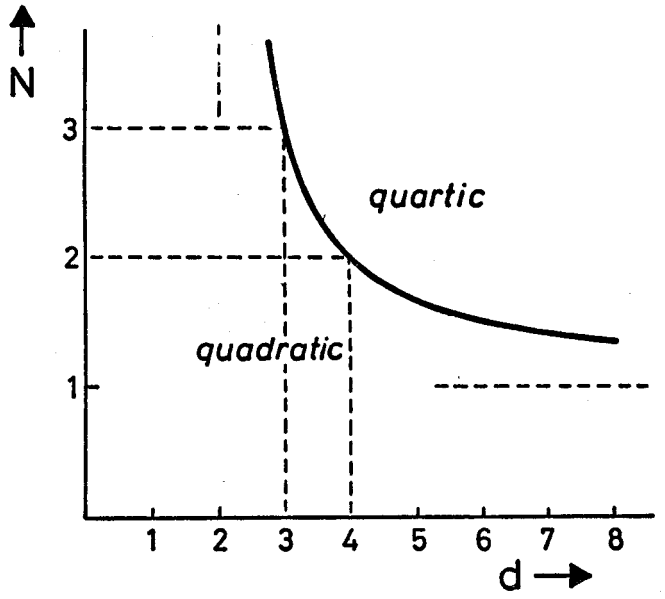

Fig. 14

it would simply vanish for the important region $B_\mu \to 0$. The gauge fixing term must be chosen such that the Q fields are forced to be zero if $S^{(0)}_{\text{g.f.}} = 0$ for *all* values of B_μ. A possible choice can be

$$S^{(0)}_{\text{g.f.}} = \tfrac{1}{2}\sum_{i,j} \Big| \sum_\mu (\mathscr{P}^-_\mu(B^{ij}) Q^{ij}_\mu + \alpha |Q^{ij}_\mu|^2) \Big| + \frac{\beta}{2} \sum_{i,j} \Big(\sum_\mu \operatorname{Im} Q^{ij}_\mu \Big)^2, \tag{12.17}$$

with α and β arbitrary positive constants. The term proportional to β was introduced to eliminate the configuration $Q^{ij}_\mu = \frac{i}{\alpha} B^{ij}_\mu$. From the gauge fixing term we can compute the ghost determinant. This determinant becomes zero on the toron itself, i.e. at $Q = 0$. This is obviously a general feature of any gauge fixing and is perfectly legal.

We are now in a position to proceed with the systematic computation. Once we have a good gauge fixing we can divide the action $\frac{1}{g_0^2} S^{(0)}_{\text{eff}}$ into a big part S_0 and a small part S_{I}. If momenta $\mathscr{P}_\mu(B)$ are $O(1)$ then S_0 will contain terms quadratic in Q and S_{I} terms cubic in Q or higher order. However, if the momenta $\mathscr{P}_\mu(B)$ become small, i.e. of the order of the Q fields, then, clearly all the terms in S_0 will all carry a total number of the $\mathscr{P}_\mu(B)$ and Q_ν factors equal to four. In this case S_{I} contains all the terms of order five or more.

The first approximation to the free energy will be obtained by integrating $\exp\left(-\frac{1}{g_0^2} S_0\right)$

over the Q variables. As we shall see for the case $d > d_c$ we can obtain the lowest approximation without formulating the systematic expansion very precisely. We need only the fact that the gauge fixing like (12.17) is possible. The necessary ingredient is that we can perform rescaling of variables and extending integrations to infinity, which we are allowed by the gauge fixing.

13. Combined contribution and finite-size effects for Wilson loops

The gauge fixing procedure for zero-momentum modes proposed in the last Section guarantees that $Q_\mu(p \neq 0) = O(g_0)$ and $Q_\mu(p = 0) = O(g_0^{1/2})$. Computation with such gauge fixing can in practice prove very difficult to handle. In this Section we shall limit ourselves to the leading order effects due to the singular torons. This case includes the physically interesting SU(3) group in $d = 4$ dimensions (but not the SU(2) group in $d = 4$ dimension, which being the critical dimension for this group is the most difficult to handle).

Consider the partition function Z, where all the Q modes (both with $p \neq 0$ and $p = 0$) were integrated out. The partition function is given by the remaining B integral:

$$Z = (g_0^2)^{\frac{1}{2}(d-1)N(N-1)(V-1)}$$

$$\times e^{-F_{\mathrm{sl}}} \int \prod_{\mu,i} d\hat{B}_\mu^i \exp\{-(\sum_{i,j} \Delta F_\varphi(\hat{B}^{ij}) + F_0(\hat{B}_\mu/L_\mu, g_0^2/V))\}$$

$$\times (1 + O(g_0^{1/2})). \tag{13.1}$$

We used the expression (10.19) for the effective action, hence the argument in the zero-momentum free energy F_0 was $B_\mu = \hat{B}_\mu/L_\mu$ and g_0^2/V instead of g_0^2. F_0 is the free energy one obtains integrating $\exp\left(-\frac{1}{g_0^2} S_{\mathrm{eff}}^{(0)}\right)$, Eq. (12.1) over the $Q_\mu^{ij}(p = 0)$ variables. All the bulk effects are in the factors in front of the integral in (13.1). The finite-size effects we are interested in are contained in the integral.

As was shown in Section 11 the contribution from the non-constant modes $\Delta F_\varphi(\hat{B})$ is regular for small $\hat{B}$. For the singular toron (cf. (12.6)) the leading order contribution will come from the region where $\hat{B} = O(g_0^{1/2})$ so to the lowest order we can forget about $\Delta F_\varphi(\hat{B})$. In the case, where regular torons dominate, $\hat{B} = O(1)$ and we would need the whole structure of $\Delta F_\varphi(\hat{B})$ to compute (13.1). For a *singular* toron the leading order of the integral in (13.1) comes from the zero-momentum part of the effective action. This leading order behaviour can be computed by considering the integral over the zero-momentum modes of $S_{\mathrm{eff}}^{(0)}$. We can use the parametrization

$$DU_\mu = \prod_a dA_\mu^a \exp(-S_{\mathrm{m}}(A_\mu)), \tag{13.2}$$

where we take

$$U_\mu = \exp(iA_\mu), \quad A_\mu = \tfrac{1}{2} \sum_a \lambda_a A_\mu^a. \tag{13.3}$$

The integral over the zero-momentum sector becomes

$$Z_0 = \int dA_\mu^a \exp\left\{-\left[S_m(A)+\frac{V}{g_0^2}S_{eff}(A)+\frac{1}{g_0^2}S_{g.f.}(A)\right]\right\}\det\left(\frac{\delta G}{\delta\omega}\right), \tag{13.4}$$

where G is a gauge fixing of the type discussed in the previous Section. We can observe here that the singular toron is itself invariant under the global gauge transformations (not so the regular torons) and we may use

$$\det\left(\frac{\delta G}{\delta\omega}\right)\int d\Omega \exp\left(-\frac{1}{g_0^2}S_{g.f.}(A)\right) = 1 \tag{13.5}$$

to eliminate ghosts and gauge fixing from (13.4). That means that to lowest order Z_0 can be written as:

$$Z_0 = \left(\frac{g_0^2}{V}\right)^{\frac{1}{4}d(N^3-1)}\int\prod_{a,\mu} d\hat{A}_\mu^a \exp\left(\sum_{\mu,\nu}\mathrm{Tr}\,[A_\mu, A_\nu]^2 + O(g_0^{1/2})\right), \tag{13.6}$$

where we scaled the A_μ potentials with the factor $g_0^{1/2}V^{-1/4}$. It can be shown that for $d > d_c$ the quartic integral in (13.6) remains finite even when the integration region is extended to infinity.

The method described above can be used to compute the leading-order zero-momentum effect on the averages of the Wilson loops. As before we limit our computation to the case $d > d_c$ and to the leading order $O(g_0^2)$. To this order, as we have shown, the contributions to the partition function Z from the zero- and non-zero-momentum sectors factorize. This means that when we compute the average of the Wilson loop in the leading order we get a sum of contributions coming from these two sectors. Contribution coming from the non--zero-momentum sector is the same as in the standard perturbation approach where one takes $B_\mu = 0$. This is because B_μ become of order $g_0^{1/2}$ and do not couple to the momentum--carrying modes in the lowest order. This contribution was computed before (Ref. [42]). Here we shall evaluate the zero-momentum contribution.

For the zero momentum sector we shall use the simple scaling result

$$\frac{1}{Z_0}\int\prod_{a,\mu} dA_\mu^a\left(\sum_{\varrho,\sigma}\mathrm{Tr}\,[A_\varrho, A_\sigma]^2\right)\exp\left(\frac{V}{g_0^2}\sum_{\mu,\nu}\mathrm{Tr}\,[A_\mu, A_\nu]^2\right)$$

$$= -\tfrac{1}{4}d(N^2-1)\frac{g_0^2}{V}. \tag{13.7}$$

Consider the expectation value of a Wilson loop of size $R\times T$ in the $\{1, 2\}$ plane. The product of the link variables along the loop we call $U(R, T)$ and we compute

$$W(R, T) \equiv \left\langle\frac{\mathrm{Tr}\,U(R, T)}{N}\right\rangle. \tag{13.8}$$

If we expand the loop (13.8) and the action in terms of fields we get for $W(R, T)$ a series

$$W(R, T) = 1 - c_1 g_0^2 N + \text{higher order in } g_0. \tag{13.9}$$

The coefficient c_1 receives contributions only from the expansion of $W(R, T)$ from $\boldsymbol{p} \neq 0$ and $\boldsymbol{p} = 0$ sectors:

$$c_1 = c_1(\boldsymbol{p} \neq 0) + c_1(\boldsymbol{p} = 0). \tag{13.10}$$

For $c_1(\boldsymbol{p} \neq 0)$ calculation is standard [42] and gives

$$c_1(\boldsymbol{p} \neq 0) = \frac{N^2-1}{N^2} \left\{ \frac{1}{4V} \sum_{\boldsymbol{p} \neq 0} \frac{1}{\mathscr{P}^2(\boldsymbol{p})} \left(|\exp(ip_1 R) - 1|^2 \right.\right.$$
$$\left.\left. \times \left| \sum_{k=0}^{T-1} \exp(ikp_2) \right|^2 + (R \leftrightarrow T, 1 \leftrightarrow 2) \right) \right\}. \tag{13.11}$$

At this order, the $\boldsymbol{p} = 0$ contribution comes from the expectation value (13.7) of the square of the commutator:

$$c_1(\boldsymbol{p} = 0) = \frac{1}{NV} \langle -\operatorname{Tr} [RA_1, TA_2]^2 \rangle$$
$$= \frac{(RT)^2}{V} \frac{1}{4(d-1)} \frac{N^2-1}{N^2}. \tag{13.12}$$

The contribution we have evaluated is indeed the finite-size effect: it has a factor $1/V$ in front, but for large loops, where R and T are of order $L/2$ it does not vanish. For even larger loops, where $R = T = L$ it is the *only* contribution which survives at this order, since $c_1(\boldsymbol{p} = 0)$ vanishes.

Computation of higher-order corrections to the Wilson loop coming from the zero--momentum sector or from the interactions between zero- and non-zero-momentum sectors is much more complicated and requires introducing systematic treatment of the gauge fixing and ghosts in the zero momentum sector.

14. Discussion

In the last Section we have derived a leading-order contribution to the averages of the Wilson loops coming from the zero-momentum sector. We can see that this contribution is not negligible, particularly for large loops. Wilson loops are not the only objects which feel the contribution from this sector. One may expect an even stronger effect for quantities, which involve correlations of plaquettes. An example of such quantity is the 0^{++} glueball correlation:

$$\Gamma_{0^{++}}(V, t) = \frac{1}{(d-1)^2(d-2)^2} \sum_{\substack{\mu,\nu \\ \varrho,\sigma \\ r}} \left\langle \frac{\operatorname{Tr} P_{\mu\nu}(r, t)}{N} \frac{\operatorname{Tr} P_{\varrho\sigma}(0, 0)}{N} \right\rangle_c, \tag{14.1}$$

$P_{\mu\nu}(n)$ are plaquette placed at a distance $\{r, t\}$ from each other. The contribution
ne zero-momentum sector [63] is to the lowest order

$$\Gamma_{0++}(V, t; p = 0) = \frac{g_0^4}{(d-1)^2(d-2)^2} \frac{V_s}{V^2} \sum_{\substack{\mu,\nu \\ \varrho,\sigma}} \left\langle \frac{\mathrm{Tr}\,[A_\mu, A_\nu]^2}{N} \frac{\mathrm{Tr}\,[A_\varrho, A_\sigma]^2}{N} \right\rangle_c$$

$$= \frac{g_0^4}{(d-1)^2(d-2)^2} \frac{V_s}{V^2} \frac{N^2-1}{4N^2}. \tag{14.2}$$

2) V_s is the spacial volume of the lattice. This last result can be obtained by differen-
log Z_0 (13.6) twice with respect to $1/g_0^2$. Γ_{0++} also gets a contribution from the $p \neq 0$
This however falls for large t either as a power (symmetric box $L_\mu = L \to \infty$) or
ntially (asymmetric box $L_0 \to \infty$, L_i fixed) [63]. In both cases for t large enough
omentum contribution starts dominating over the $p \neq 0$ contribution.
is result shows that finite-size effects can seriously contaminate measurements
elations, even casting doubt on the experimentally obtained values of masses.
ere is a simple solution which consists of introducing non-trivial boundary condi-
n a lattice. Such *twisted* boundary conditions were proposed in Ref. [62] for gauge
s in a finite periodic box. Twisted boundary conditions can be easily generalized
ttice system [44] and their effect is to eliminate a number of zero-modes and reduce
ze effects. As we have observed in Section 10 there is a strong interaction between
ysical momentum p and the group momenta $\hat{B}$. The same effect appears, when
boundary conditions are introduced: its effect is like a shift in momenta (9.3)
xed fraction of $2\pi/L$ which in particular affects the zero-momentum sector. For
roups and lattice dimensions a complete elimination of zero-modes is possible [66].
e result of our analysis suggests that using twisted boundary conditions in numerical
Carlo experiments is a simple way to reduce the finite-size effects. Commonly used
c boundary conditions seem to maximize these effects. On the other hand physically
ting quantities, like glueball mass, in the weak coupling have the same infinite volume
n the twisted and in the untwisted system.

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APPENDIX A

The global behaviour of the determinant over the non-constant modes

e shall now try to analyze the global behaviour of $F_\varphi(\hat{B})$ as a function of $\hat{B}$. We shall
trate now on a single (i, j) term in (10.20) and drop the (i, j) index on $\hat{B}$. The mo-

mentum sums, which appear in (10.20) can be expressed as

$$\sum_{p_\mu} f(e^{ip_\mu}) = \int_{0-\varepsilon}^{2\pi-\varepsilon} d\alpha \sum_{l_\mu=-\infty}^{\infty} \delta\left(\frac{2\pi l_\mu}{L_\mu} - \alpha\right) f(e^{i\alpha}). \tag{A.1}$$

We can also use the Poisson formula for periodic functions

$$\sum_{l_\mu=-\infty}^{\infty} \delta\left(l_\mu - \frac{\alpha L_\mu}{2\pi}\right) = \sum_{q_\mu=-\infty}^{\infty} \exp\left(iq_\mu \alpha L_\mu\right). \tag{A.2}$$

Applying this to the free energy $F_\varphi(\hat{\boldsymbol{B}})$ and including the $p = 0$ term gives

$$\begin{aligned} F(B) &= \frac{2F_\varphi(\hat{\boldsymbol{B}})}{d-2} + \log \mathscr{P}^2(\hat{B}) \\ &= \sum_{\{q_\mu\}} \left(\prod_\mu e^{-iq_\mu \hat{B}_\mu}\right) \tilde{F}(q), \end{aligned} \tag{A.3}$$

where

$$\tilde{F}(q) = V \int_0^{2\pi} \prod_\mu \left(\frac{d\alpha_\mu}{2\pi} e^{iq_\mu L_\mu \alpha_\mu}\right) \log\left(2 \sum_\mu (1 - \cos \alpha_\mu)\right). \tag{A.4}$$

Eqs (A.3) and (A.4) allow a systematic expansion of the determinant for large V. The zero-momentum contribution on the r.h.s. of (A.2)

$$\log \mathscr{P}^2(\boldsymbol{B}) \sim \log \sum_\mu \frac{\hat{\boldsymbol{B}}_\mu^2}{L_\mu^2} + O\left(\frac{\hat{\boldsymbol{B}}^2}{L^2}\right) \tag{A.2'}$$

is included in (A.4) and must in the end be subtracted out.

Let us distinguish between the two limits:

(*i*) Symmetric case: $L_\mu = L \to \infty$ for $\mu = 1, \ldots, d$. The leading contribution as $V = L^d \to \infty$ is given by the $\hat{\boldsymbol{B}}$-independent term ($q_1 = q_2 = \ldots = q_d = 0$) in $\tilde{F}(q)$:

$$\tilde{F}(0) = V \int_0^{2\pi} \prod_\mu \frac{d\alpha_\mu}{2\pi} \log\left(2 \sum_\mu (1 - \cos \alpha_\mu)\right). \tag{A.5}$$

$\tilde{F}(0)$ coincides with the infinite lattice free energy. The finite-volume corrections appear together with the $\hat{\boldsymbol{B}}$ dependence in the next order. To compute them, we note that when the q_μ are not all zero we may write

$$\tilde{F}(q) = -V \int_0^\infty \frac{dx}{x} \prod_\mu \left(I_{q_\mu L_\mu}(x) e^{-x}\right). \tag{A.6}$$

Using the asymptotic expansion of the modified Bessel functions I_{qL}:

$$e^{-x}I_{qL}(x) \sim \frac{\exp(-q^2L^2/2x)(1+O(1/x))}{\sqrt{2\pi x}} \tag{A.7}$$

we obtain

$$\tilde{F}(q) \sim -\left(\pi \sum_{\mu} q_\mu^2\right)^{-d/2} \Gamma(\tfrac{1}{2}d) + O\left(\frac{1}{L}\right). \tag{A.8}$$

Substituting (A.8) into (A.6) one finds a logarithmic singularity for small $\hat{B}$ canceled by the zero-momentum subtraction from the r.h.s. of (A.3).

It is useful to go back to (A.3) and perform the summation over q_μ (excluding $q_1 = q_2 = \ldots = q_d = 0$):

$$\tilde{F}(\hat{B}) - \tilde{F}(0) = -V \int_0^\infty \frac{dx}{x} \left(\prod_{\mu} H_{L_\mu}(x, \hat{B}_\mu) - (I_0(x)e^{-x})^d \right), \tag{A.9}$$

where

$$H_{L_\mu}(x, \hat{B}_\mu) \equiv \sum_{q_\mu} I_{q_\mu L_\mu}(x) e^{-iq_\mu \hat{B}_\mu} e^{-x}. \tag{A.10}$$

As L goes to infinity, the latter tends to the following expression containing the Jacobi function

$$H_L(x, \hat{B}) \simeq \frac{1}{\sqrt{2\pi x}} \vartheta_3(\hat{B}/2\pi, iL^2/2\pi x) + I_0(x)e^{-x} - \frac{1}{\sqrt{2\pi x}}. \tag{A.11}$$

Plugging (A.9) and (A.11) together we get for $L \to \infty$

$$\tilde{F}(\hat{B}) - \tilde{F}(0) = -\int_0^\infty \frac{dx}{x} (2\pi x)^{-d/2} \left(\prod_{\mu} \vartheta_3(\hat{B}_\mu/2\pi, i/2\pi x) - 1 \right). \tag{A.12}$$

To get (A.12) we changed variables $x \to xL^2$ and took the limit $L \to \infty$. Note the disappearance of the volume factor on the r.h.s. of (A.12). This formula, after removal of the logarithmically divergent term (A.2′) yields a smooth dependence of the determinant if $|\hat{B}_\mu| < 2\pi$.

(*ii*) Asymmetric limit: $L_\mu = L \to \infty$, $\mu = 1, \ldots, d_1$; $L_\alpha = I$ — finite, $\alpha = d_1+1, \ldots, d$. In this limit we find that the free energy $F(\hat{B})$ splits up in two parts. One part is proportional to the big volume $V_1 = L^{d_1}$ and depends only on the field $\hat{B}_\alpha$, $\alpha = d_1+1, \ldots d$. The other part is proportional only to the small volume $v = I^{d-d_1}$, and is exponentially suppressed in the fields $\hat{B}_\alpha$. Its behaviour in the fields B_μ, $\mu = 1, \ldots, d_1$ is smooth like in case (*i*). We write this as

$$F(\hat{B}) = V_1 v F_{(\alpha)}(\hat{B}_\alpha) + F_{(\mu)}(\hat{B}_\mu; \hat{B}_\alpha). \tag{A.13}$$

This splitting is achieved by using the following identity:

$$V(\prod_{\mu=1}^{d} H_{L_\mu}(x, \hat{B}_\mu) - I_0^d(x)e^{-dx})$$

$$\equiv V\{\prod_{\alpha=d_1+1}^{d} H_{L_\alpha}(x, \hat{B}_\alpha) - (I_0(x)e^{-x})^{d-d_1}\}(I_0(x)e^{-x})^{d_1}$$

$$+ V\{\prod_{\mu=1}^{d_1} H_{L_\mu}(x, \hat{B}_\mu) - (I_0(x)e^{-x})^{d_1}\} \prod_{\alpha=d_1+1}^{d} H_{L_\alpha}(x, \hat{B}_\alpha). \tag{A.14}$$

V is as before the total volume $V_1 v$.

Integrating the r.h.s. of (A.14) with the measure $\int_0^\infty \frac{dx}{x}$ gives for the first term indeed the first term in Eq. (A.13), the correction term in the large-L limit can be evaluated using the duality transformation for ϑ functions [65]:

$$(2\pi x)^{-1/2}\vartheta_3(\hat{B}/2\pi, i/2\pi x) = \sum_{n=-\infty}^{\infty} \exp(-\tfrac{1}{2}x(\hat{B}+2n\pi)^2) \tag{A.15}$$

to give

$$F_{(\mu)}(\hat{B}_\mu; \hat{B}_\alpha) \sim -\int_0^\infty \frac{dx}{x}(2\pi x)^{-d_1/2}\left(\prod_{\mu=1}^{d_1}\vartheta_3(\hat{B}_\mu/2\pi, i/2\pi x) - 1\right)$$

$$\times \prod_{\alpha=d_1+1}^{d} \exp(-\hat{B}_\alpha^2 L^2 x/2I^2). \tag{A.16}$$

The bulk terms $F_{(\alpha)}(\hat{B}_\alpha)$ have been computed by Weiss [59, 60] for $d = 4$, $d_1 = 3$ (finite temperature) and by Lüscher [64] for $d = 4$, $d_1 = 1$ (zero temperature).

APPENDIX B

The zero-momentum power-laws

Here we shall demonstrate inequalities for power laws used in the text.

Let us call $M(\{V\})$ the power of a given toron with partition $(V_1, \ldots, V_P)$, which means that this toron, after rescaling its variables will contribute to the partition function a factor $g^{2M(\{V\})}$.

We have

$$M(\{V\}) = \tfrac{1}{2}E_2 + \tfrac{1}{4}E_4, \tag{B.1}$$

where

$$E_2 = (d-1)(N^2 - \sum_{r=1}^{P} V_r^2), \quad E_4 = d\sum_{r=1}^{P}(V_r^2 - 1). \tag{B.2}$$

We rewrite (B.1) as

$$M(\{V\}) = S(V)d + f(V),$$

where

$$S(V) = \tfrac{1}{4}(N^2 + \Delta - P), \quad f(V) = -\tfrac{1}{2}\Delta, \quad \Delta = N^2 - \sum_{r=1}^{P} V_r^2. \tag{B.3}$$

Theorem: For a fixed SU(N) gauge group, the dominant power $M = \inf_{\{V\}} M(\{V\})$, is the power of the regular toron ($P = N$, $V_1 = V_2 = \ldots = V_N = 1$) for $d \leqslant d_c(N)$; and it is the power of the singular toron ($P = 1$, $V_1 = N$) for $d \geqslant d_c(N)$.

It is equivalent to say that the following two inequalities hold for any partition $\{V\}$:

$$\text{For } d \leqslant d_c(N) \quad M(\{V\}) \geqslant M_{\text{reg}} = \tfrac{1}{2}(d-1)N(N-1), \tag{B.4}$$

$$\text{For } d \geqslant d_c(N) \quad M(\{V\}) \geqslant M_{\text{sing}} = \tfrac{1}{4}(N^2-1)d. \tag{B.5}$$

Note that at $d = d_c$ the two right-hand sides are equal and give a power $M(d_c) = \frac{1}{2}N(N+1)$. The proof is based on the inequality

$$(P-1)N \leqslant \Delta, \tag{B.6}$$

which comes simply from

$$\Delta = (\sum V_r)^2 - \sum V_r^2 = \sum_{i \neq j} V_i V_j \geqslant \sum_{i=1}^{P} V_i \sum_{j \neq i} 1 = N(P-1). \tag{B.7}$$

Let us derive (B.4)

$$\begin{aligned} M(\{V\}) - M_{\text{reg}} &= \tfrac{1}{4}((N^2 + \Delta - P)d - 2\Delta) - \tfrac{1}{2}(d-1)(N^2-N) \\ &= \tfrac{1}{4}(\Delta(d-2) + (N-P)d + (N^2-N)(2-d)). \end{aligned}$$

From (B.6) we have

$$\begin{aligned} M(\{V\}) - M_{\text{reg}} &\geqslant \tfrac{1}{4}(N-P)(2-d)N + d) \\ &= \tfrac{1}{4}(N-P)(d-2)(N_c(d) - N). \end{aligned} \tag{B.8}$$

For any non-regular toron $P \leqslant N-1$ so that for $N \leqslant N_c$ (which is equivalent to $d \geqslant 2$ and $d \leqslant d_c$) one has $M - M_{\text{reg}} \geqslant 0$.

The proof of (B.5) is similar. From (B.2) we have

$$M(\{V\}) - M_{\text{sing}} = \tfrac{1}{4}[(d-2)\Delta - (P-1)d]$$

and (B.6) gives us

$$\begin{aligned} M(\{V\}) - M_{\text{sing}} &\geqslant \tfrac{1}{4}((d-2)N - d)(P-1) \\ &= \tfrac{1}{4}(d-2)(N - N_c(d))(P-1), \end{aligned} \tag{B.9}$$

which is positive for any non-singular toron above the critical dimension.

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