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$m^2 = 0$ LIMIT OF NONMINIMAL DESCRIPTION OF SPIN 2*

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It is shown that the theories of spin 2, equivalent in the massive case, are not equivalent in the $m^2 = 0$ limit. While the massless theory of Pauli and Fierz describes the helicities ± 2, the one based on the 3-rd rank tensor has no physical content (a pure gauge theory), and the one based on the 4-th rank tensor describes the helicity 0 (a scalar "notivarg" theory).

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1. *Introduction*

In the previous paper [1] the nonminimal descriptions of a massive field carrying spin 2 (using the 3-rd and 4-th rank tensor with symmetries proposed by Fierz [2]) have been obtained. These decriptions are equivalent to the minimal one of Pauli and Fierz [3]. All they are connected by the Legendre transformations. It is well known that the theories equivalent for $m^2 \neq 0$ need not to be equivalent in the $m^2 = 0$ limit (e.g. the "notoph" of Ogievetsky and Polubarinov [4]). The Pauli-Fierz theory in the zero mass limit describes particles with the helicities ± 2.

In the present paper we give the analysis of the zero mass limit of the nonminimal descriptions [1] in three manners:
— we solve constraints obtained from first order actions,
— we investigate a gauge invariance of the theory,
— we perform a canonical analysis.

We conclude that the 3-rd rank tensor is a pure gauge and the 4-th rank one describes the helicity 0 (the scalar "notivarg" in the terminology of Deser, Siegel and Townsend [5]).

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2. *The 3-rd rank tensor theory* ($m^2 = 0$)

2.1. The action

Let us start with the first order action [1]

$$I = \int dx \left\{ -\frac{m}{\sqrt{2}} S_{\alpha\beta\nu}[\partial^\alpha h^{\beta\nu} - \partial^\beta h^{\alpha\nu} - (g^{\alpha\nu}h^\beta - g^{\beta\nu}h^\alpha) + (g^{\alpha\nu}\partial^\beta - g^{\beta\nu}\partial^\alpha)h] \right.$$
$$\left. + \tfrac{1}{2} m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \tfrac{1}{2} m^2[(S^{\alpha\beta\nu})^2 - 2(S^\alpha)^2] \right\}, \tag{2.1}$$

where $h^{\mu\nu} = h^{\nu\mu}$, $h^\mu \equiv \partial_\nu h^{\nu\mu}$, $h \equiv h^\alpha_\alpha$. The $S^{\alpha\beta\nu}$ has the symmetry of the Fierz tensor $S^{\alpha\beta\nu} = -S^{\beta\alpha\nu}$, $\varepsilon_{\mu\nu\alpha\beta}S^{\alpha\beta\nu} = 0$; $S^\alpha \equiv S^{\alpha\beta}{}_\beta$.

From the action (2.1), after the following steps

1) $mh^{\alpha\beta} \to h^{\alpha\beta}$,
2) $m^2 \to 0$,
3) performing the point transformation

$$S^{\alpha\beta\nu} \to S^{\alpha\beta\nu} + \tfrac{1}{2}(g^{\alpha\nu}S^\beta - g^{\beta\nu}S^\alpha),$$

4) performing integration by parts

we obtain

$$I = \int dx[\sqrt{2}\, S_{\beta\nu}h^{\beta\nu} + \tfrac{1}{2}(h_{\mu\nu}h^{\mu\nu} - h^2)], \tag{2.2}$$

where $S_{\beta\nu} \equiv \partial^\alpha S_{\alpha\beta\nu}$. We see that $h^{\alpha\beta}$ is a Lagrange multiplier. The field equations following from the action (2.2) are

$$h^{\mu\nu} = -\frac{1}{\sqrt{2}}(S^{\mu\nu} + S^{\nu\mu}) + \frac{\sqrt{2}}{3} g^{\mu\nu}S, \tag{2.3a}$$

$$\partial^\alpha h^{\beta\nu} - \partial^\beta h^{\alpha\nu} = 0, \tag{2.3b}$$

where $S \equiv S^\alpha{}_\alpha$. Eliminating $h^{\alpha\beta}$ from these equations we get the field equation for $S^{\alpha\beta\nu}$

$$\partial^\alpha(S^{\beta\nu} + S^{\nu\beta}) - \partial^\beta(S^{\alpha\nu} + S^{\nu\alpha}) + \tfrac{2}{3}(g^{\alpha\nu}\partial^\beta - g^{\beta\nu}\partial^\alpha)S = 0. \tag{2.4}$$

Eq. (2.4) is (up to the point transformation of $S^{\alpha\beta\nu}$) the $m^2 = 0$ limit of the equation for the massive field $S^{\alpha\beta\nu}$ (see Eq. (2.5) of Ref. [1]). Eliminating $h^{\alpha\beta}$ from the action (2.2) we get

$$I = \int dx[-\tfrac{1}{2} S_{\mu\nu}(S^{\mu\nu} + S^{\nu\mu}) + \tfrac{1}{3} S^2]. \tag{2.5}$$

This action can be obtained from the one involving the massive field $S^{\alpha\beta\nu}$ only (see Eq. (3.3) of Ref. [1]) after performing the point transformation and taking the limit $m^2 \to 0$. Of course, the field equation (2.4) follows immediately from the action (2.5).

Let us analyse Eq. (2.3b). It can be regarded as a constraint on the field $h^{\alpha\beta}$. We deduce the general form of $h^{\alpha\beta}$

$$h^{\alpha\beta} = \partial^\alpha\partial^\beta F. \tag{2.6}$$

Substituting it to the action (2.2) we obtain after integration by parts

$$I = 0. \tag{2.7}$$

We conclude that the action (2.5) does not describe any physical degrees of freedom.

2.2. The gauge transformation

Let us look at the action (2.5) from another point of view. The action (2.5) is invariant under the following gauge transformation

$$\delta S^{\alpha\beta\nu} = (g^{\alpha\nu}\partial^\beta - g^{\beta\nu}\partial^\alpha)\lambda$$
$$+\varepsilon^{\alpha\beta\gamma\delta}\partial_\gamma\eta^\nu_\delta + \tfrac{1}{3}\,\varepsilon^{\alpha\beta\nu\sigma}(\partial_\sigma\eta^\lambda_\lambda - \partial_\lambda\eta^\lambda_\sigma), \tag{2.8}$$

where η^α_β is a general 16 component tensor.

Analysing the role of the gauge transformation (2.8) (in spirit of the second theorem of Noether) we conclude that (*i*) the part containing η^α_β restricts the functional dependence of the Lagrangian to $\partial_\alpha S^{\alpha\beta\nu}$ only, (*ii*) the part containing λ determines the relative weight of two terms in the action (2.5).

Confronting Eq. (2.8) with the general form of $S^{\alpha\beta\nu}$ (see Appendix A)

$$S^{\alpha\beta\nu} = \frac{1}{\sqrt{2}}(g^{\alpha\nu}\partial^\beta - g^{\beta\nu}\partial^\alpha)F$$
$$+\varepsilon^{\alpha\beta\gamma\delta}\partial_\gamma B^\nu_\delta + \tfrac{1}{3}\,\varepsilon^{\alpha\beta\nu\sigma}(\partial_\sigma B^\lambda_\lambda - \partial_\lambda B^\lambda_\sigma) \tag{2.9}$$

$F(x)$ is a scalar function, $B^\mu_\nu(x)$ is a general 16 component tensor field) we conclude that $S^{\alpha\beta\nu}$ is a pure gauge.

2.3. The canonical analysis

Using the decomposition (see Appendix B)

$$S^{\alpha\beta\nu} = (S^{ij}, A^i, V^i, N^{ij}, M^{ij}, Q)$$

we can rewrite the action (2.5) in the form[1]

$$I = \int dx\mathscr{L}, \tag{2.10}$$

where the Lagrangian density is

$$\mathscr{L} = -\tfrac{1}{2}\,[(\partial^0 V^i)^2 - 2\partial^0 V^i(\partial^m N_{mi} - \tfrac{1}{3}\,\partial_i Q + 3\partial^j M_{ji})$$
$$+2(\partial^0 N^{ij})^2 + 4\partial^0 N^{ij}\partial^m S^p_j \varepsilon_{mip} + 4\partial^0 N^{ij}\partial_j A_i$$
$$+(\partial_m N^{mi})^2 + \tfrac{1}{9}\,(\partial^i Q)^2 + \tfrac{2}{3}\,\partial_m N^{mi}\partial_i Q$$
$$+9(\partial_m M^{mi})^2 + 6\partial_j M^{ji}\partial^k N_{ki} + \tfrac{1}{2}\,(\varepsilon^{mip}\partial_m S^j_p$$
$$+\varepsilon^{mjp}\partial_m S^i_p + \partial^i A^j + \partial^j A^i - 2g^{ij}\partial_m A^m)^2$$
$$-\tfrac{8}{3}\,(\partial_i A^i)^2 + \tfrac{4}{3}\,(\partial_i V^i)^2 - \tfrac{8}{3}\,\partial_i A^i \partial_j V^j]. \tag{2.11}$$

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[1] Some integrations by parts are performed to remove the velocities occurring linearly only.

We observe that S^{ij}, A^i, Q and M^{ij} are Lagrange multipliers. Let us define the canonical momenta

$$\Pi^i \equiv \frac{\partial \mathscr{L}}{\partial \partial^0 V_i} = -\partial^0 V^i + \partial_m N^{mi} - \tfrac{1}{3}\partial^i Q + 3\partial_m M^{mi}, \tag{2.12}$$

$$P^{ij} \equiv \frac{\partial \mathscr{L}}{\partial \partial^0 N_{ij}} = -2\partial^0 N^{ij} + \varepsilon^{imk}\partial_m S^j_k + \varepsilon^{jmk}\partial_m S^i_k - $$
$$-(\partial^i A^j + \partial^j A^i) + \tfrac{2}{3} g^{ij}\partial_m A^m. \tag{2.13}$$

We perform the Legendre transformation[2]

$$I = \int dx(\Pi^i\partial^0 V_i + P^{ij}\partial^0 N_{ij} - \mathscr{H}), \tag{2.14}$$

where $\mathscr{H}$ is the Hamiltonian density

$$\mathscr{H} = -\tfrac{1}{2}(\Pi^i)^2 + \Pi^i\partial^m N_{mi} - 3\partial^m \Pi^i M_{mi}$$
$$+\tfrac{1}{3}(\partial_i \Pi^i - 2\partial_i\partial_j N^{ij})Q - \tfrac{1}{4}(P^{ij})^2 + \tfrac{2}{3}(\partial_i V^i)^2$$
$$+\varepsilon_{mik}\partial^m P^{ij} S^k_j + (\tfrac{4}{3}\partial_i\partial_m V^m + \partial^m P_{mi})A^i. \tag{2.15}$$

Varying the Hamiltonian

$$H = \int d\vec{x}\mathscr{H}$$

with respect to the Lagrange multipliers S^{ij}, A^i, Q and M^{ij} we get the following constraints:

$$\partial^i \Pi^j - \partial^j \Pi^i = 0, \quad \partial_m P^{mi} + \tfrac{4}{3}\partial^i\partial_m V^m = 0,$$
$$\partial_i \Pi^i - 2\partial_i\partial_j N^{ij} = 0, \quad \varepsilon_{mik}\partial^m P^i_j + \varepsilon_{mij}\partial^m P^i_k = 0.$$

They are consistent with the canonical equations

$$\partial^0 V^i = \frac{\delta H}{\delta \Pi_i}, \quad \partial^0 \Pi^i = -\frac{\delta H}{\delta V_i},$$
$$\partial^0 N^{ij} = \frac{\delta H}{\delta P_{ij}}, \quad \partial^0 P^{ij} = -\frac{\delta H}{\delta N_{ij}}.$$

To solve the constraints we use the standard decomposition of a traceless symmetric tensor (see Appendix C). We obtain

$$\Pi^i_{\mathrm{T}} = 0, \quad \Pi^i_{\mathrm{L}} = -2\partial^i N_{\mathrm{L}}, \quad P^i_{\mathrm{T}} = 0,$$
$$P_{\mathrm{L}} = \tfrac{4}{3}\partial_i V^i_{\mathrm{L}}, \quad P^{ij}(\pm 2) = 0.$$

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[2] One can regard this transformation as one from the Lagrangian to the Routhian density $\mathscr{R}(V^i, N^{ij}, \Pi^i, P^{ij}, S^{ij}, A^i, Q, M^{ij}, \partial^0 S^{ij}, \partial^0 A^i, \partial^0 Q, \partial^0 M^{ij})$. Actually, $\mathscr{R}$ does not depend on velocities. So, $\mathscr{R}$ is the Hamiltonian density $\mathscr{H}$. Such a transformation is limited to velocities $\{v^i\}$ corresponding the regular part of the Hessian $\left\| \frac{\partial^2 L}{\partial v^i \partial v^j} \right\|$ [6].

Inserting these solutions to Eqs (2.15) and (2.14) we get $\mathscr{H} = 0$ and $I = 0$. So, there is no physical degree of freedom.

3. The 4-th rank tensor theory ($m^2 = 0$)

3.1. The action

Let us start with the first order action [1]

$$\begin{aligned} I = \int dx[&-\sqrt{2}\, m(B_{\mu\nu\alpha\beta}\partial^\mu S^{\alpha\beta\nu} - \tfrac{2}{9} BS) \\ &-\sqrt{2}\, m(\tfrac{1}{2} R_{\nu\beta} S^{\nu\beta} + \tfrac{1}{2} R_{\nu\beta}\partial^\nu S^\beta - \tfrac{1}{3} RS) \\ &+\tfrac{1}{2} m^2(S_{\alpha\beta\nu}S^{\alpha\beta\nu} - 2S_\alpha S^\alpha) + \tfrac{1}{2} m^2(R_{\mu\nu\sigma\beta}B^{\mu\nu\alpha\beta} - \tfrac{2}{9} RB) \end{aligned} \tag{3.1}$$

where $S^{\alpha\beta\nu}$ has the symmetry properties as in Section 2, $R^{\mu\nu\alpha\beta} = -R^{\mu\nu\beta\alpha} = -R^{\nu\mu\alpha\beta} = R^{\alpha\beta\mu\nu}$, $\varepsilon_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = 0$, $R^{\mu\alpha} \equiv R^{\mu\beta\alpha}{}_\beta$, $R \equiv R^\alpha_\alpha$ and the same is valid for $B_{\mu\nu\alpha\beta}$.

From the action (3.1), after the following steps

1) $mS^{\alpha\beta\nu} \to S^{\alpha\beta\nu}$,
2) $m^2 \to 0$,
3) introducing the new field

$$K^{\mu\nu\alpha\beta} = B^{\mu\nu\alpha\beta} + \tfrac{1}{4}(g^{\mu\alpha}R^{\nu\beta} + g^{\nu\beta}R^{\mu\alpha} - g^{\mu\beta}R^{\nu\alpha} - g^{\nu\alpha}R^{\mu\beta}),$$

4) performing the point transformation

$$K^{\mu\nu\alpha\beta} \to K^{\mu\nu\alpha\beta} - \tfrac{1}{3}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})K,$$

where $K \equiv K^{\mu\nu}{}_{\mu\nu}$,
5) performing integration by parts

we obtain the action

$$I = \int dx[\sqrt{2}\, \partial_\mu K^{\mu\nu\alpha\beta}S_{\alpha\beta\nu} + \tfrac{1}{2}(S_{\alpha\beta\nu}S^{\alpha\beta\nu} - 2S_\alpha S^\alpha)]. \tag{3.2}$$

We see that $S^{\alpha\beta\nu}$ is a Lagrange multiplier. The field equations following from the action (3.2) are

$$S^{\alpha\beta\nu} = -\sqrt{2}\,[\partial_\mu K^{\mu\nu\alpha\beta} + \tfrac{1}{2}(g^{\alpha\nu}K^\beta - g^{\beta\nu}K^\alpha)], \tag{3.3a}$$

$$\partial^\mu S^{\alpha\beta\nu} - \partial^\nu S^{\alpha\beta\mu} + \partial^\alpha S^{\mu\nu\beta} - \partial^\beta S^{\mu\nu\alpha} = 0, \tag{3.3b}$$

where $K^\alpha \equiv \partial_\mu K^{\mu\nu\alpha}{}_\nu$. Eliminating $S^{\alpha\beta\nu}$ from these equations we obtain

$$\begin{aligned} &\partial^\mu K^{\nu\alpha\beta} - \partial^\nu K^{\mu\alpha\beta} + \partial^\alpha K^{\beta\mu\nu} - \partial^\beta K^{\alpha\mu\nu} \\ &-\tfrac{1}{2}[g^{\mu\alpha}(\partial^\nu K^\beta + \partial^\beta K^\nu) + g^{\nu\beta}(\partial^\mu K^\alpha + \partial^\alpha K^\mu) \\ &-g^{\mu\beta}(\partial^\nu K^\alpha + \partial^\alpha K^\nu) - g^{\nu\alpha}(\partial^\mu K^\beta + \partial^\beta K^\mu)] = 0, \end{aligned} \tag{3.4}$$

where $K^{\nu\alpha\beta} \equiv \partial_\mu K^{\mu\nu\alpha\beta}$. This equation cannot be obtained by the point transformation from the one for $R_{\mu\nu\alpha\beta}$ (see Eq. (2.11) in Ref. [1]) in the $m^2 = 0$ limit.

Eliminating $S^{\alpha\beta\nu}$ from the action (3.2) we get

$$I = \int dx[-(\partial_\sigma K^{\sigma\nu\alpha\beta})^2+(\partial_\sigma K^{\sigma\nu\alpha}{}_\nu)^2]. \tag{3.5}$$

This action can be obtained from the action containing $B^{\mu\nu\alpha\beta}$ and $R^{\mu\nu\alpha\beta}$ only (see Eq. (3.6) in Ref. [1]) after steps 3), 4) and 2).

Let us analyse Eq. (3.3b). It can be regarded as a constraint on the field $S^{\alpha\beta\nu}$ determining its general form

$$S^{\alpha\beta\nu} = \frac{1}{\sqrt{3}}(\partial^\nu A^{\alpha\beta}+\partial^\alpha D^{\beta\nu}-\partial^\beta D^{\alpha\nu}), \tag{3.6}$$

where $A^{\alpha\beta} = -A^{\beta\alpha}$ and $D^{\alpha\beta} = -D^{\beta\alpha}$. The factor $1/\sqrt{3}$ is chosen for further convenience. The fields $A^{\alpha\beta}$ and $D^{\alpha\beta}$ are not independent for $\varepsilon_{\mu\nu\alpha\beta}S^{\alpha\beta\nu} = 0$. So, we get

$$\varepsilon_{\mu\nu\alpha\beta}(\partial^\nu A^{\alpha\beta}+2\partial^\nu D^{\alpha\beta}) = 0. \tag{3.7}$$

From Eq. (3.7) it follows that

$$(\partial^\nu A^{\alpha\beta})^2-2(\partial_\alpha A^{\alpha\beta})^2 = 4[(\partial^\nu D^{\alpha\beta})^2-2(\partial_\alpha D^{\alpha\beta})^2].$$

In terms of $A^{\alpha\beta}$ the action (3.5) can be rewritten in the form:

$$I = -\tfrac{1}{2}\int dx[(\partial^\nu A^{\alpha\beta})^2-2(\partial_\alpha A^{\alpha\beta})^2]. \tag{3.8}$$

This is the action for the Ogievetsky — Polubarinov "notoph" [4]. So, the action (3.5) describes the helicity 0 (a scalar "notivarg").

3.2. The gauge transformation

The action (3.5) is invariant under the following gauge transformation

$$\begin{aligned}\delta K^{\mu\nu\alpha\beta} &= \varepsilon^{\mu\nu\sigma\lambda}\partial_\sigma\omega^{\alpha\beta}{}_\lambda+\varepsilon^{\alpha\beta\sigma\lambda}\partial_\sigma\omega^{\mu\nu}{}_\lambda\\ &+g^{\mu\alpha}(\partial^\nu\eta^\beta+\partial^\beta\eta^\nu)+g^{\nu\beta}(\partial^\mu\eta^\alpha+\partial^\alpha\eta^\mu)\\ &-g^{\mu\beta}(\partial^\nu\eta^\alpha+\partial^\alpha\eta^\nu)-g^{\nu\alpha}(\partial^\mu\eta^\beta+\partial^\beta\eta^\mu)\\ &-2(g^{\mu\alpha}g^{\nu\beta}-g^{\mu\beta}g^{\nu\alpha})\partial_\sigma\eta^\sigma,\end{aligned} \tag{3.9}$$

where $\omega^{\alpha\beta\nu}$ has the symmetries of the Fierz tensor $\omega^{\alpha\beta\nu} = -\omega^{\beta\alpha\nu}$, $\varepsilon_{\mu\nu\alpha\beta}\omega^{\alpha\beta\nu} = 0$ and obeys the condition $\partial_\alpha\omega^{\alpha\beta\nu} = 0$.

Analysing the role of the gauge transformation (3.9) (in spirit of the second theorem of Noether) we conclude that
(*i*) the part containing $\omega^{\alpha\beta\nu}$ restricts the functional dependence of the Lagrangian to $\partial_\sigma K^{\sigma\nu\alpha\beta}$ only,
(*ii*) the part containing η^α determines the relative weight of two terms in the action (3.5).

We adopt

$$K^{\mu\nu} = 0, \tag{3.10}$$

$$\partial_\mu\partial_\alpha K^{\mu\nu\alpha\beta} = 0 \tag{3.11}$$

as the gauge conditions. It can be verified that taking into account these conditions (*i*) one gets from the field equation (3.4)

$$\Box K^{\mu\nu\alpha\beta} = 0;$$

(*ii*) in the momentum space in the frame $p^\mu = (p, 0, 0, p)$ the tensor $K^{\mu\nu\alpha\beta}$ has five independent components $S^{ij} = S^{ji}$, $S^i_i = 0$ (see Appendix B). These components belong to $(2, 0) \oplus (0, 2)$ representation of the Lorentz group;
(*iii*) a gauge freedom is not removed completely. Only the component S^{33} is invariant under the remaining gauge transformation. It describes the helicity 0.

3.3. The canonical analysis

Using the decomposition (see Appendix B)

$$K^{\mu\nu\alpha\beta} = (T^{ij}, S^{ij}, A^i, K^{ijkl}) \tag{3.12}$$

we can rewrite the action (3.5) in the form

$$I = \int dx \mathscr{L}(T^{ij}, S^{ij}, A^i, K^{ijkl}, \partial^0 T^{ij}, \partial^0 S^{ij}), \tag{3.13}$$

where the Lagrangian density is

$$\begin{aligned}\mathscr{L} = & -2(\partial^0 T^{ki})^2 + (\partial^0 T^m_m)^2 - 4\partial^0 T^{ki}\varepsilon_{jkm}\partial^j S^m_i \\ & -8\partial^0 T^{ki}\partial_i A_k + 2(\partial^0 S^{ki})^2 - (\partial_m T^{mi})^2 \\ & -2\varepsilon^{ijp}\partial^0 S^k_p \partial^m K_{mkij} + 2(\partial^i S^{kj})^2 \\ & -4(\partial^k A^i)^2 + 4(\partial_i A^i)^2 - 8\varepsilon^{ijm}\partial^k S_{km}\partial_i A_j \\ & -(\partial_m K^{mkij})^2 + (\partial_m K^{mjk}{}_j)^2 + 2\partial_m T^{mk}\partial^i K_{ijk}{}^j.\end{aligned} \tag{3.14}$$

We see that A^i and K^{ijkl} are Lagrange multipliers.

Let us define the canonical momenta

$$\begin{aligned}\Pi^{ki} \equiv \frac{\partial \mathscr{L}}{\partial \partial^0 T_{ki}} = & -4\partial^0 T^{ki} + 2g^{ki}\partial^0 T^j_j \\ & + 2(\varepsilon^{kjm}\partial_j S^i_m + \varepsilon^{ijm}\partial_j S^k_m) - 4(\partial^k A^i + \partial^i A^k),\end{aligned} \tag{3.15}$$

$$P^{ki} \equiv \frac{\partial \mathscr{L}}{\partial \partial^0 S_{ki}} = 4\partial^0 S^{ki} - (\varepsilon^{sti}\partial_m K^{mk}{}_{st} + \varepsilon^{stk}\partial_m K^{mi}{}_{st}) + \tfrac{2}{3}\, g^{ki}\varepsilon^{stj}\partial^m K_{mjst} \tag{3.16}$$

and perform the Legendre transformation

$$I = \int dx (\Pi^{ki}\partial^0 T_{ki} + P^{ki}\partial^0 S_{ki} - \mathscr{H}). \tag{3.17}$$

The Hamiltonian density is

$$\mathscr{H} = -\tfrac{1}{8}(\Pi^{ik})^2+\tfrac{1}{8}(\Pi^i_i)^2-2\Pi^{ki}\partial_k A_i$$
$$+2\Pi^i_i\partial_m A^m+8\varepsilon^{ijm}\partial^k S_{km}\partial_i A_j+4\varepsilon^{kpm}\partial_p S^i_m\partial_i A_k-3(\partial_m S^{mi})^2+(\partial_m T^{mi})^2$$
$$-2\partial_m T^{mk}\partial^p K_{pjk}{}^j+\tfrac{1}{8}(P^{ik})^2+\tfrac{1}{2}P_{ik}\varepsilon^{jti}\partial_m K^{mk}{}_{jt}. \tag{3.18}$$

Varying the Hamiltonian

$$H = \int d\vec{x}\mathscr{H}$$

with respect to A^i and K^{ijkl} we get the following constraints

$$\partial_m\Pi^{mi}-\partial^i\Pi^m_m+2\varepsilon^{ijm}\partial_j\partial^k S_{km} = 0, \tag{3.19}$$
$$\partial^i Q^{jkl}-\partial^j Q^{ikl}+\partial^k Q^{lij}-\partial^l Q^{kij} = 0, \tag{3.20}$$

where

$$Q^{jkl} = \varepsilon^{klm}P^j_m-2(\partial_m T^{mk}g^{jl}-\partial_m T^{ml}g^{jk}).$$

These constraints do not contradict the canonical equations

$$\partial^0 T^{ij} = \frac{\delta H}{\delta\Pi_{ij}}, \quad \partial^0\Pi^{ij} = -\frac{\delta H}{\delta T_{ij}},$$
$$\partial^0 S^{ij} = \frac{\delta H}{\delta P_{ij}}, \quad \partial^0 P^{ij} = -\frac{\delta H}{\delta S_{ij}}$$

if further constraints are obeyed:

$$\partial_i\partial_j T^{ij} = 0, \tag{3.21}$$
$$\partial^i\partial^j\Pi^{kl}+\partial^k\partial^l\Pi^{ij}-\partial^i\partial^k\Pi^{jl}-\partial^j\partial^l\Pi^{ik} = 0. \tag{3.22}$$

The constraints (3.21) and (3.22) are consistent with the canonical equations. So, there are no other constraints.

To solve the constraints (3.19)–(3.22) we use the standard decomposition of a vector and a traceless symmetric tensor (see Appendix C). Let

$$\Pi^{ij} \equiv \tilde{\Pi}^{ij}+\tfrac{1}{3}g^{ij}\Pi^m_m, \quad T^{ij} \equiv \tilde{T}^{ij}+\tfrac{1}{3}g^{ij}T^m_m,$$
$$(\tilde{\Pi}^m_m = \tilde{T}^m_m = 0),$$

then we obtain

(*i*) from Eq. (3.19)

$$\tilde{\Pi}^i_{\mathrm{T}} = -2\varepsilon^{ijk}\partial_j S_{\mathrm{T}k}, \quad \tilde{\Pi}_{\mathrm{L}} = -\tfrac{2}{3}\Pi^m_m;$$

(*ii*) from Eq. (3.20)

$$P^{ij}(\pm 2) = 0, \quad \tilde{T}_{\mathrm{L}} = \tfrac{1}{3}T^m_m, \quad P^i_{\mathrm{T}} = -2\varepsilon^{ijk}\partial_j\tilde{T}_{\mathrm{T}k};$$

(*iii*) from Eq. (3.21)

$$\tilde{T}_L = \tfrac{1}{3} T^m_m;$$

(*iv*) from Eq. (3.22)

$$\tilde{\Pi}^{ij}(\pm 2) = 0, \quad \tilde{\Pi}_L = -\tfrac{2}{3} \Pi^m_m.$$

Using these solutions in the Lagrangian density (3.17) we get

$$\mathscr{L} = \tfrac{3}{2} P_L \partial^0 S_L - \mathscr{H},$$

$$\mathscr{H} = \tfrac{3}{16} P_L^2 - 3(\partial^i S_L)^2.$$

The field equation for the scalar S_L is

$$\Box S_L = 0.$$

In the momentum space in the frame $p^\mu = (p, 0, 0, p)$ we get $S_L = S^{33}$. So, we confirm the result obtained in Sections 3.1 and 3.2.

To conclude this Section we compare the action (3.5) with the one (describing also a scalar "notivarg") obtained by Deser, Siegel and Townsend [5]. This last can be rewritten (in our notation) in the form

$$I = \tfrac{1}{2} \int dx (-\tfrac{1}{3} K^\nu \partial_\nu K + K^{\nu\alpha\beta} \partial_\alpha K_{\nu\beta}).$$

There is no point transformation of $K^{\mu\nu\alpha\beta}$ connecting it with our action (3.5). So, the action (3.5) can be regarded as the alternative formulation of the scalar "notivarg" theory [7].

4. *Final remarks*

Let us briefly summarize our results. We have shown that the theories of spin 2, equivalent in the massive case, are not such in the $m^2 = 0$ limit. While the massless theory of Pauli and Fierz describes the helicities ± 2, the one based on the third rank rensor $S^{\alpha\beta\nu}$ has no physical content, and the one based on the 4-th rank tensor $K^{\mu\nu\alpha\beta}$ describes the helicity 0.

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APPENDIX A

In this Appendix we obtain the general form of $S^{\alpha\mu\nu}$ obeying Eqs (2.3a) and (2.3b)[3]. Decomposing $S^{\mu\nu}$ into the symmetric $\sigma^{\mu\nu}$ and the antisymmetric $\alpha^{\mu\nu}$ parts

$$S^{\mu\nu} = \sigma^{\mu\nu} + \alpha^{\mu\nu} \tag{A1}$$

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[3] I thank Dr. P. Kosiński for exhaustive discussion of this point.

we get from Eqs (2.3a) and (2.6)

$$\sigma^{\mu\nu} = \frac{1}{\sqrt{2}}(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)F. \tag{A2}$$

Because $\partial_\mu\partial_\alpha S^{\alpha\mu\nu} \equiv \partial_\mu S^{\mu\nu} = 0$, we conclude that

$$\partial_\mu\alpha^{\mu\nu} = 0. \tag{A3}$$

The solution of Eq. (A3) has the form

$$\alpha^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta}\partial_\alpha A_\beta,$$

where A_β is a vector function. We get

$$S^{\mu\nu} = \frac{1}{\sqrt{2}}(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)F + \varepsilon^{\mu\nu\alpha\beta}\partial_\alpha A_\beta$$

$$= \partial_\alpha\left[\frac{1}{\sqrt{2}}(g^{\mu\nu}\partial^\alpha - \tfrac{1}{2}\, g^{\mu\alpha}\partial^\nu - \tfrac{1}{2}\, g^{\nu\alpha}\partial^\mu)F + \varepsilon^{\alpha\mu\nu\beta}A_\beta\right].$$

We conclude that

$$S^{\alpha\mu\nu} = \frac{1}{\sqrt{2}}(g^{\mu\nu}\partial^\alpha - \tfrac{1}{2}\, g^{\mu\alpha}\partial^\nu - \tfrac{1}{2}\, g^{\nu\alpha}\partial^\mu)F + \varepsilon^{\alpha\mu\nu\beta}A_\beta + \Delta^{\alpha\mu\nu}, \tag{A4}$$

where

$$\partial_\alpha \Delta^{\alpha\beta\nu} = 0. \tag{A5}$$

The tensor $\Delta^{\alpha\mu\nu}$ obeys the condition (as a consequence of $S^{\alpha\mu\nu} + S^{\mu\alpha\nu} = 0$)

$$\Delta^{\alpha\mu\nu} + \Delta^{\mu\alpha\nu} + \frac{1}{\sqrt{2}}(\tfrac{1}{2}\, g^{\mu\nu}\partial^\alpha + \tfrac{1}{2}\, g^{\alpha\nu}\partial^\mu - g^{\mu\alpha}\partial^\nu)F = 0. \tag{A6}$$

We seek the solution of Eq. (A5) in the form

$$\Delta^{\alpha\mu\nu} = \Pi^{\alpha\mu\nu} + \Sigma^{\alpha\mu\nu},$$

where $\Pi^{\alpha\mu\nu}$ is the general solution obeying

$$\partial_\alpha \Pi^{\alpha\mu\nu} = 0, \quad \Pi^{\alpha\mu\nu} + \Pi^{\mu\alpha\nu} = 0,$$

and $\Sigma^{\alpha\mu\nu}$ is the special solution obeying

$$\partial_\alpha \Sigma^{\alpha\mu\nu} = 0, \tag{A7a}$$

$$\Sigma^{\alpha\mu\nu} + \Sigma^{\mu\alpha\nu} + \frac{1}{\sqrt{2}}(\tfrac{1}{2}\, g^{\mu\nu}\partial^\alpha + \tfrac{1}{2}\, g^{\alpha\nu}\partial^\mu - g^{\mu\alpha}\partial^\nu)F = 0. \tag{A7b}$$

The general solution has the form

$$\Pi^{\alpha\mu\nu} = \varepsilon^{\alpha\mu\gamma\beta}\partial_\gamma B^\nu_\beta,$$

where B^{ν}_{β} is a 16 component tensor field. We seek the special solution in the form

$$\Sigma^{\alpha\mu\nu} = (ag^{\alpha\mu}\partial^{\nu}+bg^{\alpha\nu}\partial^{\mu}+cg^{\mu\nu}\partial^{\alpha})F.$$

From Eqs (A7a) and (A7b) we get $c = 0$, $a = -b = 1/2\sqrt{2}$ and

$$\Sigma^{\alpha\mu\nu} = \frac{1}{2\sqrt{2}}(g^{\alpha\mu}\partial^{\nu}-g^{\alpha\nu}\partial^{\mu})F.$$

So,

$$S^{\alpha\mu\nu} = \frac{1}{\sqrt{2}}(g^{\mu\nu}\partial^{\alpha}-g^{\alpha\nu}\partial^{\mu})F+\varepsilon^{\alpha\mu\nu\beta}A_{\beta}+\varepsilon^{\alpha\mu\gamma\beta}\partial_{\gamma}B^{\nu}_{\beta},$$

where

$$A_{\beta} = \tfrac{1}{3}(\partial_{\beta}B^{\nu}_{\nu}-\partial_{\nu}B^{\nu}_{\beta})$$

as consequence of $\varepsilon_{\mu\nu\alpha\beta}S^{\alpha\mu\nu} = 0$.

APPENDIX B

1. Let us consider the tensor $S^{\alpha\beta\nu} = -S^{\beta\alpha\nu}$ obeying the condition

$$\varepsilon_{\mu\nu\alpha\beta}S^{\alpha\beta\nu} = 0. \tag{B1}$$

We introduce the new variables $S^{ij} = S^{ji}$, A^{i}, $B^{ij} = -B^{ji}$, V^{i}, $N^{ij} = N^{ji}$ $(N^{i}_{i} = 0)$, Q and $M^{ij} = -M^{ji}$ $(i, j = 1, 2, 3)$ defined by

$$S^{ijk} = \varepsilon^{ijm}S^{k}_{m}+g^{ik}A^{j}-g^{jk}A^{i},$$

$$S^{ij0} = B^{ij}, \qquad S^{0i0} = V^{i},$$

$$S^{0ij} = N^{ij}+\tfrac{1}{3}g^{ij}Q+M^{ij}.$$

In these variables Eq. (B1) reads

$$S^{i}_{i} = 0, \qquad B^{ij} = -2M^{ij}.$$

So, we get the following decomposition of $S^{\alpha\beta\nu}$

$$S^{\alpha\beta\nu} = (S^{ij}, A^{i}, V^{i}, N^{ij}, Q, M^{ij}),$$

where S^{ij} and N^{ij} are traceless.

2. Let us consider the tensor $K^{\mu\nu\alpha\beta} = K^{\alpha\beta\mu\nu} = -K^{\mu\nu\beta\alpha}$ obeying the condition

$$\varepsilon_{\mu\nu\alpha\beta}K^{\mu\nu\alpha\beta} = 0. \tag{B2}$$

We introduce the new variables $T^{ij} = T^{ji}$, $S^{ij} = S^{ji}$ and A^{i} $(i, j = 1, 2, 3)$ defined by

$$K^{0i0j} = T^{ij},$$

$$K^{0ijk} = \varepsilon^{jkm}S^{i}_{m}+g^{ij}A^{k}-g^{ik}A^{j}.$$

In these variables Eq. (B2) reads $S^i_i = 0$. So, we get the following decomposition of $K^{\mu\nu\alpha\beta}$

$$K^{\mu\nu\alpha\beta} = (T^{ij}, S^{ij}, A^i, K^{ijmn}),$$

where S^{ij} is traceless.

APPENDIX C

The well known decomposition of a vector into transversal and longitudinal parts is

$$V^i \equiv V^i_{\rm T} + V^i_{\rm L},$$

where

$$V^i_{\rm T} = V^i + \frac{1}{\Delta}\,\partial^i\partial_j V^j, \quad V^i_{\rm L} = -\frac{1}{\Delta}\,\partial^i\partial_j V^j, \quad \Delta = -\partial_i\partial^i.$$

The analogous decomposition of a symmetric traceless tensor a^{ij} is

$$a^{ij} \equiv a^{ij}(\pm 2) + a^{ij}(\pm 1) + a^{ij}(0),$$

where

$$a^{ij}(\pm 1) = -\frac{1}{\Delta}\,(\partial^i a^j_{\rm T} + \partial^j a^i_{\rm T}),$$

$$a^{ij}(0) = \tfrac{3}{2}\left(\frac{1}{\Delta}\,\partial^i\partial^j + \tfrac{1}{3}\,g^{ij}\right) a_{\rm L},$$

$$a^i_{\rm T} = a^i + \frac{1}{\Delta}\,\partial^i\partial_j a^j, \quad a_{\rm L} = \frac{1}{\Delta}\,\partial_i a^i,$$

$$a^i = \partial_j a^{ji}.$$

We see that

$$a^i_i(\pm 2) = a^i_i(\pm 1) = a^i_i(0) = 0,$$

$$\partial_i a^{ij}(\pm 2) = 0, \quad \partial_i a^{ij}(\pm 1) = a^j_{\rm T},$$

$$\partial_i a^{ij}(0) = -\partial^j a_{\rm L}.$$

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