AXIAL ANOMALY THROUGH ANALYTIC RENORMALIZATION

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(*Received July 15, 1986*)

The problem of axial anomaly is analyzed within the framework of analytic regularization scheme. The form of triangle anomaly is obtained.

PACS numbers: 11.10. Gh

The problem of anomalies attracts recently much attention. In spite of many efforts their meaning and consequences remain to some extent obscure. There exists now the mathematical theory which infers the very existence of anomalies from nontrivial topological structure of gauge group; however, they also do occur in topologically trivial context. From the point of view of perturbation theory the situation is also quite interesting. The form of anomalies does not depend on the regularization procedure (up to the finite renormalization) but the way they occur does. This problem was analyzed within the framework of the momentum cut-off [1], Pauli-Villars [2], point-splitting [3] as well as dimensional [4] regularization schemes. As far as one is dealing with vector gauge theories the techniques preserving explicitly the vector gauge invariance are favoured. However, when dealing with the most general coupling of vector and axial-vector fields one can choose the regularization at will, gaining subsequently the apropriate final form by the addition of suitable counterterms.

One of the best developed regularization schemes is the so-called analytic regularization/renormalization introduced by Speer [5]. It consists in replacing each propagator $P_l(p)(p^2-m_l^2+i\varepsilon)^{-1}$ by $(\mu^2)^{\lambda_l-1}\cdot P_l(p)(p^2-m_l^2+i\varepsilon)^{-\lambda_l}$, $\lambda_l \in \mathbf{C}$, with Re λ_l sufficiently large; μ is some reference mass.

The Feynman integral becomes then a meromorphic function in $\mathbf{C}^L$(L is the number of internal lines of the graph under consideration), the pole at $\lambda_l = 1$ corresponding to the divergence of the integral one has started with.

The complicated structure of the pole at $\lambda_l = 1$ corresponds to the complicated combinatorial structure of the overall divergence and the subdivergencies. To obtain the finite results the special renormalization procedure is applied consisting in using the so-

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-called generalized evaluator [5] which replaces the recursive structure of usual subtractions.

The whole scheme is quite complicated but it simplifies considerably in the case of one-loop diagrams. In this case adopting the usual Feynman procedure we arrive at the following expression for the amplitude

$$F \sim \sum_k \Gamma(\Lambda - k) \int \frac{\prod_l \alpha_l^{\lambda_l - 1} d\alpha_l \delta(1 - \sum_l \alpha_l) P_k(p, m; \alpha)}{\Delta^{\Lambda - n_k}(p, m; \alpha)}, \quad \Lambda \equiv \sum_l \lambda_l^{\lambda_l - 1}.$$

The α-integral is now convergent (provided there are no massless fields or at least the external momenta are nonexceptional) and the divergence appears as the pole in Γ-function at $\Lambda = L$.

One can then define the finite part of the amplitude by expanding the integrand in Λ at the point $\Lambda = L$, taking the first-order term in the expansion (which cancels the pole of Γ-function) and putting $\lambda_l = 1$ in other places. It is easy to show that this is indeed a renormalization — the divergent part is local; it differs in general by a finite counterterms from that used by Speer.

Let us apply this scheme to the simplest case of AVV anomaly. The well known graphs for the AVV and PVV vertices are given in Figs 1 and 2, respectively.

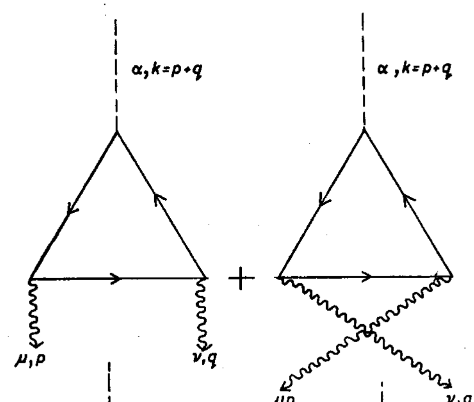

Fig.1

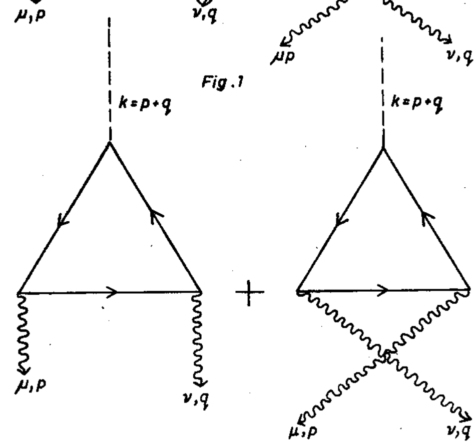

Fig.2

We write as usual

$$F^{\alpha\mu\nu}(p,q) = \Gamma^{\alpha\mu\nu}(p,q)+\Gamma^{\alpha\nu\mu}(q,p),$$

$$F^{\mu\nu}(p,q) = \Gamma^{\mu\nu}(p,q)+\Gamma^{\nu\mu}(q,p),$$

with

$$\Gamma^{\alpha\mu\nu}(p,q) = ie^2\int\frac{dr}{(2\pi)^4}\frac{\mathrm{Tr}\,\gamma^5\gamma^\alpha\gamma^a[\gamma(p+r)+m]\gamma^\mu[\gamma r+m]\gamma^\nu[\gamma(r-q)+m]}{[m^2-(p+r)^2][m^2-r^2][m^2-(r-q)^2]},$$

$$\Gamma^{\mu\nu}(p,q) = ie^2\int\frac{dr}{(2\pi)^4}\frac{\mathrm{Tr}\,\gamma^5\gamma^a[\gamma(p+r)+m]\gamma^\mu[\gamma r+m]\gamma^\nu[\gamma(r-q)+m]}{[m^2-(p+r)^2][m^2-r^2][m^2-(r+q)^2]},$$

Performing the traces we obtain the following expression for regularized form of $\Gamma^{\alpha\mu\nu}$

$$\Gamma^{\alpha\mu\nu}_{\mathrm{reg}} = 4e^2\int\frac{dr}{(2\pi)^4}\frac{N^{\alpha\mu\nu}(p,q,r)\,(\mu^2)^{\Lambda-3}}{[m^2-(r-q)^2]^{\lambda_1}[m^2-(r+p)^2]^{\lambda_2}[m^2-r^2]^{\lambda_3}}.$$

Here $\Lambda \equiv \lambda_1+\lambda_2+\lambda_3$ and

$$\begin{aligned}N^{\alpha\mu\nu}(p,q,r) = {} & -m^2\varepsilon^{\alpha\mu\nu\beta}(p-q)_\beta+(r^2-m^2)\varepsilon^{\alpha\mu\nu\beta}r_\beta\\ & -p\cdot q\varepsilon^{\alpha\mu\nu\beta}r_\beta+g^{\mu\nu}\varepsilon^{\alpha\beta\delta\gamma}p_\beta q_\delta r_\gamma-(r^\mu\varepsilon^{\alpha\nu\beta\gamma}+r^\nu\varepsilon^{\alpha\mu\beta\gamma})p_\beta q_\gamma\\ & +(p^\alpha q_\beta+q^\alpha p_\beta)\varepsilon^{\mu\nu\beta\gamma}r_\gamma+(p-q)^\lambda r_\lambda\varepsilon^{\alpha\mu\nu\beta}r_\beta\\ & +(r^\mu\varepsilon^{\alpha\nu\beta\gamma}+r^\nu\varepsilon^{\alpha\mu\beta\gamma})k_\beta r_\gamma-r^\alpha\varepsilon^{\mu\nu\beta\gamma}(p-q)_\beta r_\gamma.\end{aligned}$$

Introducing the Feynman parameters and integrating over the momentum r we arrive at the following expression

$$\begin{aligned}\Gamma^{\alpha\mu\nu}_{\mathrm{reg}}(p,q) = {} & \frac{ie^2\Gamma(\Lambda-3)}{4\pi^2\prod_{i=1}^{3}\Gamma(\lambda_i)}\int_\Delta dxdy\,\frac{x^{\lambda_1-1}y^{\lambda_2-1}(1-x-y)^{\lambda_3-1}}{D^{\Lambda-3}}\\ & \times\varepsilon^{\alpha\mu\nu\beta}(3(yp-xq)_\beta-(p-q)_\beta)\,(\mu^2)^{\Lambda-3}\\ & +\frac{ie^2\Gamma(\Lambda-2)}{4\pi^2\prod_{i=1}^{3}\Gamma(\lambda_i)}\int_\Delta dxdy\,\frac{x^{\lambda_1-1}y^{\lambda_1-1}(1-x-y)^{\lambda_3-1}N^{\alpha\mu\nu}(p,q;x,y)\,(\mu^2)^{\Lambda-3}}{D^{\Lambda-2}},\end{aligned}$$

$$D = m^2-x(1-x)q^2-y(1-y)p^2-2xyp\cdot q,$$

$$\Delta = \{(x,y)\,|x\geqslant 0, y\geqslant 0, x+y\leqslant 1\},$$

$$N^{\alpha\mu\nu}(p,q:x,y) = \varepsilon^{\alpha\mu\nu\beta}(py-qx)_\beta\,(m^2+pq+(p-q)\,(py-qx)-(py-qx)^2)$$

$$+(\varepsilon^{\alpha\mu\beta\gamma}(py-qx)^{\nu}+\varepsilon^{\alpha\nu\beta\gamma}(py-qx)^{\mu})p_{\beta}q_{\gamma}-\varepsilon^{\mu\nu\beta\gamma}(py-qx)_{\gamma}(p^{\alpha}q_{\beta}+q^{\alpha}p_{\beta})$$
$$+k_{\beta}(py-qx)_{\gamma}(\varepsilon^{\alpha\mu\beta\gamma}(py-qx)^{\nu}+\varepsilon^{\alpha\nu\beta\gamma}(py-qx)^{\mu})-m^{2}\varepsilon^{\alpha\mu\nu\beta}(p-q)_{\beta}$$
$$-\varepsilon^{\mu\nu\beta\gamma}(py-qx)^{\alpha}(py-qx)_{\gamma}(p-q)_{\beta}.$$

In the sequel we assume for simplicity that the photons are on the mass-shell, i.e. $p^2 = q^2 = 0$.

The integral representing $\Gamma^{\alpha\mu\nu}$ is superficially linearly divergent. However, for dimensional reason it can diverge at most logarithmically. This divergence appears now as a pole in Γ-function at $\Lambda = 3$. It can be easily checked that the residue of the pole is actually zero, i.e. as in other regularization schemes the triangle diagram is convergent. We can now remove the regularization according to our prescription to get

$$\Gamma^{\alpha\mu\nu}(p,q) = \frac{ie^2}{4\pi^2}\int_{\Delta} dxdy\,\varepsilon^{\alpha\mu\nu\beta}((p-q)_{\beta}-3(yp-xq)_{\beta})\ln\frac{D}{\mu^2}$$
$$+\frac{ie^2}{4\pi^2}\int_{\Delta} dxdy\,\frac{N^{\alpha\mu\nu}(p,q;x,y)}{D}.$$

Let us observe that the μ^2-dependence of the first integral is spurious. $\Gamma^{\mu\nu}(p,q)$ is convergent and need not be regularized. The corresponding expression reads

$$\Gamma^{\mu\nu}(p,q) = \frac{ime^2}{4\pi^2}\varepsilon^{\mu\nu\beta\gamma}p_{\beta}q_{\gamma}\int\frac{dxdy}{D}.$$

Let us now contract $\Gamma^{\alpha\mu\nu}(p,q)$ with k_{α}. After some rearrangements we get

$$k_{\alpha}\Gamma^{\alpha\mu\nu}(p,q) = \frac{ie^2}{4\pi^2}\int_{\Delta} dxdy\,\varepsilon^{\mu\nu\alpha\beta}q_{\alpha}p_{\beta}(2-3(x+y))\ln\frac{D}{\mu^2}$$
$$+\frac{ie^2}{4\pi^2}\int_{\Delta} dxdy\,\varepsilon^{\mu\nu\alpha\beta}q_{\alpha}p_{\beta}(x+y)$$
$$-\frac{2ime^2}{4\pi^2}\int_{\Delta} dxdy\,\frac{\varepsilon^{\mu\nu\alpha\beta}q_{\alpha}p_{\beta}}{D}$$
$$+\frac{ie^2}{4\pi^2}\int_{\Delta} dxdy\,\frac{4pq\varepsilon^{\mu\nu\alpha\beta}q_{\alpha}p_{\beta}xy(x+y-1)}{D}.$$

Now it can be checked that the fourth integral on the right-hand side cancels against the first one and we find

$$k_{\alpha}\Gamma^{\alpha\mu\nu}(p,q) = 2m\Gamma^{\mu\nu}(p,q)-\frac{ie^2}{12\pi^2}\varepsilon^{\mu\nu\alpha\beta}p_{\alpha}q_{\beta}$$

or

$$k_\alpha F^{\alpha\mu\nu}(p, q) = 2mF^{\mu\nu}(p, q) - \frac{ie^2}{6\pi^2}\varepsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta. \quad (1)$$

Let us, in turn, examine the vector-current anomaly. By adding two contributions from $\Gamma^{\alpha\mu\nu}$ and $\Gamma^{\alpha\nu\mu}$ we get after some simple calculations

$$p_\mu F^{\alpha\mu\nu}(p, q) = -\frac{ie^2}{6\pi^2}\varepsilon^{\alpha\nu\mu\beta}p_\mu q_\beta. \quad (2)$$

Let us conclude with some remarks. We see from Eqs (1), (2) that both currents are treated symmetrically in the analytic regularization approach. By adding to $F^{\alpha\mu\nu}$ the finite counterterm

$$R^{\alpha\mu\nu} = \frac{ie^2}{6\pi^2}\varepsilon^{\alpha\mu\nu\beta}(p-q)_\beta$$

we obtain the modified amplitude $\tilde{F}^{\alpha\mu\nu}$ such that

$$p_\mu \tilde{F}^{\alpha\mu\nu} = 0,$$

$$k_\alpha \tilde{F}^{\alpha\mu\nu} = 2mF^{\mu\nu} - \frac{ie^2}{2\pi^2}\varepsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta.$$

This is the standard form of the axial-current anomaly. We can also repeat the analysis of Bardeen [3] within our framework. There one considers the general theory consisting of quantized spinor field with arbitrary internal degrees of freedom having arbitrary nonderivative couplings to external scalar, pseudoscalar, vector and axial-vector fields. The renormalized vacuum-to-vacuum amplitude is defined and its behaviour under the general gauge transformation is investigated.

I am grateful to Prof. J. Rembieliński and Drs S. Giler, P. Maślanka and L. Szymanowski for helpful discussions.

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