KINEMATICS OF RELATIVE MOTION OF CHARGED TEST PARTICLES IN GENERAL RELATIVITY. I. THE FIRST ELECTROMAGNETIC DEVIATION

By S. L. Bażański* and N. N. Kostyukovich**

Institute of Theoretical Physics, University of Warsaw***

(*Received October 22, 1986*)

This is the first of two articles containing the definition and a detailed mathematical analysis of the concept of electromagnetic deviation in pseudo-Riemannian geometry. It presents a study of both the Lorentz equations of motion and the notion of the first electromagnetic deviation together with its equations of evolution, whereas the second article, which will follow soon, will be dealing with the notion of the second electromagnetic deviation. An interrelation of properties of the initial value problems for the Lorentz and the deviation equations is investigated. The analysis makes use of a formalism that is reparametrisation covariant and invariant under gauges of deviations defined in the paper. It is shown that by specializations of gauges one can obtain particular kinds of deviation, some of which were not discussed so far.

PACS numbers: 04.20.Cv

Introduction

From a physical point of view, the equations of geodesic deviation along a given geodesic world line Γ represent a law of evolution of the information which enables the observer whose history is represented by Γ to describe the geodesic world lines in the neighbourhood of Γ. Depending on the size of this neighbourhood, this description requires introducing the concept of geodesic deviations of higher orders. A general scheme of defining such higher deviations, as well as their geometrical interpretation was formulated in [1].

The present paper is a generalization of [1]. It consists of two parts published separately: Part I, appearing as this article, and Part II that will come out soon. The theoretical scheme worked out for geodesic lines in [1] is now applied to a more general case of Lorentz-

* Supported by the Polish Research Project CPBP 01.03.

** Present address: Laboratory of Theoretical Physics, Institute of Physics of the BSSR Academy of Science, Minsk, USSR.

*** Address: Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Hoża 69, 00-681 Warszawa, Poland.

ian world lines representing motions of charged test particles in given electromagnetic and gravitational fields of general relativity, as well as to their deviations called here the electromagnetic (e.m.) deviations.

The main approach accepted here is based on a single Lorentzian world line Γ defined by an appropriate system of differential equations and on several laws of transport of e.m. deviation vectors, of the first and the second order, along this line. In the case of geodesic deviations that were discussed in [1], a method was formulated of how to define a one-to-one correspondence between sets of deviation vectors and world lines from a neighbourhood of Γ that are described by these sets. This method, besides its practical purpose, explains the geometrical meaning of all the deviation vectors and reveals the necessity of dealing with similar vectors of higher order. An interpretation analogous to that worked out in [1] exists also for the e.m. deviations as its straightforward modification which is not worthwhile to be repeated here. Its details could easily be worked out for the single line approach just by imitating the geometrical interpretation of the geodesic deviation vectors in terms of the covariant Taylor theorem (cf. [2]), similarly as it was done in [1].

Besides the main, single line, approach we discuss in the paper also an auxiliary approach which makes use of one-parametric families of Lorentzian world lines. This, called here the Σ-approach, helps us to formulate the transport laws for deviation vectors of several orders and to find the geometric interpretation of the constraints that exists in the problem.

The primary objective of our consideration is to reveal the logical and geometrical structure behind the notion of e.m. deviation. In the paper this objective is realized explicitly for the e.m. deviations of the first and the second order, but the same scheme of reasoning applies evidently to deviations of any higher order. Since in relativity the physical meaning of a timelike world line is attached to a curve that ought to be understood as a one-dimensional manifold and not to a parametrized curve, also a concept of deviation should basically be treated from such a point of view. A notion of an e.m. deviation of this kind is thus given by a multiplicity of solutions (forming an equivalence class) of an underdetermined system of differential equations written along a given Lorentzian world line and being reparametrization covariant. The procedure of solving the underdetermined system requires introduction of arbitrary functions which label members of the family of sets of differential equations such that for each fixed function the corresponding set of equations has a unique solution which is just a single representative of the class of equivalence being the e.m. deviation. Two special choices of the arbitrary functions are discussed in some detail. In the case of the first e.m. deviation vector they lead to two types of e.m. deviation equations, one of which has already been discussed in the literature (see [3, 4]), and the other one is a special case of the general nongeodesic deviation equations (i.e. for world lines with a general force law) that were formulated by Ehlers [5] and later used e.g. in [6–8]. From the analysis done in the paper it follows however that in order to obtain a geometrically well defined law of transport of the e.m. deviation vectors along a Lorentzian world line Γ, the traditional e.m. deviation equations, as well as many others which one could obtain for other special choices of the arbitrary functions, must be supplemented by some constraint conditions. In particular, these conditions reduce the number of independent degrees of freedom in the dynamics of e.m. deviations, which will be a subject of a subsequent publication of the authors. In our opinion, the significance of the constraint conditions supplementing the e.m. and the geodesic deviation equations was commonly not taken into account in the literature so far, with the only exception of Ref. [1] which can be considered as a particular case, for vanishing charges, of that discussed now.

The general scheme described in the foregoing paragraph is used in the paper on three levels: in Sect. 1 on the level of a single Lorentzian world line, which could be treated as being the e.m. deviation of the zeroth order, and then in Sects 2–5 in Part I and 6–9 in Part II for the first and the second e.m. deviation vectors correspondingly. One can note that several properties of the objects discussed here, which were emerging on a certain level, have their consequences also on subsequent levels. Besides of material being just a review, every of the Sections contains results obtained recently by the authors. These results are explicitly summarized in the last Section of Part II of the paper.

This is the first from a series of papers now in preparation. The forthcoming papers will be dealing with some dynamical properties of e.m. deviations, their conservation laws, and methods of solving the e.m. deviation equations in some spacetimes of general relativity.

1. The Lorentz equations

Let V_n be a n-dimensional pseudo-Riemannian manifold endowed with a coordinate system $\{x^\alpha\}$, $\alpha = 1, \ldots, n$, admissible in a region $\Omega \subset V_n$. In such a coordinate system an arbitrary world line $\Gamma : R \to V_n$ parametrized by a scalar parameter $\tau \in [a, b] = I \subset R$ is described by a set of n functions $\xi^\alpha : I \to R$, where $\xi^\alpha(\tau) = x^\alpha \circ \Gamma(\tau)$. We shall consider Lorentzian curves which for $n = 4$ represent the world lines of charged test particles in a gravitational field described by the metric tensor $g_{\alpha\beta}$ of V_n and in an electromagnetic field defined by a given antisymmetric tensor field $F_{\alpha\beta}$ on V_n. The general relativistic Lorentz equations of motion in terms of an arbitrary parameter τ, as a result of an appropriate variational principle (see [9]), take one of the two equivalent forms

$$\frac{D}{d\tau}\left(\frac{u^\alpha}{\sqrt{|u_\lambda u^\lambda|}}\right) = \sigma F^\alpha{}_\beta u^\beta \tag{1.1}$$

or

$$L[\xi^\alpha] := \frac{h^\alpha{}_\beta}{\sqrt{|u_\lambda u^\lambda|}} \frac{Du^\beta}{d\tau} - \sigma F^\alpha{}_\beta u^\beta = 0,$$

where $\frac{D}{d\tau}(\cdot) := (\cdot)_{;\lambda} u^\lambda$ denotes the absolute derivative along Γ, $u^\alpha := \frac{d\xi^\alpha}{d\tau}$ is a vector tangent to Γ, $u_\alpha = g_{\alpha\beta} u^\beta$, $h^\alpha{}_\beta := \delta^\alpha_\beta - \frac{u^\alpha u_\beta}{u_\lambda u^\lambda}$ is a projection tensor, $\sigma = q/\mu c^2$ is a constant, and $|u_\lambda u^\lambda| \neq 0$ is assumed anywhere along Γ. It is also assumed that the quantities $g_{\alpha\beta}$, $\Gamma^\alpha{}_{\beta\gamma}$ and $F_{\alpha\beta}$ entering Eqs (1.1) are given functions of τ.

It may be simply checked that $u_\alpha L[\xi^\alpha] \equiv 0$ is a strong identity, i.e. is valid for any ξ^α. Hence, the n equations (1.1) are not independent and together with the initial conditions

$$\xi^\alpha(\tau_0) = \xi_0^\alpha, \quad \frac{d\xi^\alpha}{d\tau}(\tau_0) = u_0^\alpha \tag{1.2}$$

do not determine the n functions ξ^α uniquely. This fact may be stated more precisely:

PROPOSITION 1.1. If a set of n functions $\xi^\alpha : I \to R$ is a solution of the Lorentz equations (1.1), then

(*i*) the set of composite functions $\xi^\alpha \circ f$, for any C_2 function f: $[a, b] \to [a', b']$ such that $f' \neq 0$, is also a solution of Eqs (1.1);

(*ii*) any solution $\tilde{\xi}^\alpha$ of Eqs (1.1), which satisfies the same initial conditions as ξ^α, can be represented as $\tilde{\xi}^\alpha = \xi^\alpha \circ f$, where $f \in C_2$ is uniquely determined by $\tilde{\xi}^\alpha$, ξ^α.

Proof. Part (*i*) follows from an immediate computation. The proof of part (*ii*) consists in constructing a differential equation for f. This equation must, of course, be supplemented by some initial conditions which can be taken in one of the two possible forms.

First, we may assume that ξ^α and $\tilde{\xi}^\alpha$ fulfil for $\tau = \tau_0$ the same initial conditions (1.2). The freedom of the choice of f is then reduced to C_2 functions such that

$$f(\tau_0) = \tau_0, \quad f'(\tau_0) = 1. \tag{1.3}$$

Second, we might require more generally that ξ^α and $\tilde{\xi}^\alpha$ describe the same geometrical locus of points, i.e. for a certain $\tilde{\tau}_0$ (not necessarily equal to τ_0)

$$\tilde{\xi}^\alpha(\tilde{\tau}_0) = \xi_0^\alpha, \quad \frac{d\tilde{\xi}^\alpha}{d\tau}(\tilde{\tau}_0) = k u_0^\alpha, \tag{1.4}$$

where k is a nonzero and otherwise arbitrary constant. In this second case we are left with somewhat greater freedom of choice of f:

$$f(\tilde{\tau}_0) = \tau_0, \quad f'(\tilde{\tau}_0) = k. \tag{1.5}$$

The proof of part (*ii*) requires yet the following lemma.

LEMMA 1.1. Any solution of Eqs (1.1) which satisfies (1.2) is completely specified by a choice of a continuous function $\lambda : I \to R$.

Proof. A given solution ξ^α defines the function

$$\lambda(\tau) := \frac{d}{d\tau} \ln \sqrt{\left| g_{\alpha\beta} \frac{d\xi^\alpha}{d\tau} \frac{d\xi^\beta}{d\tau} \right|}, \tag{1.6}$$

where in accordance with our assumption $|u_\lambda u^\lambda| \neq 0$. Inversely, if a function $\lambda : I \to R$ is given, ξ^α is uniquely determined by the initial conditions (1.2) and the differential equations (cf. 1.1))

$$\frac{Du^\alpha}{d\tau} = \lambda(\tau) u^\alpha + \sigma \sqrt{|u_\lambda u^\lambda|}\, F^\alpha{}_\beta u^\beta. \tag{1.7}$$

For such ξ^α Eqs (1.6) are satisfied as a weak identity and therefore ξ^α is also a solution of Eqs (1.1). This completes the proof of the lemma.

COROLLARY 1.1. There exists a one-to-one map F of the set of all solutions of Eqs (1.1), corresponding to some fixed initial data in (1.2), onto the set of all continuous functions $\lambda: I \to R$. The map is defined by (1.6) as $\xi^\alpha(\tau) \mapsto F(\xi^\alpha(\tau)) = \lambda(\tau)$ and its inverse is determined by the solution of the initial value problem (1.2) for the differential equations (1.7).

Now, if ξ^α is solution of Eqs (1.1) fulfilling the initial conditions (1.2), then $\tilde{\xi}^\alpha := \xi^\alpha \circ f$ is, in accordance with part (*i*) of Prop. 1.1, also a solution. Let $F(\xi^\alpha) = \lambda$, then $F(\tilde{\xi}^\alpha) = \tilde{\lambda}$ is defined, as it follows from (1.6), to be

$$\tilde{\lambda}(\tau) = \frac{f''(\tau)}{f'(\tau)} + f'(\tau)\lambda(f(\tau)). \tag{1.8}$$

Thus, if ξ^α and $\tilde{\xi}^\alpha$ are any two given solutions of Eqs (1.1), fulfilling respectively (1.2) or (1.4), i.e. $F(\xi^\alpha) = \lambda$ and $F(\tilde{\xi}^\alpha) = \tilde{\lambda}$ are given functions, then solving the differential equation (1.8) together with (1.3) or (1.5) respectively, one finds a unique function f such that $\tilde{\xi}^\alpha = \xi^\alpha \circ f$. This ends the proof of part (*ii*) and of the whole Prop. 1.1.

We shall now discuss the question of how to formulate a system of differential equations together with its initial value problem that would, firstly, be well-posed in the sense of mathematical physics (i.e. would assure a one-to-one correspondence between solutions and sets of initial conditions), and, secondly, be in a certain sense, which is precisely formulated in what follows, equivalent to Eqs (1.1) with the initial conditions (1.2). Having this purpose in mind, let us recall the following definition from differential geometry.

DEFINITION 1.1. Two descriptions, by ξ^α and $\tilde{\xi}^\alpha$, respectively, of a curve Γ in a coordinate system $\{x^\alpha\}$ are equivalent iff there is such a C_2 function $f: [a, b] \to [a', b']$ that (*i*) $f'(\tau) \neq 0$ in the whole domain $\Omega \subset V_n$, (*ii*) $\tilde{\xi}^\alpha = \xi^\alpha \circ f$.

Therefore, Prop. 1.1 states that Eqs (1.1) together with (1.2) uniquely determine an equivalence class of descriptions of a Lorentzian world line Γ in a coordinate system $\{x^\alpha\}$. Each member of this class is defined by Eqs (1.7) with a fixed function λ. In other words, from an analytic point of view, we have on one side a system of n differential equations (1.1) that is analytically underdetermined and admits a multiplicity of solution ξ^α, and on the other side a family of sets of n differential equations (1.7) (each member of the family being labelled by a function λ) and every of these sets together with appropriate initial conditions uniquely determines a solution ξ^α, but in general the solutions of two different sets (i.e. for $\lambda_1(\tau) \neq \lambda_2(\tau)$) from the family (1.7) are analytically different from each other. From a geometric point of view, however, in accordance with Prop. 1.1 and Def. 1.1, such two solutions of two different sets of the family (1.7) are just two different descriptions of the same curve Γ (understood as a locus of points) which is uniquely determined by Eqs (1.1) and the initial conditions (1.2). For any practical purpose, one does not need the whole multiplicity of representatives of the world line Γ, but just only one. Hence, to find a solution of Eqs (1.1) fulfilling (1.2) with some given initial data $\{\xi_0^\alpha, u_0^\alpha\}$, it is sufficient to solve (1.7) with the possibly simplest choice of the function λ. Such a choice consists, for instance, in taking $\lambda = 0$ for any $\tau \in I$.

With this choice, any set $\{\xi_0^\alpha, u_0^\alpha\}$ of initial data in (1.2) and the equations

$$\frac{Du^\alpha}{d\pi} = \sigma\sqrt{|u_\lambda u^\lambda|}\, F^\alpha{}_\beta u^\beta \tag{1.9}$$

determine in a neighbourhood of initial point a unique Lorentzian world line Γ with a unique parametrization by a "preferred" parameter π. This parametrization resembles the affine parametrization of geodesics in that Eqs (1.9) are invariant under the transformation $\tilde{\pi} = A\pi + B$ of the parameter, where A and B are constant.

From a geometric point of view the preferred parametrization is distinguished by the properties of the law of transport of the tangent vector $u^\alpha = \frac{d\xi^\alpha}{d\pi}$ along Γ defined by Eqs (1.9). To see the geometric meaning of this law, let us consider two points $p = \Gamma(\pi)$ and $q = \Gamma(\pi+\Delta\pi)$ on the world line Γ which are so close to each other that the arc $\widehat{pq}$ can be considered as an arc of a geodesic line. Then up to an approximation by terms of the order $(\Delta\pi)^{-2}$ the derivative $\frac{Du^\alpha}{d\pi}(p)$ can be replaced by the differences quotient

$$\frac{Du^\alpha}{d\pi}(p) \approx \frac{u^\alpha_{\|p}(q) - u^\alpha(p)}{\Delta\pi}, \tag{1.10}$$

where $u^\alpha_{\|p}(q)$ is the vector $u^\alpha(q)$ parallelly transported to the point p. Substituting (1.10) into (1.9) and performing a parallel transport of all the vectors from p to q along $\widehat{pq}$, we obtain

$$u^\alpha(q) = (\delta^\alpha_\beta + \sigma\sqrt{|u_\lambda(q)u^\lambda(q)|}\,\Delta\pi F^\alpha_\beta(q))u^\beta_{\|q}(p), \tag{1.11}$$

where $u^\alpha_{\|q}(p)$ is the result of the parallel transport of $u^\alpha(p)$ to the point q. Thus, the vector tangent to Γ at q is a result of superposition of the parallel transport of the tangent vector from p to q and of an infinitesimal Lorentz transformation determined by the electromagnetic field at q. In principle, this statement is a relativistic generalization of Larmor's theorem (cf. [10]), since in the particular case of a purely magnetic field the infinitesimal transformation in (1.11) reduces to an infinitesimal spatial rotation. Let us observe that due to (1.9) the term $\sqrt{|u_\lambda u^\lambda|}\Delta\pi$ in the infinitesimal part of the Lorentz transformation is parameter independent. Moreover, the transformation (1.11) is an infinitesimal isometry, whereas in the general case of an arbitrary parameter τ an analogous transport based on (1.7) would instead be a conformal transformation. It is just the appearance of isometry that geometrically distinguishes the preferred parametrization both for $\sigma \neq 0$ as well as for $\sigma = 0$.

The correspondence between the initial data and the classes of equivalence of solution ξ^α of (1.9) is, however, still not a one-to-one, because two different sets of initial data might lead to equivalent solutions of (1.9), i.e. might render two different descriptions, characterized by two different preferred parametrizations, of the same Lorentzian world line Γ. This fact can be restated in the form of a proposition.

PROPOSITION 1.2. Two sets of initial data, $\{\xi_0^\alpha, u_0^\alpha\}$ for $\pi = \pi_0$ and $\{\xi_0^\alpha, \tilde{u}_0^\alpha\}$ for $\pi = \tilde{\pi}_0$, will lead to two equivalent solutions ξ^α and $\tilde{\xi}^\alpha = \xi^\alpha \circ f$ ($f' \neq 0$) of Eqs. (1.9) iff

$$\text{a) } \tilde{u}_0^\alpha = k u_0^\alpha, \quad k = \text{const} \neq 0; \qquad \text{b) } f(\pi) = k(\pi - \tilde{\pi}_0) + \pi_0. \tag{1.12}$$

The proof is obvious because Eq. (1.8) is in accordance with Eqs (1.9) iff $f'(\pi) = \text{const}$.

Thus, Eqs (1.12a) for an arbitrary $k \neq 0$ define an equivalence relation between sets of initial data. The freedom of choice of these data may be limited by putting on them a constraint condition which selects only one member from each class of equivalence and which should be added to the differential equations (1.9) in order to assure a one-to-one correspondence between the Lorentzian world line Γ parametrized by a preferred parameter and the sets of initial data (1.2). For example, a universal choice of this kind is

$$|\overset{\circ}{g}_{\alpha\beta} u_0^\alpha u_0^\beta| = 1, \tag{1.13}$$

where $\overset{\circ}{g}_{\alpha\beta} = g_{\alpha\beta}(\xi^\gamma(\pi_0))$. If the quantity $|\overset{\circ}{g}_{\alpha\beta} u_0^\alpha u_0^\beta| > 0$ in a set of initial data is not exactly equal to one, i.e. does not satisfy (1.13), it always can be brought to a correct form by a transformation of the type (1.12).

As a final observation, we remark that Eqs (1.9) satisfy the following property:

PROPOSITION 1.3. In an arbitrary pseudo-Riemannian manifold V_n, Eqs (1.9) admit the first integral

$$g_{\alpha\beta} u^\alpha u^\beta = C_1 = \text{const}. \tag{1.14}$$

The proof is obvious. Clearly, Eq. (1.14) is a weak identity.

The existence of the first integral (1.14) assures that the condition (1.13) is satisfied not only for the initial value of the preferred parameter, but also for all the other values from its admissible domain. In other words, condition (1.13) implies that

$$|g_{\alpha\beta} u^\alpha u^\beta| = 1 \tag{1.15}$$

anywhere along Γ. The condition (1.13) is thus not only assuring a one-to-one correspondence between initial data and the Lorentzian world lines Γ being the solution of (1.9), but also introducing a universal, natural parametrization along all such lines. In terms of the natural parameter s, the Lorentz equations accept the customary form

$$\frac{Du^\alpha}{ds} = \sigma F^\alpha{}_\beta u^\beta. \tag{1.16}$$

It should however be emphasized that for the reasons already discussed in this Section, the differential equations (1.16) must be always considered in conjunction with the constraint relation (1.13) or, equivalently, with (1.15), since otherwise they would also admit solutions parametrized by preferred parameters different from s. The natural parametrization in terms of s is obviously induced by the metric structure of the manifold. In general, any condition of the type (1.14) fixes the unit of the preferred parameter scale. It still leaves, as is customary and convenient, the freedom of choice of the origin π_0 of this scale.

2. *The general first e.m. deviation*

Similarly like in the case of the geodesic deviation (cf. [1]), it is possible to formulate two different approaches to the e.m. deviation.

(*i*) *The Σ-approach.* Let us consider a one-parametric family Σ of Lorentzian curves each of which satisfies Eqs (1.1) in the region $\Omega \subset V_n$. Let each member Γ_ε of the family Σ be labelled by a value $\varepsilon \in [c, d] = J \subset R$ and the points on Γ_ε be parametrized by $\tau \in [a, b] = I \subset R$. Thus, in a coordinate system $\{x^\alpha\}$, the coordinates of points belonging to any Lorentzian curve of the family Σ are defined as $\xi^\alpha(\tau, \varepsilon) := x^\alpha \circ \Gamma_\varepsilon(\tau)$. It is assumed that these n functions $\xi^\alpha : R^2 \to R$ are at least of class C_2.

The set of points

$$\Sigma := \{p \in V_n | x^\alpha(p) = \xi^\alpha(\tau, \varepsilon); \quad (\tau, \varepsilon) \in I \times J\}$$

forms a two-cube in V_n. To any pair (τ, ε) one assigns two vectors, u^α and r^α, from the tangent space of V_n at a point p:

$$u^\alpha(\tau, \varepsilon) := \frac{\partial \xi^\alpha}{\partial \tau}(\tau, \varepsilon), \qquad r^\alpha(\tau, \varepsilon) := \frac{\partial \xi^\alpha}{\partial \varepsilon}(\tau, \varepsilon). \tag{2.1}$$

These vector valued functions u^α and r^α will be called here vector fields on Σ parametrized by (τ, ε). Since the Lorentzian world lines may intersect, they need not be a restriction to Σ of any vector field in V_n. There is obviously the following relation valid[1]

$$\frac{Du^\alpha}{\partial \varepsilon} = \frac{Dr^\alpha}{\partial \tau}. \tag{2.2}$$

As is known, any vector field t^α on Σ satisfies the Ricci identity

$$\frac{D^2 t^\alpha}{\partial \varepsilon \partial \tau} - \frac{D^2 t^\alpha}{\partial \tau \partial \varepsilon} = R^\alpha{}_{\beta\gamma\delta} t^\beta r^\gamma u^\delta. \tag{2.3}$$

Let us choose $t^\alpha = \dfrac{u^\alpha}{\sqrt{|u_\lambda u^\lambda|}}$. Then, making use of Eqs (1.1), we get from (2.3) the identity

$$\frac{D}{d\tau}\left(\frac{h^\alpha{}_\beta}{\sqrt{|u_\lambda u^\lambda|}} \frac{Dr^\beta}{d\tau}\right) + \frac{1}{\sqrt{|u_\lambda u^\lambda|}} R^\alpha{}_{\beta\gamma\delta} u^\beta r^\gamma u^\delta = \sigma\left(F^\alpha{}_{\beta;\gamma} u^\beta r^\gamma + F^\alpha{}_\beta \frac{Dr^\beta}{d\tau}\right), \tag{2.4}$$

---

[1] This relation, equivalent to $\pounds_u r^\alpha = 0$, is of course a consequence of the holonomic definitions (2.1). However, also the case of a nonholonomic vector field r^α given on Σ, for which $\pounds_u r^\alpha \neq 0$, can easily be reduced to the holonomic one as a result of an appropriate reparametrization $\tau \mapsto \tilde{\tau}(\tau, \varepsilon)$ along every of the Lorentzian world lines. Thus, within the approach based on arbitrary parametrization, we need not distinguish between the holonomic and nonholonomic cases, contrary to that what is sometimes done in the literature (cf. e.g. [7, 8]).

which after setting $\sigma = 0$ coincides with Eqs (2.5) quoted in [1]. Using the n-dimensional Maxwell equations $F_{[\alpha\beta;\gamma]} = 0$, it can also be rewritten in the form

$$L_1[r^\alpha] := \frac{D}{d\tau}\left(\frac{h^\alpha{}_\beta}{\sqrt{|u_\lambda u^\lambda|}}\frac{Dr^\beta}{d\tau} - \sigma F^\alpha{}_\beta r^\beta\right)$$

$$+\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left(R^\alpha{}_{\beta\gamma\delta} + \frac{k\sigma}{\sqrt{|u_\lambda u^\lambda|}} F_{\beta\gamma}{}^{;\alpha} u_\delta\right) u^\beta r^\gamma u^\delta = 0,$$

where the positive sign of $k = \pm 1$ corresponds to timelike Lorentzian world lines.

Obviously, Eqs (2.4) can, in principle, be written for a fixed value of ε, say $\varepsilon = 0$, e.g. along the Lorentzian world line Γ_0. If we repeat the procedure analogous to that above, but immersing now Γ_0 in another one-parametric family $\tilde{\Sigma}$ of Lorentzian world lines, with another field $\tilde{r}^\alpha(\tau, \varepsilon)$, we will still get the same Eqs (2.4) for $\tilde{r}^\alpha(\tau, 0)$ along Γ_0. Therefore, it is possible to formulate the following definition.

DEFINITION 2.1. Two one-parametric families Σ and $\tilde{\Sigma}$ of Lorentzian world lines which contain the same curve Γ_0 are equivalent iff $r^\alpha(\tau, 0) = \tilde{r}^\alpha(\tau, 0)$. The corresponding class of equivalence is called the first e.m. deviation vector field along Γ_0.

It is also possible to base the definition of the e.m. deviation directly on Eqs (2.4) without any immediate appeal to the family Σ of curves Γ. This second possibility we shall call the single line approach.

(ii) The single line approach. Let us assume that a single parametrized Lorentzian world line Γ is given. We define along Γ a vector field r (determined at $p(\tau) \in \Gamma$ by $r^\alpha(\tau)$) as a solution of the differential equations (2.4) fulfilling the initial conditions

$$r^\alpha(\tau_0) = r_0^\alpha, \quad \frac{Dr^\alpha}{d\tau}(\tau_0) = v_0^\alpha. \tag{2.5}$$

It is assumed here that the quantities $g_{\alpha\beta}$, $\Gamma^\alpha{}_{\beta\gamma}$, $R^\alpha{}_{\beta\gamma\delta}$, u^α, $F^\alpha{}_\beta$ and $F^\alpha{}_{\beta;\gamma}$ enter Eqs (2.4) being evaluated at the point $p(\tau)$ and therefore are given functions of τ.

DEFINITION 2.2. Any solution $r^\alpha(\tau)$ of Eqs (2.4), with the initial conditions (2.5), defined along a single Lorentzian world line Γ is called the first e.m. deviation vector field along Γ.

This second definition is, of course, more general than Def. 2.1.

It may be checked by a straightforward calculation that $u_\alpha L_1[r^\alpha] \equiv 0$ is a strong identity, i.e. is valid for any ξ^α and r^α. Therefore, the n equations (2.4) are not independent and their solution admits the freedom of introducing an arbitrary function. This can be restated as in the next two propositions.

PROPOSITION 2.1. If a set of n functions $r^\alpha : I \to R$ is a solution of Eqs (2.4) taken along a Lorentzian world line Γ, then $r^\alpha \circ f$, for any $f \in C_2$ and $f' \neq 0$, are also a solution of (2.4) along the same Γ, but now parametrized by $f(\tau)$.

The proof follows from inspection.

The necessity of reparametrization of r^α stated above is rather a consequence of the properties of Eqs (1.1) and only indirectly of Eqs (2.4). The underdeterminacy of Eqs

(2.4) for a fixed description of the basic world line Γ is manifested instea dby another property of the first e.m. deviation vector.

PROPOSITION 2.2. If a set of n functions $r^\alpha : I \to R$ is a solution of Eqs (2.4) taken along a Lorentzian world line Γ described by functions ξ^α, then

(*i*) the functions

$$\tilde{r}^\alpha = r^\alpha + \kappa(\tau) u^\alpha, \tag{2.6}$$

where $\kappa : I \to R$ is an arbitrary C_2 function, are also a solution of (2.4) along the same curve Γ with the same parametrization;

(*ii*) any solution $\tilde{r}^\alpha$ of Eqs (2.4), which satisfies the same initial conditions (2.5) as r^α, can be represented in the form (2.6), where $\kappa \in C_2$ is uniquely determined by these two solutions $\tilde{r}^\alpha$, r^α and the conditions $\kappa(\tau_0) = \kappa'(\tau_0) = 0$.

Proof. The proof of part (*i*) follows from inspection. The proof of part (*ii*) requires the next lemma.

LEMMA 2.1. A solution of Eqs (2.4) which satisfies (2.5) is completely specified by a choice of a continuous function $\mu : I \to R$.

Proof. A given solution r^α (with known ξ^α) defines the function

$$\mu(\tau) := \frac{d}{d\tau}\left(\frac{u_\alpha}{u_\lambda u^\lambda} \frac{Dr^\alpha}{d\tau}\right). \tag{2.7}$$

Inversely, for a given function $\mu : I \to R$, the vector r^α fulfilling (2.5) as initial conditions is defined as the unique solution of the system of differential equations

$$\frac{D^2 r^\alpha}{d\tau^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta r^\gamma u^\delta = \lambda(\tau) \frac{Dr^\alpha}{d\tau} + \mu(\tau) u^\alpha$$

$$+ \sigma \sqrt{|u_\lambda u^\lambda|} \left[F^\alpha{}_{\beta;\gamma} u^\beta r^\gamma + F^\alpha{}_\beta \frac{Dr^\beta}{d\tau} + F^\alpha{}_\beta u^\beta \left(\frac{u_\gamma}{u_\lambda u^\lambda} \frac{Dr^\gamma}{d\tau}\right)\right], \tag{2.8}$$

where the function λ is given by (1.6). For such r^α, Eq. (2.7) is satisfied as a weak identity and therefore r^α solves also (2.4), and this completes the proof of the lemma.

COROLLARY 2.1. There exists a one-to-one map G of the set of all solutions of Eqs (2.4), corresponding to some given initial data in (2.5), onto the set of all continuous functions $\mu : I \to R$. The map is defined by (2.7) as $r^\alpha(\tau) \to G(r^\alpha(\tau)) = \mu(\tau)$, and its inverse is determined by the solution of the initial value problem (2.5) for the differential equations (2.8).

Let r^α and $\tilde{r}^\alpha$ be two solutions of Eqs (2.4) such that $G(r^\alpha) = \mu$ and $G(\tilde{r}^\alpha) = \tilde{\mu}$, then κ in (2.6) is defined (because of (2.7)) as the unique solution of the differential equation

$$\tilde{\mu}(\tau) = \mu(\tau) + \frac{d}{d\tau}\left(\frac{d\kappa}{d\tau}(\tau) + \lambda(\tau)\kappa(\tau)\right) \tag{2.9}$$

with the initial data $\kappa(\tau_0) = \kappa'(\tau_0) = 0$. This ends the proof of Prop. 2.2.

Remark 2.1. Prop. 2.1 states obviously that Eqs (2.4) are covariant under arbitrary reparametrization of Γ and of the field r^α, whereas Prop. 2.2 can be understood as a statement that Eqs (2.4) are invariant under arbitrary gauge transformations of r^α which are defined by Eqs (2.6).

Let us note that contrary to Eqs (2.4), Eqs (2.8) for a chosen function μ can be uniquely solved with respect to the derivatives of the highest order and admit therefore a well-posed initial value problem. Thus for each choice of μ, Eqs (2.8) determine a law of transport of r^α along Γ.

The geometrical interpretation of the fact stated in Prop. 2.2 follows at once from the Σ-approach leading to the relation (2.4) considered as an identity. Because of the freedom of reparametrization $\tau \mapsto f(\tau, \varepsilon)$, where $f\colon R^2 \to R$ is a C_2 function, a different (in general) parametrization may be introduced on each Lorentzian world line Γ_ε. In particular, one can require that the parametrization along the basic curve $\Gamma_0(\varepsilon = 0)$ remains unchanged, but changes on those with $\varepsilon \neq 0$ so that

$$f(\tau, 0) = \tau, \qquad \frac{df}{d\tau}(\tau, 0) = \kappa(\tau), \tag{2.10}$$

where κ is given. Then, defining $r^\alpha(\tau, \varepsilon)$ as in (2.1) and $\tilde{r}^\alpha(\tau, \varepsilon)$ as

$$\tilde{r}^\alpha(\tau, \varepsilon) := \frac{\partial \xi^\alpha}{\partial \varepsilon}(f(\tau, \varepsilon), \varepsilon) \tag{2.11}$$

one easily derives that $r^\alpha(\tau, 0)$ and $\tilde{r}^\alpha(\tau, 0)$, i.e. both on Γ_0, satisfy the relation (2.6). The multiplicity of solutions of Eqs (2.4), described in Prop. 2.2, can therefore be interpreted as a possibility of introducing new parametrization on neighbouring Lorentzian world lines, while keeping fixed the parametrization on the basic line Γ_0.

3. *The natural first e.m. deviation*

Let us return to the approach with a single Lorentzian world line Γ. Similarly like in Sect. 1, we have again on one side a system of differential equations (2.4) with a multiplicity of solutions described by (2.6), and on the other, a family of systems of differential equations (2.8), each member of which is labelled by a function μ. If μ is given[2], Eqs (2.8) determine a unique solution r^α that satisfies the initial conditions (2.5). Such a solution is simultaneously a solution of Eqs (2.4). All these solutions form an equivalence class of the relation defined by (2.6). Every equivalence class is distinguished from the others by the values of the initial data $\{r_0^\alpha, v_0^\alpha\}$ in (2.5) and from a geometric point of view it describes a Lorentzian world line in the "first neighbourhood" of the basic line Γ_0. (A rigorous geometric inter-

---

[2] In Eqs (2.8), like in all the other equations in Sects 3–5 it is assumed that λ is fixed and imposed by the accepted and fixed parametrization along Γ which is described by Eqs (1.7) whose solution satisfies the weak identity (1.6). In particular, all statements of Sects 3–5 can easily be rephrased for $\lambda = 0$, i.e. for the case when along Γ we have a preferred or even, after adding the constraint condition (1.15), the natural parametrization.

pretation in terms of a covariant Taylor theorem [2] could be repeated here along similar lines as it was done in [1].) For any practical purpose, we need only one first e.m. deviation vector to describe the equivalence class to which it belongs, and it is reasonable to choose a possibly simplest function μ in Eqs (2.8). The first obvious choice consists in taking $\mu = 0$ for any $\tau \in I$. The corresponding Eqs (2.8) take then the form

$$\frac{D^2 r^\alpha}{d\tau^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta r^\gamma u^\delta = \lambda(\tau) \frac{Dr^\alpha}{d\tau} + \sigma \sqrt{|u_\lambda u^\lambda|} \Bigg[F^\alpha{}_{\beta;\gamma} u^\beta r^\gamma + F^\alpha{}_\beta \frac{Dr^\beta}{d\tau} + F^\alpha{}_\beta u^\beta \left(\frac{u_\gamma}{u_\lambda u^\lambda} \frac{Dr^\gamma}{d\tau} \right) \Bigg]. \tag{3.1}$$

For each set of initial data in (2.5) Eqs (3.1) have a unique solution. However two different sets of initial data might still lead to equivalent solutions, as it follows from the next proposition.

PROPOSITION 3.1. Two sets, $\{r_0^\alpha, v_0^\alpha\}$ and $\{\tilde{r}_0^\alpha, \tilde{v}_0^\alpha\}$, of initial data in (2.5) will render two solutions r^α and $\tilde{r}^\alpha$ of (3.1) equivalent in the sense of (2.6) iff

$$\text{a)}\ \tilde{r}_0^\alpha = r_0^\alpha + \frac{a u_0^\alpha}{\sqrt{|\mathring{g}_{\beta\gamma} u_0^\beta u_0^\gamma|}}, \qquad \tilde{v}_0^\alpha = v_0^\alpha + (b\delta_\beta^\alpha + a\sigma \mathring{F}^\alpha{}_\beta) u_0^\beta;$$

$$\text{b)}\ \kappa(\tau) = \frac{1}{\sqrt{|u_\lambda u^\lambda|}} \left(b \int_{\tau_0}^{\tau} \sqrt{|u_\lambda u^\lambda|}\, d\tau + a \right), \tag{3.2}$$

where a and b are arbitrary constants, $\mathring{g}_{\alpha\beta} = g_{\alpha\beta}(\xi^\gamma(\tau_0))$, $\mathring{F}_{\alpha\beta} = F_{\alpha\beta}(\xi^\gamma(\tau_0))$ and u_0^α are the initial data for Γ fulfilling (1.7).

The proof is straightforward and will be omitted.

Thus, Eqs (3.2a) establish an equivalence relation of initial data for (3.1). A single representative from each class of equivalence can be determined by choosing some definite values of the constants a and b in (3.2). These conditions may be determined, for instance, by means of the following proposition.

PROPOSITION 3.2. In an arbitrary pseudo-Riemannian manifold V_n the system of equations (1.7) and (3.1) admits the following first integral

$$\frac{u_\alpha}{u_\lambda u^\lambda} \frac{Dr^\alpha}{d\tau} = C_2 = \text{const.} \tag{3.3}$$

The proof is a result of contracting Eqs (3.1) with u_α, making use of Eqs (1.7) and of the weak identity (1.6). Clearly, Eq. (3.3) is also a weak identity.

The requirement $\mu = 0$ on Γ_0 has a simple geometric interpretation which can be found by means of the Σ-approach. Each of the Lorentzian world lines Γ_ε from the family Σ can be parametrized by a different parameter τ_ε specified by a regular function $C_1(\tau, \varepsilon)$ in the form

$$g_{\alpha\beta} u^\alpha(\tau, \varepsilon) u^\beta(\tau, \varepsilon) = C_1(\tau, \varepsilon). \tag{3.4}$$

Differentiating this with respect to ε and taking into account (2.2), one obtains

$$\frac{u_\alpha}{u_\lambda u^\lambda}\frac{Dr^\alpha}{d\tau} = \frac{\partial}{\partial\varepsilon}(\ln\sqrt{|C_1(\tau,\varepsilon)|}). \tag{3.5}$$

Let us observe that in the Σ-approach Eq. (1.6) along a fixed line Γ_ε reads as

$$\lambda(\tau,\varepsilon) = \frac{\partial}{\partial\tau}(\ln\sqrt{|u_\lambda(\tau,\varepsilon)u^\lambda(\tau,\varepsilon)|}).$$

Differentiating this with respect to ε, due to Eq. (2.2) we obtain

$$\frac{\partial\lambda}{\partial\varepsilon}(\tau,\varepsilon) = \frac{\partial}{\partial\tau}\left(\frac{u_\alpha}{u_\lambda u^\lambda}\frac{Dr^\alpha}{d\tau}\right) = \mu(\tau,\varepsilon). \tag{3.6}$$

Assuming temporarily that $\mu(\tau,\varepsilon) = \mu(\varepsilon)$, we can integrate the last equation in (3.6)

$$\frac{u_\alpha}{u_\lambda u^\lambda}\frac{Dr^\alpha}{d\tau} = \tau\mu(\varepsilon)+C_2(\varepsilon), \tag{3.7}$$

and obtain from here and from Eq. (3.5) a differential equation on $C_1(\tau,\varepsilon)$:

$$\frac{\partial}{\partial\varepsilon}(\ln\sqrt{|C_1(\tau,\varepsilon)|}) = \tau\mu(\varepsilon)+C_2(\varepsilon), \tag{3.8}$$

which gives

$$C_1(\tau,\varepsilon) = C_1(\tau,0)\exp\{2\int_0^\varepsilon[\tau\mu(\varepsilon)+C_2(\varepsilon)]d\varepsilon\}, \tag{3.9}$$

where the function $C_1(\tau, 0)$ is determined by the choice of some parametrization along the basic world line Γ_0. Thus, the requirement $\mu = \mu(\varepsilon)$ means that all the Lorentzian world lines from the family Σ are parametrized by different parameters τ_ε specified by the function (3.9). Let us observe that the first integral (3.3) exists along Γ_0 only for functions $\mu(\varepsilon)$ such that $\mu(0) = 0$. Then C_2 in (3.3) is equal to $C_2(0)$ in (3.7). So, the requirement $\mu = 0$ in the Σ-approach can be interpreted as a restriction $C_2(0) = C_2$ imposed on the arbitrary function $C_2(\varepsilon)$ in (3.9).

Passing in Eq. (3.3) from a solution r^α to $\tilde{r}^\alpha$ given by (2.6), with κ defined by (3.2a), results in adding b to the constant C_2 in this equation. To fix b, it is therefore sufficient to fix the value of C_2. The simplest choice is to require that

$$u_\alpha\frac{Dr^\alpha}{d\tau} = 0, \tag{3.10}$$

and because of Prop. 3.2 it is sufficient to impose this constraint on the initial data only. As a consequence of (3.10), Eqs. (3.1) take form

$$\frac{D^2r^\alpha}{d\tau^2}+R^\alpha{}_{\beta\gamma\delta}u^\beta r^\gamma u^\delta = \lambda(\tau)\frac{Dr^\alpha}{d\tau}+\sigma\sqrt{|u_\lambda u^\lambda|}\left(F^\alpha{}_{\beta;\gamma}u^\beta r^\gamma+F^\alpha{}_\beta\frac{Dr^\beta}{d\tau}\right), \tag{3.11}$$

and it is a simple task to prove that (3.3) is a first integral of the system of Eqs. (1.7) and (3.11), provided (1.6) is taken into account.

The condition (3.10), because of $\mu(0) = 0$ and Eq. (3.7), is equivalent to $C_2(0) = C_2 = 0$. From $\mu(0) = C_2(0) = 0$ and Eq. (3.9) it follows that $\left(\frac{\partial C_1}{\partial \varepsilon}(\tau, \varepsilon)\right)_{\varepsilon=0} = 0$, which geometrically means that all the Lorentzian world lines from the "first neighbourhood" of the basic line Γ_0 are parametrized by the same parameter τ, specified for every ε by the function $C_1(\tau, \varepsilon)$. In virtue of (1.8), even if the function λ is kept fixed, i.e. $\tilde{\lambda} = \lambda$ which is equivalent to $\tilde{C}_1(\tau, \varepsilon) = C_1(\tau, \varepsilon)$, there still remains the freedom of reparametrization such that $f'(\tau) = 1$, which means that the freedom of choice of an initial value of the parameter τ along each of the world lines is still left at our disposal.

DEFINITION 3.1. A vector field r^α which is a solution of Eqs. (3.11) evaluated along a Lorentzian world line Γ parametrized by an arbitrary parameter τ and which satisfies (3.10) as a constraint condition is called the natural first e.m. deviation vector.

There remains still the freedom of choosing the value of the constant a in (3.2). In the case of the geodesic deviation it meant the freedom of fixing the scalar product of the vectors u^α and r^α, which did not depend on the parameter τ [1]. Now the situation is slightly more involved. In virtue of (1.6), (1.7) and (3.10), one can obtain

$$\frac{d}{d\tau}\left(\frac{u_\alpha r^\alpha}{\sqrt{|u_\lambda u^\lambda|}}\right) = \sigma F_{\alpha\beta} r^\alpha u^\beta$$

or

$$\frac{u_\alpha r^\alpha}{\sqrt{|u_\lambda u^\lambda|}} = \sigma \int_{\tau_0}^{\tau} F_{\alpha\beta} r^\alpha u^\beta d\tau + \text{const.} \tag{3.12}$$

Since the condition (3.10) requires that $b = 0$, therefore due to Prop. 3.1 Eqs. (3.11) admit still equivalent solutions of the form

$$\tilde{r}^\alpha = r^\alpha + \frac{a u^\alpha}{\sqrt{|u_\lambda u^\lambda|}}. \tag{3.13}$$

Hence, replacing r^α by $\tilde{r}^\alpha$ results in adding the constant a to the right hand side of (3.12), and the choice of a is therefore equivalent to specifying the value of the integration constant in (3.12).

Remark 3.1. Let us note that Eq. (3.12) can also be represented as a first integral of the system of equations (1.7) and (3.11), but only if an appropriate gauge condition for the e.m. 4-potential A_α is accepted. This follows from the identity

$$F_{\alpha\beta} r^\alpha u^\beta = u^\alpha \underset{r}{\pounds} A_\alpha - \frac{d}{d\tau}(A_\alpha r^\alpha)$$

which can easily be checked directly by making use of the definition $F_{\alpha\beta} := A_{\beta;\alpha} - A_{\alpha;\beta}$ and of the definition of the Lie derivative. Accepting now the gauge condition

$$u^\alpha \underset{r}{\pounds} A_\alpha = 0,$$

we obtain from (3.12) the relation

$$r_\alpha \left(\frac{u^\alpha}{\sqrt{|u_\lambda u^\lambda|}} + \sigma A^\alpha \right) = \text{const}$$

which is a first integral of Eqs. (1.7) and (3.11) under the conditions mentioned above.

4. *The first separation vector*

In the case of the geodesic deviation (i.e. for $\sigma = 0$) the constraint condition (3.10), which is used to determine the natural deviation, implies at once $u_\alpha r^\alpha = \text{const}$. In other words, in that case the natural deviation, which can be interpreted as an infinitesimal, parametrization preserving mapping of a geodesic line Γ_0 onto such another line from the "first neighbourhood" of Γ_0, satisfies the conservation condition $u_\alpha r^\alpha = \text{const}$. The inverse implication is also true: any solution of the equations corresponding to (3.1) in the case $\sigma = 0$, which satisfies the constraint condition $u_\alpha r^\alpha = \text{const}$, must necessarily be a natural geodesic deviation vector. In the general case, the natural first e.m. deviation vector, which is still a parametrization preserving mapping onto neighbouring Lorentzian world lines, changes during its evolution along the basic Lorentzian world line Γ_0 its inclination $u_\alpha r^\alpha$ with Γ_0. The general equations (2.8) are however flexible enough and permit of such a transformation of them, being the result of a suitable choice of the function μ, after which they uniquely determine an e.m. deviation vector that preserves its inclination $u_\alpha r^\alpha$ with the basic line Γ_0. In certain applications, some of which will be discussed in another paper of ours, now in preparation, it is just a first e.m. deviation vector of this second kind that is needed. It is therefore worthwhile to study also this second case in some detail.

Let us observe that Prop. 3.2 is really a consequence of our choice $\mu = 0$ in the weak identity (2.7) when passing from Eqs. (2.8) to (3.1). Now the question is whether it is possible to find such a function μ that substituted into Eqs. (2.8) will give us the first e.m. deviation equations preserving the product $u_\alpha r^\alpha$, or a simple function of it. The answer follows from an immediate transformation of Eq. (2.7) to the form

$$\mu(\tau) = \frac{d}{d\tau}\left[\frac{k}{\sqrt{|u_\lambda u^\lambda|}}\,\frac{d}{d\tau}\left(\frac{u_\alpha r^\alpha}{\sqrt{|u_\lambda u^\lambda|}}\right)\right] + \frac{d}{d\tau}\left(\frac{k\sigma F_{\alpha\beta}u^\alpha r^\beta}{\sqrt{|u_\lambda u^\lambda|}}\right),$$

where use was made of (1.7) and (1.6). Thus, if one accepts that

$$\mu(\tau) := \frac{d}{d\tau}\left(\frac{k\sigma F_{\alpha\beta}u^\alpha r^\beta}{\sqrt{|u_\lambda u^\lambda|}}\right), \tag{4.1}$$

then Eqs. (2.8) transform into the form

$$\frac{D^2 r^\alpha}{d\tau^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta r^\gamma u^\delta = \lambda(\tau)\frac{Dr^\alpha}{d\tau} + u^\alpha \frac{d}{d\tau}\left(\frac{k\sigma F_{\beta\gamma} u^\beta r^\gamma}{\sqrt{|u_\lambda u^\lambda|}}\right)$$

$$+\sigma\sqrt{|u_\lambda u^\lambda|}\left[F^\alpha{}_{\beta;\gamma} u^\beta r^\gamma + F^\alpha{}_\beta \frac{Dr^\beta}{d\tau} + F^\alpha{}_\beta u^\beta \left(\frac{u_\gamma}{u_\lambda u^\lambda}\frac{Dr^\gamma}{d\tau}\right)\right]. \tag{4.2}$$

Now, the system of equations (1.7) and (4.2) admits the desired first integral.

PROPOSITION 4.1. In an arbitrary pseudo-Riemannian manifold V_n any solution of the system of equations (1.7) and (4.2), with λ given by (1.6), satisfies the relation

$$\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\frac{d}{d\tau}\left(\frac{u_\alpha r^\alpha}{\sqrt{|u_\lambda u^\lambda|}}\right) = C_3 = \text{const.} \tag{4.3}$$

For the same reason as before, for each set of initial data in (2.5), Eqs. (4.2) have a unique solution. But two different sets of initial data may still lead to solutions equivalent in the sense of (2.6), as Prop. 3.1 can be proved for the case of Eqs. (4.2). Thus, Eqs. (3.2a) establish an equivalence relation of initial data for the differential equations (4.2). To select a single representative from each class of equivalence, one has to choose some definite values of the constants a and b in (3.2). This can again be done by adding to Eqs. (4.2) a constraint condition on the initial data. Due to the existence of the conservation law, one can impose (4.3) on the initial data and this requirement will be compatible with the evolution governed by Eqs. (1.7) and (4.2).

Let us investigate the result of the transformation (3.2a) on a constraint condition of the form (4.3). The transformation (2.6) from a solution r^α to $\tilde{r}^\alpha$ with κ defined by (3.2b) results in adding b to the constant C_3 in (4.3). Therefore to fix b, it is again sufficient to fix the value of this constant. The simplest choice is to require that the constant in (4.3) vanishes which results in

$$\frac{u_\alpha r^\alpha}{\sqrt{|u_\lambda u^\lambda|}} = \text{const.} \tag{4.4}$$

Because of Prop. 4.1 it is sufficient to impose this constraint on the initial data only. Since the condition (4.4) requires that $b = 0$, due to Prop. 3.1, Eqs. (4.2) still admit equivalent solutions of the form (3.13). Hence, replacing r^α by $\tilde{r}^\alpha$ results in adding a to the constant in (4.4). The choice of the constant a is therefore equivalent to specifying the value of the integration constant in (4.4). It is convenient to accept

$$u_\alpha r^\alpha = 0. \tag{4.5}$$

This condition automatically implies (4.3), with the constant being zero, and fixes both a and b in (3.2). Let us however observe that Eq. (4.5) is compatible with the first integral

(4.3) iff the relative velocity $\frac{Dr^\alpha}{d\tau}$ satisfies the condition

$$\frac{u_\alpha}{\sqrt{|u_\lambda u^\lambda|}} \frac{Dr^\alpha}{d\tau} = \sigma F_{\alpha\beta} u^\alpha r^\beta. \tag{4.6}$$

Such requirements have an obvious geometric interpretation and justify the introduction of the following definition.

DEFINITION 4.1. A vector field r^α which is a solution of Eqs. (4.2) evaluated along a Lorentzian world line Γ parametrized by an arbitrary parameter τ and which satisfies (4.5) and (4.6) as constraint conditions is called the first e.m. separation vector.

It should be noted that the e.m. separation vector, like any deviation vector defined by Eqs. (2.8), is a mapping of the basic Lorentzian world line Γ_0 onto such another line in its infinitesimal neighbourhood. This mapping does not however preserve the parametrization of the world line, as it follows from a discussion analogous to that preceding Eq. (3.9). But in the special case of the geodesic deviation (i.e. for $\sigma = 0$), the two sets of equations Eqs. (3.11) and Eqs. (4.2), are identical, and the two kinds of deviation vectors merge a single concept of the natural geodesic deviation vector.

5. *Relationship between the two kinds of first e.m. deviation*

The whole scheme developed in previous sections indicates that there must be a relation between the natural first e.m. deviation and the first e.m. separation vectors. As a corollary of Prop. 2.2 it follows that if we have two members of the family of systems of differential equations (2.8), corresponding to two functions $\tilde{\mu}$ and μ respectively, then their solutions must satisfy the relation (2.6) in which the function κ is uniquely determined by Eq. (2.9) with appropriate initial conditions. Hence, taking $\tilde{\mu} = 0$ and μ in the form (4.1), by solving Eq. (2.9) we obtain that

$$\kappa(\tau) = \left[\int_{\tau_0}^{\tau} \left(c - \frac{k\sigma F_{\alpha\beta} u^\alpha r^\beta}{\sqrt{|u_\lambda u^\lambda|}}\right) \exp\left(\int_{\tau_0}^{\tau'} \lambda(\tau'') d\tau''\right) d\tau' + d\right] \exp\left(-\int_{\tau_0}^{\tau} \lambda(\tau') d\tau'\right),$$

where c and d are integration constants. If one now takes into account the weak identity (1.6), the expression for κ can be rewritten as

$$\kappa(\tau) = \frac{1}{\sqrt{|u_\lambda u^\lambda|}} \left[\int_{\tau_0}^{\tau} (c\sqrt{|u_\lambda u^\lambda|} - k\sigma F_{\alpha\beta} u^\alpha r^\beta) d\tau + d\right]. \tag{5.1}$$

The assumptions $\tilde{\mu} = 0$ and (4.1) mean that the formula (2.6) with κ given by the expression (5.1) transforms a vector r^α which satisfies Eqs. (4.2) with a constraint condition (4.3) into a vector $\tilde{r}^\alpha$ being a solution of Eqs (3.1) and constrained by Eq. (3.3); the constants C_2 and C_3 in the constraint conditions are arbitrary, but fixed for the solutions considered.

Let us meanwhile assume that $\{\tilde{r}_0^\alpha, \tilde{v}_0^\alpha\}$ are the data for $\tilde{r}^\alpha$ in the initial value problem (2.5) that are constrained by (3.3) and $\{r_0^\alpha, v_0^\alpha\}$ are those for r^α restricted by (4.3). Making use of the Lorentz equations (1.1) in Eqs. (4.3), we can eliminate the term $\sigma E_{\alpha\beta}u^\alpha r^\beta$ from (5.1) and write the relation (2.6) with κ given by (5.1) in the form

$$\tilde{r}^\alpha(\tau) = r^\alpha(\tau) + \frac{u^\alpha}{\sqrt{|u_\lambda u^\lambda|}}(\tau)\left\{\int_{\tau_0}^{\tau}\left[(c+kC_3)\sqrt{|u_\lambda u^\lambda|} - \frac{ku_\beta}{\sqrt{|u_\lambda u^\lambda|}}\frac{Dr^\beta}{d\tau}\right]d\tau + d\right\}. \tag{5.2}$$

Substituting here $\tau = \tau_0$, we obtain

$$\tilde{r}_0^\alpha = r_0^\alpha + \frac{du_0^\alpha}{\sqrt{|\mathring{g}_{\beta\gamma}u_0^\beta u_0^\gamma|}} \tag{5.3}$$

and

$$d = \frac{k\mathring{g}_{\alpha\beta}u_0^\alpha(\tilde{r}_0^\beta - r_0^\beta)}{\sqrt{|\mathring{g}_{\lambda\nu}u_0^\lambda u_0^\nu|}}.$$

The relation (5.3) then reads

$$\mathring{h}^\alpha{}_\beta(\tilde{r}_0^\beta - r_0^\beta) = 0, \tag{5.4}$$

where $\mathring{h}^\alpha{}_\beta = h^\alpha{}_\beta(\xi^\gamma(\tau_0))$. Differentiating (5.2) and taking $\tau = \tau_0$ gives

$$\tilde{v}_0^\alpha = \mathring{h}^\alpha{}_\beta v^\beta + d\sigma\mathring{F}^\alpha{}_\beta u_0^\beta + (c+kC_3)u_0^\alpha. \tag{5.5}$$

From (5.5) and (3.3) it follows that

$$c = C_2 - kC_3. \tag{5.6}$$

Eqs. (5.3) and (5.5), with (5.6), define a transformation $\{r_0^\alpha, v_0^\alpha\} \mapsto \{\tilde{r}_0^\alpha, \tilde{v}_0^\alpha\}$ of initial data that must accompany the transformation (2.6) with κ given by (5.1). The transformation of initial data contains one arbitrary parameter d, while C_2 and C_3 are fixed. Observing that due to (2.6) $F_{\alpha\beta}u^\alpha r^\beta = F_{\alpha\beta}u^\alpha\tilde{r}^\beta$, and making use of (4.3) and (4.6), one finds the inverse transformation to (5.5):

$$v_0^\alpha = \tilde{v}_0^\alpha - d\sigma\mathring{F}^\alpha{}_\beta u_0^\beta + u_0^\alpha\left[(kC_3 - C_2) + \frac{k\sigma\mathring{F}_{\beta\gamma}u_0^\beta\tilde{r}_0^\gamma}{\sqrt{|\mathring{g}_{\lambda\nu}u_0^\lambda u_0^\nu|}}\right]. \tag{5.7}$$

The foregoing consideration can be summarized in a form of a proposition.

PROPOSITION 5.1. A solution r^α of Eqs. (4.2) along a given Lorentzian world line Γ, satisfying the constraint condition (5.3) with C_3 being fixed, can be transformed into the vector

$$\tilde{r}^\alpha = r^\alpha + \frac{u^\alpha}{\sqrt{|u_\lambda u^\lambda|}}\left\{\int_{\tau_0}^{\tau}[(C_2 - kC_3)\sqrt{|u_\lambda u^\lambda|} - k\sigma F_{\beta\gamma}u^\beta r^\gamma]d\tau + d\right\} \tag{5.8}$$

being a solution of Eqs. (3.1) along the same line and satisfying the constraint condition (3.3) with a fixed value of C_2; d in (5.8) is a constant. The inverse transformation $\tilde{r}^\alpha \mapsto r^\alpha$

can be obtained from replacing $F_{\alpha\beta}u^\alpha r^\beta$ in the integrand by $F_{\alpha\beta}u^\alpha \tilde{r}^\beta$. The initial data for one of the vectors determine the data for the other by means of Eqs (5.3), (5.5) and (5.7).

COROLLARY 5.1. If $\tilde{r}^\alpha$ is a natural first e.m. deviation and r^α a first e.m. separation vector field along a given Lorentzian world line Γ, all the equations (5.1)–(5.8) remain valid after substituting into them $c = C_2 = C_3 = r^\alpha u_\alpha = \mathring{g}_{\alpha\beta}u_0^\alpha r_0^\beta = 0$. Since r^α satisfies two and $\tilde{r}^\alpha$ only one constraint conditions, the transformations $\tilde{r}^\alpha \mapsto \tilde{r}^\alpha$ and $\{r_0^\alpha, v_0^\alpha\} \mapsto \{\tilde{r}_0^\alpha, \tilde{v}_0^\alpha\}$ involve necessarily one free parameter d representing an additional degree of freedom. In the inverse transformation $\tilde{r}^\alpha \mapsto r^\alpha$, one ought to substitute $d = k\mathring{g}_{\alpha\beta}u_0^\alpha \tilde{r}_0^\beta \times (|\mathring{g}_{\lambda\nu}u_0^\lambda u_0^\nu|)^{-1/2}$, and the transformation $\{\tilde{r}_0^\alpha, \tilde{v}_0^\alpha\} \mapsto \{r_0^\alpha, v_0^\alpha\}$ does not contain any free parameter, since from (5.4) in the case considered now one obtains $r_0^\alpha = \mathring{h}^\alpha{}_\beta \tilde{r}_0^\beta$.

Let us note that Eqs (2.4) are invariant under arbitrary reparametrization of the basic world line Γ_0, whereas every one of the systems of Eqs (2.8) taken separately, labelled by a fixed function μ, does not enjoy such an invariance property. However the two systems of equations, Eqs (3.11) and Eqs (4.2) respectively, are again reparametrization invariant. To see it, one ought to transform each of the two systems to a form, which is manifestly invariant.

If one eliminates the function λ from Eqs (3.11) by means of (1.6), taking into account the constraint condition (3.10) and the n-dimensional Maxwell equations $F_{[\alpha\beta;\gamma]} = 0$, one obtains the differential equations for a vector field r_n^α along an arbitrarily parametrized Lorentzian world line Γ:

$$\frac{D}{d\tau}\left(\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\frac{Dr_n^\alpha}{d\tau} - \sigma F^\alpha{}_\beta r_n^\beta\right) + \frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left(R^\alpha{}_{\beta\gamma\delta} + \frac{k\sigma}{\sqrt{|u_\lambda u^\lambda|}}F_{\beta\gamma;}{}^\alpha u_\delta\right)u^\beta r_n^\gamma u^\delta = 0, \tag{5.9}$$

which must necessarily be considered in conjunction with the constraint condition (3.10), i.e.

$$u_\alpha \frac{Dr_n^\alpha}{d\tau} = 0,$$

since otherwise their solution might have not been a natural first e.m. deviation vector, for Eqs (5.9) admit (3.3) and not (3.10) as their first integral. The manifest invariance of Eqs (5.9) and of the constraint condition is evident. Besides, these equations are solvable with respect to the highest derivatives and admit therefore a well-posed initial value problem. All this means that along a given Lorentzian world line Γ, understood as a locus of points, Eqs (5.9) together with the constraint condition (3.10) and the initial value problem (2.5) uniquely determine a natural first e.m. deviation vector field r_n^α, which therefore has a parametrization independent, geometrical meaning. In particular, when Γ is parametrized by the natural parameter s, one must substitute into (5.9) the relation (1.15), which results in the form of the first e.m. deviation equations given in [3] or in an equivalent form

$$\frac{D^2 r_n^\alpha}{ds^2} + R^\alpha{}_{\beta\gamma\delta}u^\beta r_n^\gamma u^\delta = \sigma\left(F^\alpha{}_{\beta;\gamma}u^\beta r_n^\gamma + F^\alpha{}_\beta \frac{Dr_n^\beta}{ds}\right) \tag{5.10}$$

given in [4].

A similar elimination by means of (1.6) of the function λ from Eqs (4.2) shows that the first e.m. separation vector r_s^α satisfies the equations

$$\frac{D}{d\tau}\left[\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\frac{Dr_s^\alpha}{d\tau}-\sigma\left(g^{\alpha\beta}+\frac{u^\alpha u^\beta}{u_\gamma u^\gamma}\right)F_{\beta\delta}r_s^\delta\right]$$
$$+\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left(R^\alpha{}_{\beta\gamma\delta}+\frac{k\sigma}{\sqrt{|u_\lambda u^\lambda|}}F_{\beta\gamma}{}^{;\alpha}u_\delta\right)u^\beta r_s^\gamma u^\delta=0 \tag{5.11}$$

which must be supplemented by the two constraint conditions (4.5) and (4.6), i.e.

$$u_\alpha r_s^\alpha=0,\qquad \frac{u_\alpha}{\sqrt{|u_\lambda u^\lambda|}}\frac{Dr_s^\alpha}{d\tau}=\sigma F_{\alpha\beta}u^\alpha r_s^\beta.$$

The initial value problem is here again a well-posed one and the first e.m. separation vector field r_s^α has a parametrization independent, geometrical meaning, although in general it differs from the vector field r^α. In particular, Eqs (5.11) and the constraint conditions can be easily rewritten in the case when Γ is parametrized by the natural parameter s, or in the equivalent form

$$\frac{D^2r_s^\alpha}{ds^2}+R^\alpha{}_{\beta\gamma\delta}u^\beta r_s^\gamma u^\delta=\sigma\left(F^\alpha{}_{\beta;\gamma}u^\beta r_s^\gamma+F^\alpha{}_\beta\frac{Dr_s^\beta}{ds}\right)$$
$$+\frac{D}{ds}(\sigma u^\alpha F_{\beta\gamma}u^\beta r_s^\gamma). \tag{5.12}$$

From (5.8) and (4.5) it follows immediately that the first e.m. separation vector can always be represented in the form $r_s^\alpha = h^\alpha{}_\beta r_n^\beta \equiv r_\perp^\alpha$, where r_n^α is a natural first e.m. deviation vector. Therefore, Eqs (5.12) are a particular case (for the Lorentz force) of the general nongeodesic deviation equations for $r_\perp^\alpha$ formulated in [5] and later considered in [6–8].

REFERENCES

[1] S. L. Bażański, *Ann. Inst. Henri Poincaré* **A27**, 115 (1977).

[2] S. L. Bażański, *J. Math. Phys.* **17**, 217 (1976).

[3] J. Plebański, *Acta Phys. Pol.* **28**, 141 (1965).

[4] S. Prakash, S. R. Roy, *Lett. Nuovo Cimento* **37**, 525 (1983).

[5] J. Ehlers, *Abh. Math.-Naturw. Kl. Akad. Wiss. Lit. Mainz* **1961**, 791 (1961).

[6] S. W. Hawking, G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge U. P., Cambridge 1974, p. 81.

[7] S. Manoff, *Gen. Relativ. Gravitation* **11**, 189 (1979).

[8] N. S. Swaminarayan, J. L. Safko, *J. Math. Phys.* **24**, 883 (1983).

[9] S. L. Bażański, N. N. Kostyukovich, *Dynamics of Relative Motion of Charged Test Particles in General Relativity* (in preparation).

[10] L. D. Landau, E. M. Lifshitz, *The Classical Theory of Fields*, 3rd English Ed. Pergamon Press, Oxford 1971.

[11] S. L. Bażański, N. N. Kostyukovich, *Acta Phys. Pol.* **B18** (1987) in print.

[12] S. L. Bażański, N. N. Kostyukovich *Acta Phys. Pol.* **B18** (1987) in print.