KINEMATICS OF RELATIVE MOTION OF CHARGED TEST PARTICLES IN GENERAL RELATIVITY. II. THE SECOND ELECTROMAGNETIC DEVIATION

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This is Part II of an article by the authors (Acta Phys. Pol. **B18**, 601 (1987)), on generalization of the concept of geodesic deviation to the case in which an electromagnetic field is present. Whereas in Part I the consideration was limited to the first electromagnetic (e.m.) deviation, which as was indicated is an approximation, Part II extends the construction introduced in Part I to the case of the second e.m. deviation being a successive approximation to the notion considered in the two parts. The last section contains a summary of both Part I and II of the article.

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6. *The general second e.m. deviation*

As in the case of geodesic deviation equations (cf. [1]), also a solution of Eqs (2.4)[1] describes, in accordance with its interpretation, only approximately the behaviour of a Lorentzian world line from a neighbourhood of a basic curve Γ_0 along which the equations have been evaluated. To improve this approximation one should explore the possibility of generalizing the concept of the first e.m. deviation to higher orders. Such a generalized notion of a second e.m. deviation may be introduced in close analogy with the results for the second geodesic deviation obtained in [1].

(*i*) *The Σ-approach.* Let us consider again a one-parametric family Σ of Lorentzian world lines defined by Eqs (1.1), and let us additionally suppose that for each curve Γ_ε from this family the first e.m. deviation equations (2.4) have been solved with some arbitrarily

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[1] Formulas and Sections with numbers smaller than 6 refer to Part I of this paper [13].

given initial conditions

$$r^\alpha(\tau_0, \varepsilon) = r_0^\alpha(\varepsilon), \qquad \frac{Dr^\alpha}{d\tau}(\tau_0, \varepsilon) = c_0^\alpha(\varepsilon)$$

continuously parametrized by ε. Now, if $r^\alpha = r^\alpha(\tau, \varepsilon)$ is any solution of this initial value problem, such functions r^α determine an additional to u^α vector field on the two-cube Σ, namely

$$w^\alpha(\tau, \varepsilon) := \frac{Dr^\alpha}{\partial\varepsilon}(\tau, \varepsilon). \tag{6.1}$$

To obtain for the vector field w^α equations analogous to (2.4) for r^α, let us substitute the field $t^\alpha = \frac{h^\alpha{}_\beta}{\sqrt{|u_\lambda u^\lambda|}} \frac{Dr^\beta}{d\tau}$ into the Ricci identity (2.3). After applying this identity once again under the $\frac{D}{d\tau}$-differentiation in the second term of so obtained equality and taking into account Eqs (1.1), (2.1)–(2.4), (6.1) and the symmetry properties of $R^\alpha{}_{\beta\gamma\delta}$, we obtain the identity

$$\begin{aligned}
&\frac{D}{d\tau}\left(\frac{h^\alpha{}_\beta}{\sqrt{|u_\lambda u^\lambda|}} \frac{Dw^\beta}{d\tau}\right) + \frac{1}{\sqrt{|u_\lambda u^\lambda|}} R^\alpha{}_{\beta\gamma\delta} u^\beta w^\gamma u^\delta = \frac{1}{\sqrt{|u_\lambda u^\lambda|}}\Big\{(R^\alpha{}_{\beta\gamma\delta;\varepsilon} \\
&\quad + R^\alpha{}_{\varepsilon\gamma\delta;\beta}) u^\beta u^\gamma r^\delta r^\varepsilon + 4R^\alpha{}_{\beta\gamma\delta} \frac{Dr^\beta}{d\tau} u^\gamma r^\delta + 2\frac{Dr^\alpha}{d\tau} \frac{d}{d\tau}\left(\frac{u_\beta}{u_\lambda u^\lambda} \frac{Dr^\beta}{d\tau}\right) \\
&\quad + u^\alpha \frac{d}{d\tau}\left[\frac{1}{u_\lambda u^\lambda}\left(h_{\beta\gamma} \frac{Dr^\beta}{d\tau} \frac{Dr^\gamma}{d\tau} + R_{\beta\gamma\delta\varepsilon} u^\beta r^\gamma r^\delta u^\varepsilon\right) - \left(\frac{u_\beta}{u_\lambda u^\lambda} \frac{Dr^\beta}{d\tau}\right)^2\right]\Big\} \\
&\quad + \sigma\Big\{F^\alpha{}_{\beta;\gamma\delta} u^\beta r^\gamma r^\delta + F^\alpha{}_{\beta;\gamma}\left[u^\beta w^\gamma + 2\frac{Dr^\beta}{d\tau} r^\gamma + 2u^\beta r^\gamma \left(\frac{u_\delta}{u_\lambda u^\lambda} \frac{Dr^\delta}{d\tau}\right)\right] \\
&\quad + (F^\alpha{}_\beta R^\beta{}_{\gamma\delta\varepsilon} - R^\alpha{}_{\gamma\delta\beta} F^\beta{}_\varepsilon) r^\gamma r^\delta u^\varepsilon + F^\alpha{}_\beta\left[\frac{Dw^\beta}{d\tau} + 2\frac{Dr^\beta}{d\tau}\left(\frac{u_\gamma}{u_\lambda u^\lambda} \frac{Dr^\gamma}{d\tau}\right)\right. \\
&\quad \left.+ \frac{u^\beta}{u_\lambda u^\lambda}\left(h_{\gamma\delta} \frac{Dr^\gamma}{d\tau} \frac{Dr^\delta}{d\tau} + R_{\gamma\delta\varepsilon\tau} u^\gamma r^\delta r^\varepsilon u^\tau\right)\right]\Big\}
\end{aligned} \tag{6.2}$$

which transforms to Eqs (3.2) from [1] after setting $\sigma = 0$. It can be also rewritten in the form

$$\begin{aligned}
L_2[w^\alpha] :=\ & \frac{D}{d\tau}\left\{\frac{h^\alpha{}_\beta}{\sqrt{|u_\lambda u^\lambda|}}\left[\frac{Dw^\beta}{d\tau} - 2\frac{Dr^\beta}{d\tau}\left(\frac{u_\gamma}{u_\lambda u^\lambda} \frac{Dr^\gamma}{d\tau}\right)\right]\right. \\
& - \frac{u^\alpha}{\sqrt{|u_\lambda u^\lambda|}}\left[\frac{1}{u_\lambda u^\lambda}\left(h_{\beta\gamma} \frac{Dr^\beta}{d\tau} \frac{Dr^\gamma}{d\tau} + R_{\beta\gamma\delta\varepsilon} u^\beta r^\gamma r^\delta u^\varepsilon\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{2}{\sqrt{|u_\lambda u^\lambda|}} R^\alpha{}_{\beta\gamma\delta} r^\beta u^\gamma r^\delta - \sigma(F^\alpha{}_\beta w^\beta + F^\alpha{}_{\beta;\gamma} r^\beta r^\gamma)\Big\} + \frac{1}{\sqrt{|u_\lambda u^\lambda|}} \\
&\times \left(R_{\varepsilon\beta\gamma\delta}{}^{;\alpha} + \frac{k\sigma}{\sqrt{|u_\lambda u^\lambda|}} F_{\beta\gamma;\varepsilon}{}^{;\alpha} u_\delta \right) u^\beta r^\gamma u^\delta r^\varepsilon + \frac{1}{\sqrt{|u_\lambda u^\lambda|}} \\
&\times \left(R^\alpha{}_{\beta\gamma\delta} + \frac{k\sigma}{\sqrt{|u_\lambda u^\lambda|}} F_{\beta\gamma}{}^{;\alpha} u_\delta \right) \left(u^\beta w^\gamma + 2 \frac{Dr^\beta}{d\tau} r^\gamma \right) u^\delta - \frac{2}{\sqrt{|u_\lambda u^\lambda|}} R^\alpha{}_{\beta\gamma\delta} \Bigg[r^\beta \frac{Dr^\gamma}{d\tau} \\
&+ u^\beta r^\gamma \left(\frac{u_\varepsilon}{u_\lambda u^\lambda} \frac{Dr^\varepsilon}{d\tau} \right) \Bigg] u^\delta + \sigma \left(F_{\beta\gamma}{}^{;\alpha} \frac{Dr^\gamma}{d\tau} + R^\alpha{}_{\gamma\delta\varepsilon} F^\varepsilon{}_\beta u^\gamma r^\delta \right) r^\beta = 0,
\end{aligned}
$$

where the positive sign of $k = \pm 1$ corresponds to timelike world lines.

The set of Eqs (1.1), (2.4) and (6.2) can, in particular, be written along a selected Lorentzian world line Γ_0 labelled by $\varepsilon = 0$ and for a selected solution $r^\alpha(\tau)$ of (2.4) for $\varepsilon = 0$. We can immerse Γ_0 in another family $\tilde{\Sigma}$, extend $r^\alpha(\tau)$ to a vector field $\tilde{r}^\alpha(\tau, \varepsilon)$ of solutions of (2.4) on the new $\tilde{\Sigma}$ in such a way that $r^\alpha(\tau) = \tilde{r}^\alpha(\tau, 0)$, and accept the following definition.

DEFINITION 6.1. Two one-parametric families Σ and $\tilde{\Sigma}$ of Lorentzian world lines which contain the same curve Γ_0 are equivalent iff $w^\alpha(\tau, 0) = \tilde{w}^\alpha(\tau, 0)$. The corresponding class of equivalence is called the second e.m. deviation vector field along Γ_0.

It is also possible to give another, more general definition of the second e.m. deviation directly based on Eqs (6.2).

(ii) The single line approach. Let us suppose that a single parametrized Lorentzian world line Γ is given, and that along this line Eqs (2.4) have been solved for the same initial conditions (2.5); let this solution be denoted by $r^\alpha(\tau)$ too. We can all the functions in (6.2) like u^α, r^α, $g_{\alpha\beta}$, $F_{\alpha\beta}$, etc., evaluate along Γ for this selected solution $r^\alpha(\tau)$ and turn Eqs (6.2) into ordinary differential equations of the second order for $w^\alpha(\tau)$ fulfilling the initial conditions

$$
w^\alpha(\tau_0) = w_0^\alpha, \qquad \frac{Dw^\alpha}{d\tau}(\tau_0) = t_0^\alpha. \tag{6.3}
$$

DEFINITION 6.2. Any solution $w^\alpha(\tau)$ of Eqs (6.2), with initial conditions (6.3), defined along a single Lorentzian world line Γ is called the second e.m. deviation vector field along Γ.

It can simply be checked that $u_\alpha L_2[w^\alpha] \equiv 0$ is again a strong identity, i.e. is valid for any ξ^α and w^α. Therefore, the n differential equations (6.2) are not independent and the next two propositions take place.

PROPOSITION 6.1. If a set of n functions $w^\alpha : I \to R$ is a solution of Eqs (6.2) taken along a Lorentzian world line Γ, described in a coordinate system $\{x^\alpha\}$ by functions $\xi^\alpha : I \to R$, and for a solution of Eqs (2.4) described by functions $r^\alpha : I \to R$, then

(*i*) the composite functions $w^\alpha \circ f$, for any C_2 function f such that $f' \neq 0$, are also a solution of (6.2) along the same Lorentzian world line Γ described now by $\tilde{\xi}^\alpha = \xi^\alpha \circ f$ and for a solution of (2.4) determined by $\tilde{r}^\alpha = r^\alpha \circ f$;

(*ii*) the functions

$$\tilde{w}^\alpha = w^\alpha + 2\kappa \frac{Dr^\alpha}{d\tau} + \kappa^2 \left(\sigma \sqrt{|u_\lambda u^\lambda|}\, F^\alpha{}_\beta u^\beta + u^\alpha \frac{d}{d\tau} \ln \sqrt{|u_\lambda u^\lambda|} \right) \tag{6.4}$$

for any C_2 function $\kappa: I \to R$, form also a solution of (6.2) along the same Lorentzian world line Γ described now by ξ^α and for a solution of (2.4) determined by $\tilde{r}^\alpha = r^\alpha + \kappa u^\alpha$.

The proof follows from inspection.

Prop. 6.1 describes the consequences of reparametrizations of the basic line and of changes of the transport law of r^α along it. The intrinsic underdeterminacy of Eqs (6.2) for a fixed description of both Γ and r^α along Γ follows from another property of Eqs (6.2).

PROPOSITION 6.2. If any set of functions $w^\alpha: I \to R$ is a solution of (6.2) taken along a given Lorentzian world line Γ, described by the equations $x^\alpha = \xi^\alpha(\tau)$, and for a given solution of Eqs (2.4) determined as $r^\alpha = r^\alpha(\tau)$, then

(*i*) the set of functions

$$\tilde{w}^\alpha = w^\alpha + \psi u^\alpha, \tag{6.5}$$

where $\psi: I \to R$ is an arbitrary C_2 function, is also a solution of (6.2) along the same curve Γ, with the same parametrization, and for the same r^α;

(*ii*) any solution $\tilde{w}^\alpha$ of (6.2) fulfilling the same initial conditions (6.3) as w^α can be represented in the form of (6.5), where $\psi \in C_2$ is uniquely determined by these solutions $\tilde{w}^\alpha$, w^α and the conditions $\psi(\tau_0) = \psi'(\tau_0) = 0$.

Proof. The proof of part (*i*) follows from inspection. The proof of part (*ii*) is based on the following lemma.

LEMMA 6.1. A solution of Eqs. (6.2) which satisfies (6.3) as initial conditions is completely specified by a choice of a continuous function $\nu: I \to R$.

Proof. A given solution w^α (with known ξ^α and r^α) defines the function

$$\nu(\tau) := \frac{d}{d\tau}\left\{ \frac{1}{u_\lambda u^\lambda} \left[h_{\alpha\beta} \frac{Dr^\alpha}{d\tau} \frac{Dr^\beta}{d\tau} + R_{\alpha\beta\gamma\delta} u^\alpha r^\beta r^\gamma u^\delta + u_\alpha \frac{Dw^\alpha}{d\tau} - \frac{1}{u_\lambda u^\lambda} \left(u_\alpha \frac{Dr^\alpha}{d\tau} \right)^2 \right] \right\}. \tag{6.6}$$

Inversely, for a given function $\nu: I \to R$, the vector w^α fulfilling (6.3) as initial conditions is defined as the unique solution of the system of differential equations

$$\frac{D^2 w^\alpha}{d\tau^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta w^\gamma u^\delta = (R^\alpha{}_{\beta\gamma\delta;\varepsilon} + R^\alpha{}_{\varepsilon\gamma\delta;\beta}) u^\beta u^\gamma r^\delta r^\varepsilon + 4R^\alpha{}_{\beta\gamma\delta} \frac{Dr^\beta}{d\tau} u^\gamma r^\delta$$

$$+ \lambda(\tau) \frac{Dw^\alpha}{d\tau} + 2\mu(\tau) \frac{Dr^\alpha}{d\tau} + \nu(\tau) u^\alpha + \sigma \sqrt{|u_\lambda u^\lambda|} \Big\{ F^\alpha{}_{\beta;\gamma\delta} u^\beta r^\gamma r^\delta$$

$$+ F^\alpha{}_{\beta;\gamma} \left[u^\beta w^\gamma + 2 \frac{Dr^\beta}{d\tau} r^\gamma + 2u^\beta r^\gamma \left(\frac{u_\delta}{u_\lambda u^\lambda} \frac{Dr^\delta}{d\tau} \right) \right] + (F^\alpha{}_\beta R^\beta{}_{\gamma\delta\varepsilon}$$

$$- R^{\alpha}{}_{\gamma\delta\beta}F^{\beta}{}_{\varepsilon})r^{\gamma}r^{\delta}u^{\varepsilon} + F^{\alpha}{}_{\beta}\left[\frac{Dw^{\beta}}{d\tau} + 2\frac{Dr^{\beta}}{d\tau}\left(\frac{u_{\gamma}}{u_{\lambda}u^{\lambda}}\frac{Dr^{\gamma}}{d\tau}\right) + \frac{u_{\beta}}{u_{\lambda}u^{\lambda}}\right.$$
$$\left.\left.\times\left(h_{\gamma\delta}\frac{Dr^{\gamma}}{d\tau}\frac{Dr^{\delta}}{d\tau} + R_{\gamma\delta\varepsilon\tau}u^{\gamma}r^{\delta}r^{\varepsilon}u^{\tau} + u_{\gamma}\frac{Dw^{\gamma}}{d\tau}\right)\right]\right\}, \tag{6.7}$$

where the functions λ and μ are given by (1.6) and (2.7). Such w^{α} solves also (6.2), because Eq. (6.6) is satisfied as a weak identity, and this completes the proof of the lemma.

COROLLARY 6.1. There exists a one-to-one map H of the set of all solutions of Eqs (6.2), corresponding to some given initial data in (6.3), onto the set of all continuous functions $v : I \to R$. The map is defined by (6.6) as $w^{\alpha}(\tau) \to H(w^{\alpha}(\tau)) = v(\tau)$, and its inverse is determined by the solution of the initial value problem (6.3) for the differential eq uations (6.7).

Let w^{α} and $\tilde{w}^{\alpha}$ be two solutions of (6.2) such that $H(w^{\alpha}) = v$ and $H(\tilde{w}^{\alpha}) = \tilde{v}$. Then ψ in (6.5) is defined (because of (6.6)) as the unique solution of the differential equation

$$\tilde{v}(\tau) = v(\tau) + \frac{d}{d\tau}\left(\frac{d\psi}{d\tau}(\tau) + \lambda(\tau)\psi(\tau)\right) \tag{6.8}$$

with the initial data $\psi(\tau_0) = \psi'(\tau_0) = 0$. This ends the proof of Prop. 6.2.

Remark 6.1. Prop. 6.1 (*i*) states obviously that Eqs. (6.2) are covariant under arbitrary reparametrization of Γ and of the fields r^{α} and w^{α}, whereas Props. 6.1 (*ii*) and 6.2 can be understood as the statements that Eqs. (6.2) are also invariant under arbitrary gauge transformations of r^{α} and w^{α} which are defined by Eqs. (2.6), (6.4) and (6.5).

It may be noted that contrary to Eqs. (6.2), Eqs. (6.7) can in general be uniquely solved with respect to the derivatives of the highest order and admit therefore a well-posed initial value problem. Thus Eqs. (6.7) determine a law of transport of w^{α} along Γ.

The multiplicity of solutions of Eqs. (6.2), described in Prop. 6.1 (*ii*) and in Prop. 6.2, can again be interpreted geometrically as a possibility of introducing in the next, i.e. now in the second approximation a new arbitrary parametrization on neighbouring Lorentzian lines, keeping the parametrization on the basic line unchanged and, in the case of Prop. 6.2, keeping also fixed the selected solution $r^{\alpha}(\tau, \varepsilon)$. To show this, we should complete Eqs. (2.10) by the condition

$$\frac{\partial^2 f}{\partial \varepsilon^2}(\tau, 0) = \psi(\tau)$$

and (2.11) by

$$\tilde{w}^{\alpha} := \frac{D\tilde{r}^{\alpha}}{\partial \varepsilon}(\tau, \varepsilon).$$

7. *The natural second e.m. deviation*

Let us return to the single line approach. Similarly like in Sects 1–2, we have again on one side a system of differential equations (6.2) with a multiplicity of solutions described by (6.5), an on the other, a family of systems of differential equations (6.7), each member of which is labelled by a function v. If v is given[2], Eqs (6.7) determine a unique solution w^α that satisfies the initial conditions (6.3). Obviously, such a solution is simultaneously a solution of equations (6.2). All these solutions form an equivalence class of the relation defined by (6.5). Every equivalence class is distinguished from the others by the values of the initial data $\{w_0^\alpha, t_0^\alpha\}$ in (6.3) and from a geometrical point of view it describes a Lorentzian world line in the "second neighbourhood" of the basic line Γ_0 (see [1] for details). It is sufficient to have only a single second e.m. deviation vector to describe the equivalence class to which it belongs. Therefore, it is reasonable to choose again a possibly simplest function v in Eqs (6.7). Such a choice consists in taking $v = 0$ foı any $\tau \in I$. The corresponding Eqs (6.7) take then the form

$$
\begin{aligned}
&\frac{D^2 w^\alpha}{d\tau^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta w^\gamma u^\delta = (R^\alpha{}_{\beta\gamma\delta;\varepsilon} + R^\alpha{}_{\varepsilon\gamma\delta;\beta}) u^\beta u^\gamma r^\delta r^\varepsilon \\
&+4R^\alpha{}_{\beta\gamma\delta} \frac{Dr^\beta}{d\tau} u^\gamma r^\delta + \lambda(\tau) \frac{Dw^\alpha}{d\tau} + 2\mu(\tau) \frac{Dr^\alpha}{d\tau} + \sigma \sqrt{|u_\lambda u^\lambda|} \\
&\times \Bigg\{ F^\alpha{}_{\beta;\gamma\delta} u^\beta r^\gamma r^\delta + F^\alpha{}_{\beta;\gamma} \left[u^\beta w^\gamma + 2 \frac{Dr^\beta}{d\tau} r^\gamma + 2u^\beta r^\gamma \left(\frac{u_\delta}{u_\lambda u^\lambda} \frac{Dr^\delta}{d\tau} \right) \right] \\
&+ (F^\alpha{}_\beta R^\beta{}_{\gamma\delta\varepsilon} - R^\alpha{}_{\gamma\delta\beta} F^\beta{}_\varepsilon) r^\gamma r^\delta u^\varepsilon + F^\alpha{}_\beta \left[\frac{Dw^\beta}{d\tau} + 2 \frac{Dr^\beta}{d\tau} \left(\frac{u_\gamma}{u_\lambda u^\lambda} \frac{Dr^\gamma}{d\tau} \right) \right. \\
&\left. + \frac{u^\beta}{u_\lambda u^\lambda} \left(h_{\gamma\delta} \frac{Dr^\gamma}{d\tau} \frac{Dr^\delta}{d\tau} + R_{\gamma\delta\varepsilon\tau} u^\gamma r^\delta r^\varepsilon u^\tau + u_\gamma \frac{Dw^\gamma}{d\tau} \right) \right] \Bigg\} . \qquad (7.1)
\end{aligned}
$$

For each set of initial data in (6.3) Eqs (7.1) have a unique solution. But two different sets of initial data might still lead to equivalent solutions, as it follows from the next proposition.

PROPOSITION 7.1. Two different sets, $\{w_0^\alpha, t_0^\alpha\}$ and $\{\tilde{w}_0^\alpha, \tilde{t}_0^\alpha\}$, of initial data in (6.3) will render two solutions w^α and $\tilde{w}^\alpha$ of (7.1) equivalent in the sense of (6.5) iff

(a) $$\tilde{w}_0^\alpha = w_0^\alpha + \frac{e u_0^\alpha}{\sqrt{|\overset{\circ}{g}_{\beta\gamma} u_0^\beta u_0^\gamma|}}, \qquad \tilde{t}_0^\alpha = t_0^\alpha + (f\delta_\beta^\alpha + e\sigma \overset{\circ}{F}{}^\alpha{}_\beta) u_0^\beta;$$

(b) $$\psi(\tau) = \frac{1}{\sqrt{|u_\lambda u^\lambda|}} \left(f \int_{\tau_0}^{\tau} \sqrt{|u_\lambda u^\lambda|}\, d\tau + e \right), \qquad (7.2)$$

---

[2] In Eqs (6.7), like in all the other equations in Sects 7–9 it is assumed that λ and μ are fixed and imposed by the accepted and fixed parametrization along Γ, which is described by Eqs (1.7) whose solution satisfies the weak identity (1.6), and by the accepted and fixed transport law of the vector r^α along Γ, which is defined by Eqs (2.8) whose solution satisfies the weak identity (2.7). In particular, all statements of Sects 7–9 can easily be rephrased for $\lambda = 0$ (i.e. for preferred or natural parametrizations) and for constraints of the type (3.3), (3.10) or (4.3), (4.5).

where e and f are arbitrary constants, $\overset{\bullet}{g}_{\alpha\beta} = g_{\alpha\beta}(\xi^\gamma(\tau_0))$, $\overset{\bullet}{F}_{\alpha\beta} = F_{\alpha\beta}(\xi^\gamma(\tau_0))$ and u_0^α are the initial data for Γ fulfilling (1.7).

The proof is straightforward and will be omitted.

Thus, Eqs (7.2a) establish an equivalence relation of initial data for (7.1). A single representative from each class of equivalence can be determined by choosing some definite values of e and f. These conditions may be determined, for instance, by means of the following proposition.

PROPOSITION 7.2. In an arbitrary pseudo-Riemannian manifold V_n the system of equations (1.7), (2.8) and (7.1) admits the following first integral

$$\frac{1}{u_\lambda u^\lambda}\left[h_{\alpha\beta}\frac{Dr^\alpha}{d\tau}\frac{Dr^\beta}{d\tau}+R_{\alpha\beta\gamma\delta}u^\alpha r^\beta r^\gamma u^\delta+u_\alpha\frac{Dw^\alpha}{d\tau}-\frac{1}{u_\lambda u^\lambda}\left(u_\alpha\frac{Dr^\alpha}{d\tau}\right)^2\right] = C_4 = \text{const.} \tag{7.3}$$

The proof is a result of contracting Eqs (7.1) with u_α and making use of Eqs (1.7), (2.8) and the weak identities (1.6), (2.7). Clearly, Eq. (7.3) is also a weak identity.

The requirement $\nu = 0$ on Γ_0 has a simple geometric interpretation which can be found by means of the Σ-approach. Let us first note that differentiating (3.6) once again with respect to ε, we obtain

$$\frac{\partial}{\partial\varepsilon}\mu(\tau,\varepsilon) = \nu(\tau,\varepsilon), \tag{7.4}$$

where for $\nu(\tau,\varepsilon)$ we can write along Γ_ε an expression analogous to (6.6). Assuming temporarily that $\nu(\tau,\varepsilon) = \nu(\varepsilon)$, we can integrate (7.4) and obtain

$$\frac{1}{u_\lambda u^\lambda}\left[h_{\alpha\beta}\frac{Dr^\alpha}{d\tau}\frac{Dr^\beta}{d\tau}+R_{\alpha\beta\gamma\delta}u^\alpha r^\beta r^\gamma u^\delta+u_\alpha\frac{Dw^\alpha}{d\tau}-\frac{1}{u_\lambda u^\lambda}\left(u_\alpha\frac{Dr^\alpha}{d\tau}\right)^2\right] = \tau\nu(\varepsilon)+C_4(\varepsilon). \tag{7.5}$$

The integration function $C_4(\varepsilon)$ can easily be interpreted if we impose along each curve Γ_ε the condition (3.4), and differentiate (3.5) with respect to ε, obtaining

$$\frac{1}{u_\lambda u^\lambda}\left[h_{\alpha\beta}\frac{Dr^\alpha}{d\tau}\frac{Dr^\beta}{d\tau}+R_{\alpha\beta\gamma\delta}u^\alpha r^\beta r^\gamma u^\delta+u_\alpha\frac{Dw^\alpha}{d\tau}-\frac{1}{u_\lambda u^\lambda}\left(u_\alpha\frac{Dr^\alpha}{d\tau}\right)^2\right] = \frac{\partial^2}{\partial\varepsilon^2}\ln\sqrt{|C_1(\tau,\varepsilon)|}\,. \tag{7.6}$$

On the other hand, we can differentiate (3.8) with respect to ε once again and compare the result with (7.5) and (7.6), concluding that $C_4(\varepsilon) = \frac{dC_2}{d\varepsilon}(\varepsilon)$. Because of (7.5) the first integral (7.3) exists along Γ_0 iff $\nu(0) = 0$ and C_4 from (7.3) is equal to $\frac{dC_2}{d\varepsilon}(0)$. Thus, the requirement $\nu = 0$ imposes a restriction on the first derivative of the function $C_4(\varepsilon)$ which determines by Eq. (3.9) the parametrization on the lines Γ_ε from the family Σ.

Passing in Eq. (7.3) from a solution w^α to $\tilde{w}^\alpha$ given by (6.5), with φ defined by (7.2b), results in adding f to the constant C_4 in this equation. To fix f, it is therefore sufficient to fix

C_4. The simplest choice is to require that

$$h_{\alpha\beta}\frac{Dr^\alpha}{d\tau}\frac{Dr^\beta}{d\tau}+R_{\alpha\beta\gamma\delta}u^\alpha r^\beta r^\gamma u^\delta+u_\alpha\frac{Dw^\alpha}{d\tau}-\frac{1}{u_\lambda u^\lambda}\left(u_\alpha\frac{Dr^\alpha}{d\tau}\right)^2=0 \tag{7.7}$$

and because of Prop. 7.2 it is sufficient to impose this constraint on initial data only. As a consequence of (7.7), Eqs (7.1) take the more simple form

$$\frac{D^2w^\alpha}{d\tau^2}+R^\alpha{}_{\beta\gamma\delta}u^\beta w^\gamma u^\delta=(R^\alpha{}_{\beta\gamma\delta;\varepsilon}+R^\alpha{}_{\varepsilon\gamma\delta;\beta})u^\beta u^\gamma r^\delta r^\varepsilon$$

$$+4R^\alpha{}_{\beta\gamma\delta}\frac{Dr^\beta}{d\tau}u^\gamma r^\delta+\lambda(\tau)\frac{Dw^\alpha}{d\tau}+2\mu(\tau)\frac{Dr^\alpha}{d\tau}+\sigma\sqrt{|u_\lambda u^\lambda|}$$

$$\times\left\{F^\alpha{}_{\beta;\gamma\delta}u^\beta r^\gamma r^\delta+F^\alpha{}_{\beta;\gamma}\left[u^\beta w^\gamma+2\frac{Dr^\beta}{d\tau}r^\gamma+2u^\beta r^\gamma\left(\frac{u_\delta}{u_\lambda u^\lambda}\frac{Dr^\delta}{d\tau}\right)\right]\right.$$

$$\left.+(F^\alpha{}_\beta R^\beta{}_{\gamma\delta\varepsilon}-R^\alpha{}_{\gamma\delta\beta}F^\beta{}_\varepsilon)r^\gamma r^\delta u^\varepsilon+F^\alpha{}_\beta\left[\frac{Dw^\beta}{d\tau}+2\frac{Dr^\beta}{d\tau}\left(\frac{u^\gamma}{u_\lambda u^\lambda}\frac{Dr^\gamma}{d\tau}\right)\right]\right\} \tag{7.8}$$

and it is a straightforward exercise to prove that (7.3) is a first integral of (1.7), (2.8) and (7.8).

The condition (7.7), due to (7.5) and $\nu(0)=0$ implies $\frac{dC_2}{d\tau}(0)=0$. Differentiating Eq. (3.7) and taking into account that $\mu(0)=\nu(0)=C_2(0)=C_2'(0)=0$, we obtain that also $\left(\frac{\partial^2C_1}{\partial\varepsilon^2}(\tau,\varepsilon)\right)_{\varepsilon=0}=0$, which geometrically means that all the Lorentzian world lines from the "second neighbourhood" of the basic line Γ_0 are parametrized by the same parameter τ, specified by the function $C_1(\tau,\varepsilon)$. Even if $C_1(\tau,\varepsilon)$ is kept fixed, there remains a possibility of reparametrization with $f'(\tau)=1$ in (1.8), i.e. there is still left at our disposal the freedom of choice of the initial value τ_0 of the parameter along each of the world lines.

This result justifies the following definition.

DEFINITION 7.1. A vector field w^α which is a solution of Eqs (7.8) evaluated along a Lorentzian world line Γ parametrized by an arbitrary τ and which satisfies (7.7) as a constraint condition is called the natural second e.m. deviation vector.

There remains still the freedom of choosing the value of the constant e in (7.2). In virtue of (1.7) and (7.7) one can obtain that

$$\frac{d}{d\tau}\left(\frac{u_\alpha w^\alpha}{\sqrt{|u_\lambda u^\lambda|}}\right)=-\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left[h_{\alpha\beta}\frac{Dr^\alpha}{d\tau}\frac{Dr^\beta}{d\tau}\right.$$

$$\left.+R_{\alpha\beta\gamma\delta}u^\alpha r^\beta r^\gamma u^\delta-\frac{1}{u_\lambda u^\lambda}\left(u_\alpha\frac{Dr^\alpha}{d\tau}\right)^2\right]+\sigma F_{\alpha\beta}w^\alpha u^\beta$$

or

$$\frac{u_\alpha w^\alpha}{\sqrt{|u_\lambda u^\lambda|}} = -\int_{\tau_0}^{\tau} \left\{ \frac{1}{\sqrt{|u_\lambda u^\lambda|}} \left[h_{\alpha\beta} \frac{Dr^\alpha}{d\tau} \frac{Dr^\beta}{d\tau} + R_{\alpha\beta\gamma\delta} u^\alpha r^\beta r^\gamma u^\delta \right.\right.$$

$$\left.\left. - \frac{1}{u_\lambda u^\lambda} \left(u_\alpha \frac{Dr^\alpha}{d\tau} \right)^2 \right] - \sigma F_{\alpha\beta} w^\alpha u^\beta \right\} d\tau + \text{const.} \tag{7.9}$$

Since the condition (7.7) requires that $f = 0$, due to Prop. 7.1. Eqs (7.8) admit still equivalent solutions of the form

$$\tilde{w}^\alpha = w^\alpha + \frac{eu^\alpha}{\sqrt{|u_\lambda u^\lambda|}}. \tag{7.10}$$

Hence, replacing w^α by $\tilde{w}^\alpha$ results in adding e to the right hand side of (7.9), and the choice of the constant e is therefore equivalent to specifying the value of the integration constant in (7.9).

8. The second e.m. separation vector

Similarly like in the case of the natural first e.m. deviation (cf. Sect. 3), the natural second e.m. deviation can be interpreted as an infinitesimal, parametrization preserving mapping of the Lorentzian world line Γ_0 onto such another line from the "second neighbourhood" of Γ_0. However, even in the case of the natural second geodesic deviation (i.e. for $\sigma = 0$), the second deviation vector which corresponds to (7.7) during its evolution along the basic world line Γ_0 changes its inclination $u_\alpha w^\alpha$ with Γ_0. But the general equations (6.7) are flexible enough and permit for such a transformation of them, being the result of a suitable choice of the function v, after which they will uniquely determine a second e.m. deviation vector that preserves its inclination $u_\alpha w^\alpha$ with the basic Lorentzian world line Γ_0. We shall now study this case in some detail.

Let us observe that in analogy to a remark made in Sect. 4 Prop. 7.2 is also a consequence of our choice $v = 0$ in the weak identity (6.6) when passing from Eqs (6.7) to (7.1). Now, a similar question arises whether is it possible to find such a function v that substituted into Eqs (6.7) will give us the second e.m. deviation equations preserving the product $u_\alpha w^\alpha$, or a simple function of it. The answer follows from an immediate transformation of Eq. (6.6) to the form

$$v(\tau) = \frac{d}{d\tau} \left\{ \frac{1}{u_\lambda u^\lambda} \left[h_{\alpha\beta} \frac{Dr^\alpha}{d\tau} \frac{Dr^\beta}{d\tau} + R_{\alpha\beta\gamma\delta} u^\alpha r^\beta r^\gamma u^\delta - \frac{1}{u_\lambda u^\lambda} \left(u_\alpha \frac{Dr^\alpha}{d\tau} \right)^2 \right] \right.$$

$$\left. + \frac{k\sigma F_{\alpha\beta} u^\alpha w^\beta}{\sqrt{|u_\lambda u^\lambda|}} \right\} + \frac{d}{d\tau} \left[\frac{k}{\sqrt{|u_\lambda u^\lambda|}} \frac{d}{d\tau} \left(\frac{u_\alpha w^\alpha}{\sqrt{|u_\lambda u^\lambda|}} \right) \right],$$

where use was made of (1.7) and (1.6). Thus, if one accepts that

$$v(\tau) := \frac{d}{d\tau}\left\{\frac{1}{u_\lambda u^\lambda}\left[h_{\alpha\beta}\frac{Dr^\alpha}{d\tau}\frac{Dr^\beta}{d\tau} + R_{\alpha\beta\delta\gamma}u^\alpha r^\beta r^\gamma u^\delta - \frac{1}{u_\lambda u^\lambda}\left(u_\alpha\frac{Dr^\alpha}{d\tau}\right)^2\right] + \frac{k\sigma F_{\alpha\beta}u^\alpha w^\beta}{\sqrt{|u_\lambda u^\lambda|}}\right\}, \tag{8.1}$$

then Eqs (6.7) transform to the form

$$\begin{aligned}\frac{D^2 w^\alpha}{d\tau^2} &+ R^\alpha{}_{\beta\gamma\delta}u^\beta w^\gamma u^\delta = (R^\alpha{}_{\beta\gamma\delta;\varepsilon} + R^\alpha{}_{\varepsilon\gamma\delta;\beta})u^\beta u^\gamma r^\delta r^\varepsilon + 4R^\alpha{}_{\beta\gamma\delta}\frac{Dr^\beta}{d\tau}u^\gamma r^\delta \\ &+ \lambda(\tau)\frac{Dw^\alpha}{d\tau} + 2\mu(\tau)\frac{Dr^\alpha}{d\tau} + u^\alpha\frac{d}{d\tau}\left\{\frac{1}{u_\lambda u^\lambda}\left[h_{\beta\gamma}\frac{Dr^\beta}{d\tau}\frac{Dr^\gamma}{d\tau} + R_{\beta\gamma\delta\varepsilon}u^\beta r^\gamma r^\delta u^\varepsilon\right.\right. \\ &\left.\left. - \frac{1}{u_\lambda u^\lambda}\left(u_\beta\frac{Dr^\beta}{d\tau}\right)^2\right] + \frac{k\sigma F_{\beta\gamma}u^\beta w^\gamma}{\sqrt{|u_\lambda u^\lambda|}}\right\} + \sigma\sqrt{|u_\lambda u^\lambda|}\left\{F^\alpha{}_{\beta;\gamma\delta}u^\beta r^\gamma r^\delta\right. \\ &+ F^\alpha{}_{\beta;\gamma}\left[u^\beta w^\gamma + 2\frac{Dr^\beta}{d\tau}r^\gamma + 2u^\beta r^\gamma\left(\frac{u_\delta}{u_\lambda u^\lambda}\frac{Dr^\delta}{d\tau}\right)\right] + (F^\alpha{}_\beta R^\beta{}_{\gamma\delta\varepsilon} \\ &- R^\alpha{}_{\gamma\delta\beta}F^\beta{}_\varepsilon)r^\gamma r^\delta u^\varepsilon + F^\alpha{}_\beta\left[\frac{Dw^\beta}{d\tau} + 2\frac{Dr^\beta}{d\tau}\left(\frac{u_\gamma}{u_\lambda u^\lambda}\frac{Dr^\gamma}{d\tau}\right) + \frac{u^\beta}{u_\lambda u^\lambda}\left(h_{\gamma\delta}\frac{Dr^\gamma}{d\tau}\frac{Dr^\delta_\delta}{d\tau}\right.\right. \\ &\left.\left.\left. + R_{\gamma\delta\varepsilon\tau}u^\gamma r^\delta r^\varepsilon u^\tau + u_\gamma\frac{Dw^\gamma}{d\tau}\right)\right]\right\}.\end{aligned} \tag{8.2}$$

The system of equations (1.7), (2.8) and (8.2) admits already the desired first integral.

PROPOSITION 8.1. In an arbitrary pseudo-Riemannian manifold V_n any solution of the system of equations (1.7), (2.8) and (8.2), with λ and μ given by (1.6) and (2.7) correspondingly, satisfies the relation

$$\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\frac{d}{d\tau}\left(\frac{u_\alpha w^\alpha}{\sqrt{|u_\lambda u^\lambda|}}\right) = C_5 = \text{const.} \tag{8.3}$$

For the same reason as before, for each set of initial data in (6.3), Eqs (8.2) have a unique solution. But two different sets of initial data may still lead to solutions equivalent in the sense of (6.5), as Prop. 7.1 can also be proved for the case of Eqs (8.2). Hence, Eqs (7.2a) establish an equivalence relation of initial data for (8.2). To select a single representative from each class of equivalence, one has to choose some definite values of the constants e and f in (7.2). This can again be done by adding to Eqs (8.2) a constraint condition which can be imposed in the form of the conservation law (8.3) and this requirement will be compatible with the evolution governed by Eqs (1.7), (2.8) and (8.2).

The transformation (6.5) from a solution w^α to $\tilde{w}^\alpha$ with ψ defined by (7.2b) results in adding f to the constant C_5 in (8.3). Therefore to fix f, it is sufficient to fix the value of C_5. The simplest choice is to require that

$$\frac{u_\alpha w^\alpha}{\sqrt{|u_\lambda u^\lambda|}} = \text{const}, \tag{8.4}$$

and because of Prop. 8.1, it is sufficient to impose this constraint on the initial data only. Since the condition (8.4) requires that $f = 0$, due to Prop. 7.1 Eqs (8.2) admit still equivalent solutions of the form (7.10). Hence, replacing w^α by $\tilde{w}^\alpha$ results in adding e to the constant in (8.4). It is convenient to accept

$$u_\alpha w^\alpha = 0 \tag{8.5}$$

as a constraint specifying the value of the constants in (8.4) and (7.2). Let us however observe that Eq. (8.5) is compatible with the first integral (8.3) iff $\frac{Dw^\alpha}{d\tau}$ satisfies the condition

$$\frac{u_\alpha}{\sqrt{|u_\lambda u^\lambda|}} \frac{Dw^\alpha}{d\tau} = \sigma F_{\alpha\beta} u^\alpha w^\beta. \tag{8.6}$$

Such requirements have an obvious geometric interpretation and justify the introduction of the following definition.

DEFINITION 8.1. A vector field w^α which is a solution of Eqs (8.2) evaluated along a Lorentzian world line Γ parametrized by an arbitrary parameter τ and which satisfies (8.5) and (8.6) as constraint conditions is called the second e.m. separation vector.

It should be noted that the second e.m. separation vector, like any second e.m. deviation vector defined by Eqs (6.7), is a mapping of the basic Lorentzian world line Γ_0 onto such another line in its infinitesimal neighbourhood. This mapping does not however preserve the parametrization of the world line, as it follows from a discussion analogous to that preceding Eq. (7.6). Moreover, contrary to the case of the first e.m. deviation, even in the case of the second geodesic deviation (i.e. for $\sigma = 0$) the two sets of differential equations, Eqs (7.1) and Eqs (8.2), are not identical, and the two kinds of the second e.m. deviation vector do not merge into a single concept of the natural second geodesic deviation vector.

9. *Relationship between the two kinds of second e.m. deviation*

The whole scheme developed in Sects 6–8 indicates that there must be also a relation between the natural second e.m. deviation and the second e.m. separation vectors. As a corollary of Prop. 6.2 it follows that if we have two members of the family of systems of differential equations (6.7), corresponding to the two functions $\tilde{\nu}$ and ν respectively, then their solutions must satisfy the relation (6.5) in which the function ψ is uniquely determined by Eq. (6.8) with an appropriate initial conditions. Hence, taking $\tilde{\nu} = 0$ and ν in the form (8.1), by solving Eq. (6.8) we obtain that

$$\psi(\tau) = \left\{\int_{\tau_0}^{\tau}\left[g - \frac{1}{u_\lambda u^\lambda}\left(h_{\alpha\beta}\frac{Dr^\alpha}{d\tau}\frac{Dr^\beta}{d\tau} + R_{\alpha\beta\gamma\delta}u^\alpha r^\beta r^\gamma u^\delta\right)\right.\right.$$

$$\left.\left. + \left(\frac{u_\alpha}{u_\lambda u^\lambda}\frac{Dr^\alpha}{d\tau}\right)^2 - \frac{k\sigma F_{\alpha\beta}u^\alpha w^\beta}{\sqrt{|u_\lambda u^\lambda|}}\right]_{\tau=\tau'} \exp\left(\int_{\tau_0}^{\tau'}\lambda(\tau'')d\tau''\right)d\tau' + j\right\}$$

$$\times \exp\left(-\int_{\tau_0}^{\tau}\lambda(\tau')d\tau'\right),$$

where g and j are integration constants. If one now takes into account the weak identity (1.6), the expression for ψ can be rewritten as

$$\psi(\tau) = \frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left\{\int_{\tau_0}^{\tau}\left[g\sqrt{|u_\lambda u^\lambda|} - \frac{k}{\sqrt{|u_\lambda u^\lambda|}}\left(h_{\alpha\beta}\frac{Dr^\alpha}{d\tau}\frac{Dr^\beta}{d\tau}\right.\right.\right.$$

$$\left.\left.\left. + R_{\alpha\beta\gamma\delta}u^\alpha r^\beta r^\gamma u^\delta\right) + \sqrt{|u_\lambda u^\lambda|}\left(\frac{u_\alpha}{u_\lambda u^\lambda}\frac{Dr^\alpha}{d\tau}\right)^2 - k\sigma F_{\alpha\beta}u^\alpha w^\beta\right]d\tau + j\right\}. \tag{9.1}$$

The assumptions $\tilde{v} = 0$ and (8.1) mean that the formula (6.5) with ψ given by (9.1) transforms a vector w^α which satisfies Eqs (8.2) with a constraint condition (8.3) into a vector $\tilde{w}^\alpha$ being a solution of Eqs (7.1) and constrained by Eq. (7.3); the constants C_4 and C_5 in the constraint conditions are arbitrary, but fixed for the solutions considered.

Let us meanwhile assume that $\{\tilde{w}^\alpha_0, \tilde{t}^\alpha_0\}$ are the data for $\tilde{w}^\alpha$ in the initial value problem (6.3) that are constrained by (7.3), and $\{w^\alpha_0, t^\alpha_0\}$ are those for w^α restricted by (8.3). Making use of the Lorentz equations in (8.3), we can eliminate the term $\sigma F_{\alpha\beta}u^\alpha w^\beta$ from (9.1) and write the relation (6.5) with ψ given by (9.1) in the form

$$\tilde{w}^\alpha(\tau) = w^\alpha(\tau) + \frac{u_\alpha}{\sqrt{|u_\lambda u^\lambda|}}(\tau)\left\{\int_{\tau_0}^{\tau}\left[(g+kC_5)\sqrt{|u_\lambda u^\lambda|}\right.\right.$$

$$- \frac{k}{\sqrt{|u_\lambda u^\lambda|}}\left(h_{\beta\gamma}\frac{Dr^\beta}{d\tau}\frac{Dr^\gamma}{d\tau} + R_{\beta\gamma\delta\varepsilon}u^\beta r^\gamma r^\delta u^\varepsilon + u_\beta\frac{Dw^\beta}{d\tau}\right)$$

$$\left.\left. - \sqrt{|u_\lambda u^\lambda|}\left(\frac{u_\beta}{u_\lambda u^\lambda}\frac{Dr^\beta}{d\tau}\right)^2\right]d\tau + j\right\}. \tag{9.2}$$

Substituting here $\tau = \tau_0$, we obtain

$$\tilde{w}^\alpha_0 = w^\alpha_0 + \frac{ju^\alpha_0}{\sqrt{|\overset{\circ}{g}_{\beta\gamma}u^\beta_0 u^\gamma_0|}} \tag{9.3}$$

and

$$j = \frac{k\overset{\bullet}{g}_{\alpha\beta}u_0^\alpha(\tilde{w}_0^\beta - w_0^\beta)}{\sqrt{|\overset{\bullet}{g}_{\lambda\nu}u_0^\lambda u_0^\nu|}}.$$

The relation (9.3) then reads

$$\overset{\bullet}{h}{}^\alpha{}_\beta(\tilde{w}_0^\beta - w_0^\beta) = 0. \tag{9.4}$$

Differentiating (9.2) and putting $\tau = \tau_0$ gives

$$\tilde{t}_0^\alpha = \overset{\bullet}{h}{}^\alpha{}_\beta t_0^\beta + j\sigma\overset{\bullet}{F}{}^\alpha{}_\beta u_0^\beta + u_0^\alpha\Bigg\{(g+kC_5) - \frac{1}{u_\lambda u^\lambda}$$
$$\times\left[h^\beta{}_\gamma\frac{Dr^\beta}{d\tau}\frac{Dr^\gamma}{d\tau} + R_{\beta\gamma\delta\varepsilon}u^\beta r^\gamma r^\delta u^\varepsilon - \frac{1}{u_\lambda u^\lambda}\left(u_\beta\frac{Dr^\beta}{d\tau}\right)^2\right]\Bigg\}_{\tau=\tau_0} \tag{9.5}$$

From (9.5) and (7.3) it follows that

$$g = C_4 - kC_5. \tag{9.6}$$

Eqs (9.3) and (9.5), with (9.6), define a transformation $\{\tilde{w}_0^\alpha, \tilde{t}_0^\alpha\} \to \{\tilde{w}_0^\alpha, \tilde{t}_0^\alpha\}$ of initial data that must accompany the transformation (6.5) with ψ given by (9.1). The transformation of initial data contains one arbitrary parameter j, while C_4 and C_5 are fixed. Observing that due to (6.5) $F_{\alpha\beta}u^\alpha w^\beta = F_{\alpha\beta}u^\alpha\tilde{w}^\beta$ and making use of (7.3) and (8.6), one finds the inverse transformation to (9.5):

$$t_0^\alpha = \tilde{t}_0^\alpha - j\sigma\overset{\bullet}{F}{}^\alpha{}_\beta u_0^\beta + u_0^\alpha\Bigg\{(kC_5 - C_4) + \frac{k\sigma\overset{\bullet}{F}_{\beta\gamma}u_0^\beta\tilde{w}_0^\gamma}{\sqrt{|\overset{\bullet}{g}_{\lambda\nu}u_0^\lambda u_0^\nu|}}$$
$$+\frac{1}{\overset{\bullet}{g}_{\lambda\nu}u_0^\lambda u_0^\nu}\left[h_{\beta\gamma}\frac{Dr^\beta}{d\tau}\frac{Dr^\gamma}{d\tau} + R_{\beta\gamma\delta\varepsilon}u^\beta r^\gamma r^\delta u^\varepsilon - \frac{1}{u_\lambda u^\lambda}\left(u_\beta\frac{Dr^\beta}{d\tau}\right)^2\right]_{\tau=\tau_0}\Bigg\} \tag{9.7}$$

The foregoing considerations can be summarized in a form of a proposition.

PROPOSITION 9.1. A solution w^α of Eqs (8.2) along a given Lorentzian world line Γ, satisfying the constraint condition (8.3) with C_5 being fixed, can be transformed into the vector

$$\tilde{w}^\alpha = w^\alpha + \frac{u^\alpha}{\sqrt{|u_\lambda u^\lambda|}}\Bigg\{\int_{\tau_0}^{\tau}\Bigg[(C_4 - kC_5)\sqrt{|u_\lambda u^\lambda|} - \frac{k}{\sqrt{|u_\lambda u^\lambda|}}\left(h_{\beta\gamma}\frac{Dr^\beta}{d\tau}\frac{Dr^\gamma}{d\tau}\right.$$
$$\left. + R_{\beta\gamma\delta\varepsilon}u^\beta r^\gamma r^\delta u^\varepsilon\right) + \sqrt{|u_\lambda u^\lambda|}\left(\frac{u_\beta}{u_\lambda u^\lambda}\frac{Dr^\beta}{d\tau}\right)^2 - k\sigma F_{\beta\gamma}u^\beta w^\gamma\Bigg]d\tau + j\Bigg\} \tag{9.8}$$

being a solution of Eqs (7.1) along the same line Γ and satisfying the constraint condition (7.3) with a fixed value of C_4; j in (9.8) is a constant. The inverse transformation

$\tilde{w}^\alpha \to w^\alpha$ can be obtained from replacing $F_{\alpha\beta}u^\alpha w^\beta$ in the integrand by $F_{\alpha\beta}u^\alpha \tilde{w}^\beta$. The initial data for one of the vectors determine these for the other by means of Eqs (9.3), (9.5) and (9.7).

COROLLARY 9.1. If $\tilde{w}^\alpha$ is a natural second e.m. deviation vector and w^α a second e.m. separation vector all the equations (9.1)–(9.8) remain valid after substituting into them $g = C_4 = C_5 = u_\alpha w^\alpha = \mathring{g}_{\alpha\beta} u_0^\alpha w_0^\beta = 0$. Since w^α satisfies two and $\tilde{w}^\alpha$ only one constraint conditions, the transformations $w^\alpha \to \tilde{w}^\alpha$ and $\{w_0^\alpha, t_0^\alpha\} \to \{\tilde{w}_0^\alpha, \tilde{t}_0^\alpha\}$ involve necessarily one free parameter j representing an additional degree of freedom. In the inverse transformation $\tilde{w}^\alpha \to w^\alpha$, one ought to substitute $j = k\mathring{g}_{\alpha\beta} u_0^\alpha \tilde{w}_0^\beta (|\mathring{g}_{\lambda\nu} u_0^\lambda u_0^\nu|)^{-1/2}$, and the transformation $\{\tilde{w}_0^\alpha, \tilde{t}_0^\alpha\} \to \{w_0^\alpha, t_0^\alpha\}$ does not contain any free parameter, since from (9.4) in the case considered now we obtain $w_0^\alpha = \mathring{h}^\alpha{}_\beta \tilde{w}_0^\beta$.

Let us also note that Eqs (6.2) are invariant under arbitrary reparametrization of the basic world line Γ_0, whereas every one of the systems of Eqs (6.7), labelled by a fixed function ν, taken separately does not enjoy such an invariance property. However the two systems of equations, Eqs (7.8) and Eqs (8.2) respectively, are again reparametrization invariant. To see it, one ought to transform each of the two systems to a form which is manifestly invariant.

If one eliminates the functions λ and μ from Eqs (7.8) by means of (1.6) and (2.7), taking into account the constraint condition (7.7) and the symmetry properties of $F_{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}$, one obtains the following differential equations for a vector field w_n^α along an arbitrary parametrized Lorentzian world line Γ and for an arbitrary e.m. deviation vector r^α along this line:

$$\frac{D}{d\tau}\left\{\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left[\frac{Dw_n^\alpha}{d\tau} - 2h^\alpha{}_\beta \frac{Dr^\beta}{d\tau}\left(\frac{u_\gamma}{u_\lambda u^\lambda}\frac{Dr^\gamma}{d\tau}\right)\right] - \frac{u^\alpha}{\sqrt{|u_\lambda u^\lambda|}}\left(\frac{u_\beta}{u_\lambda u^\lambda}\frac{Dr^\beta}{d\tau}\right)^2\right.$$
$$\left. - \frac{2}{\sqrt{|u_\lambda u^\lambda|}} R^\alpha{}_{\beta\gamma\delta} r^\beta u^\gamma r^\delta - \sigma(F^\alpha{}_\beta w_n^\beta + F^\alpha{}_{\beta;\gamma} r^\beta r^\gamma)\right\} + \frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left(R_{\varepsilon\beta\delta}{}^{;\alpha}\right.$$
$$\left. + \frac{k\sigma}{\sqrt{|u_\lambda u^\lambda|}} F_{\beta\gamma;\varepsilon}{}^{;\alpha} u_\delta\right) u^\beta r^\gamma u^\delta r^\varepsilon + \frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left(R^\alpha{}_{\beta\gamma\delta}\right.$$
$$\left. + \frac{k\sigma}{\sqrt{|u_\lambda u^\lambda|}} F_{\beta\gamma}{}^{;\alpha} u_\delta\right)\left(u^\beta w_n^\gamma + 2\frac{Dr^\beta}{d\tau} r^\gamma\right) u^\delta = \frac{2}{\sqrt{|u_\lambda u^\lambda|}} R^\alpha{}_{\beta\gamma\delta}\left[r^\beta \frac{Dr^\gamma}{d\tau}\right.$$
$$\left. + u^\beta r^\gamma \left(\frac{u_\varepsilon}{u_\lambda u^\lambda}\frac{Dr^\varepsilon}{d\tau}\right)\right] u^\delta - \sigma\left(F_{\beta\gamma}{}^{;\alpha}\frac{Dr^\gamma}{d\tau} + R^\alpha{}_{\gamma\delta\varepsilon} F^\varepsilon{}_\beta u^\gamma r^\delta\right) r^\beta \tag{9.9}$$

which must necessarily be considered in conjunction with the constraint condition (7.7), i.e.

$$h_{\alpha\beta}\frac{Dr^\alpha}{d\tau}\frac{Dr^\beta}{d\tau} + R_{\alpha\beta\gamma\delta} u^\alpha r^\beta r^\gamma u^\delta + u_\alpha \frac{Dw_n^\alpha}{d\tau} - \frac{1}{u_\lambda u^\lambda}\left(u_\alpha \frac{Dr^\alpha}{d\tau}\right)^2 = 0,$$

since otherwise it might have happened that their solution had not been a natural second e.m. deviation vector, for Eqs (9.9) admit (7.3) and not (7.7) as their first integral. The manifest invariance of Eqs (9.9) and of the constraint condition is evident. Besides, these equations are solvable with respect to the highest derivatives and admit therefore a well-posed initial value problem. All this means that along a given Lorentzian world line Γ, understood as a locus of points, and for a given field r^α Eqs (9.9) together with the constraint condition (7.7) and the initial value problem (6.3) uniquely determine a natural second e.m. deviation vector field w_n^α, which therefore has a parametrization independent, geometrical meaning. In particular, when Γ is parametrized by the natural parameter s and r_n^α is a natural first e.m. deviation vector, one must substitute into (9.9) the relations (1.15) and (3.10), which result in a simplified version of (9.9) or in the following form of the second e.m. deviation equations:

$$\frac{D^2 w_n^\alpha}{ds^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta w_n^\gamma u^\delta = (R^\alpha{}_{\beta\gamma\delta;\varepsilon} + R^\alpha{}_{\varepsilon\gamma\delta;\beta}) u^\beta u^\gamma r_n^\delta r_n^\varepsilon$$
$$+4R^\alpha{}_{\beta\gamma\delta} \frac{Dr_n^\beta}{ds} u^\gamma r_n^\delta + \sigma \Bigg[F^\alpha{}_{\beta;\gamma\delta} u^\beta r_n^\gamma r_n^\delta + F^\alpha{}_{\beta;\gamma} \left(u^\beta w_n^\gamma + 2 \frac{Dr_n^\beta}{ds} r_n^\gamma \right)$$
$$+ (F^\alpha{}_\beta R^\beta{}_{\gamma\delta\varepsilon} - R^\alpha{}_{\gamma\delta\beta} F^\beta{}_\varepsilon) r_n^\gamma r_n^\delta u^\varepsilon + F^\alpha{}_\beta \frac{Dr_n^\beta}{ds} \Bigg] \tag{9.10}$$

which after setting $\sigma = 0$ coincide with Eqs (3.10) quoted in [1].

A similar elimination by means of (1.6) and (2.7) of the functions λ and μ from Eqs (8.2) shows that the second e.m. separation vector w_s^α satisfies the equations

$$\frac{D}{d\tau} \Bigg\{ \frac{1}{\sqrt{|u_\lambda u^\lambda|}} \Bigg[\frac{Dw_s^\alpha}{d\tau} - 2h^\alpha_{\ \beta} \frac{Dr^\beta}{d\tau} \left(\frac{u_\gamma}{u_\lambda u^\lambda} \frac{Dr^\gamma}{d\tau} \right) - \frac{u^\alpha}{u_\lambda u^\lambda} \Bigg(h_{\beta\gamma} \frac{Dr^\beta}{d\tau} \frac{Dr^\gamma}{d\tau}$$
$$+ R_{\beta\gamma\delta\varepsilon} u^\beta r^\gamma r^\delta u^\varepsilon \Bigg) \Bigg] - \frac{2}{\sqrt{|u_\lambda u^\lambda|}} R^\alpha{}_{\beta\gamma\delta} r^\beta u^\gamma r^\delta - \sigma \Bigg[\left(g^{\alpha\beta} + \frac{u^\alpha u^\beta}{u_\lambda u^\lambda} \right) F_{\beta\delta} w_s^\delta$$
$$+ F^\alpha{}_{\beta;\gamma} r^\beta r^\gamma \Bigg] \Bigg\} + \frac{1}{\sqrt{|u_\lambda u^\lambda|}} \left(R_{\varepsilon\beta\gamma\delta}{}^{;\alpha} + \frac{k\sigma}{\sqrt{|u_\lambda u^\lambda|}} F_{\beta\gamma;\varepsilon}{}^{;\alpha} u_\delta \right) u^\beta r^\gamma u^\delta r^\varepsilon$$
$$+ \frac{1}{\sqrt{|u_\lambda u^\lambda|}} \left(R^\alpha{}_{\beta\gamma\delta} + \frac{k\sigma}{\sqrt{|u_\lambda u^\lambda|}} F_{\beta\gamma}{}^{;\alpha} u_\delta \right) \left(u^\beta w_s^\gamma + 2 \frac{Dr^\beta}{d\tau} r^\gamma \right) u^\delta$$
$$= \frac{2}{\sqrt{|u_\lambda u^\lambda|}} R^\alpha{}_{\beta\gamma\delta} \left[r^\beta \frac{Dr^\gamma}{d\tau} + u^\beta r^\gamma \left(\frac{u_\varepsilon}{u_\lambda u^\lambda} \frac{Dr^\varepsilon}{d\tau} \right) \right] u^\delta$$
$$- \sigma \left(F_{\beta\gamma}{}^{;\alpha} \frac{Dr^\gamma}{d\tau} + R^\alpha{}_{\gamma\delta\varepsilon} F^\varepsilon{}_\beta u^\gamma r^\delta \right) r^\beta \tag{9.11}$$

which must be supplemented by two constraint conditions (8.5) and (8.6), i.e.

$$u_\alpha w_s^\alpha = 0, \qquad \frac{u_\alpha}{\sqrt{|u_\lambda u^\lambda|}} \frac{Dw_s^\alpha}{d\tau} = \sigma F_{\alpha\beta} u^\alpha w_s^\beta.$$

The initial value problem is here again a well-posed one and the second e.m. separation vector field w_s^α has a parametrization independent, geometrical meaning, although in general it differs from the vector field w_n^α. In particular, Eqs (9.11) and the constraint conditions can be easily written in the case when Γ is parametrized by the natural parameter s and r_s^α is a first e.m. separation vector, or in the equivalent form

$$\frac{D^2 w_s^\alpha}{ds^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta w_s^\gamma u^\delta = (R^\alpha{}_{\beta\gamma\delta;\varepsilon} + R^\alpha{}_{\varepsilon\gamma\delta;\beta}) u^\beta u^\gamma r_s^\delta r_s^\varepsilon$$

$$+4R^\alpha{}_{\beta\gamma\delta} \frac{Dr_s^\beta}{ds} u^\gamma r_s^\delta + 2 \frac{Dr_s^\alpha}{ds} \frac{d}{ds} (\sigma F_{\beta\gamma} u^\beta r_s^\gamma) + u^\alpha \frac{d}{ds} \Bigg[h_{\beta\gamma} \frac{Dr_s^\beta}{ds} \frac{Dr_s^\gamma}{ds}$$

$$+ R_{\beta\gamma\delta\varepsilon} u^\beta r_s^\gamma r_s^\delta u^\varepsilon + \sigma F_{\beta\gamma} u^\beta w_s^\gamma - (\sigma F_{\beta\gamma} u^\beta r_s^\gamma)^2 \Bigg] + \sigma \Bigg\{ F^\alpha{}_{\beta;\gamma\delta} u^\beta r_s^\gamma r_s^\delta$$

$$+ F^\alpha{}_{\beta;\gamma} \Bigg[u^\beta w_s^\gamma + 2 \frac{Dr_s^\beta}{ds} r_s^\gamma + 2u^\beta r_s^\gamma (\sigma F_{\delta\varepsilon} u^\delta r_s^\varepsilon) \Bigg]$$

$$+ (F^\alpha{}_\beta R^\beta{}_{\gamma\delta\varepsilon} - R^\alpha{}_{\gamma\delta\beta} F^\beta{}_\varepsilon) r_s^\gamma r_s^\delta u^\varepsilon + F^\alpha{}_\beta \Bigg[\frac{Dw_s^\beta}{ds} + 2 \frac{Dr_s^\beta}{ds} (\sigma F_{\gamma\delta} u^\gamma r_s^\delta)$$

$$+ u^\beta \left(h_{\gamma\delta} \frac{Dr_s^\gamma}{ds} \frac{Dr_s^\delta}{ds} + R_{\gamma\delta\varepsilon\tau} u^\gamma r_s^\delta r_s^\varepsilon u^\tau + \sigma F_{\gamma\delta} u^\gamma w_s^\delta \right) \Bigg] \Bigg\}. \tag{9.12}$$

Eqs (9.12) after setting $\sigma = 0$ and taking into account (8.6) coincide with Eqs (4.2) quoted in [1].

As a final observation let us remark that from (9.8) and (8.5) it follows immediately that the second e.m. separation vector can always be represented in the form $w_s^\alpha = h^\alpha{}_\beta w_n^\beta \equiv w_\perp^\alpha$, where w_n^α is a natural second e.m. deviation vector.

10. *Summary*

Concluding this paper, we would like to supplement the general characteristic of the subject which was partly presented already in the introduction by a retrospection of some of the results obtained here.

Perhaps the largest difference between the approach accepted in this paper and some other approaches that can be found in the literature is in the assumption of a general parametrization along all the world lines. This kind of approach is motivated by our conviction that in relativity it is the world line understood as sets of points and not as a parametrized curve which is of physical significance.

True enough, in Sect. 1 among others, the concept of the affine parameter along a geodesic is generalized to the case of a Lorentzian world line. This generalization is related to the possibility of expressing in a specific form the law of transport of the tangent vector to a Lorentzian world line. This specific law of transport is a relativistic generalization of the classical Larmor's theorem, and is of course of some physical significance.

However, we start the study of first and second e.m. deviations, not from this preferred parametrization, but from a general one. An additional reason for doing this is not only an attempt to obtain a more general formalism, which very often has also a practical significance as e.g. in the case of the Kerr solution, where one usually considers the parametrization along the infalling geodesics in terms of the coordinate r and not of the affine parameter, but it is mainly due to a larger flexibility of the concept of deviation based on such an approach. The general, reparametrization covariant e.m. deviation, determined by Eqs (2.4) in the case of the first and by Eqs (6.2) for the second e.m. deviation, admits the freedom of certain gauge transformations (cf. Prop. 2.2 and Props. 6.1 (*ii*), 6.2, respectively). Choosing appropriately the gauges, by fixing the function μ in Eqs (2.8) or ν in Eqs (6.7) respectively, one can obtain several different e.m. deviation fields both for the case of the first and of the second order.

One of the results connected with this approach is the identification of two types of e.m. deviation vector fields, of the natural e.m. deviation vector and of the e.m. separation vector, as well as the formulation of their laws of evolution and of the constraint conditions by which these laws must be supplemented. In the limiting case of geodesics, the concepts of the natural first geodesic deviation r_n^α and of the first geodesic separation vector r_s^α are nearly identical (the only difference follows from the constraint conditions $u_\alpha r_n^\alpha = \text{const}$ and $u_\alpha r_s^\alpha = 0$, respectively), but they still differ in the case of the second geodesic deviation, as it was already noted in [1].

Another result, related to that mentioned above is the determination of corresponding gauges, which in the limiting cases of the first geodesic deviation and the natural second geodesic deviation reduce to known results, but in the case of the second e.m. separation vector are new in the limiting case as well.

In Sects. 3, 4 and 7, 8, we give the geometric interpretation of the two types of e.m. deviations and in a future papers of ours [11] and [12] some important examples of the two types will be considered. Therefore, it is interesting to know the explicit gauge transformations which map the two types on each other. Such transformations have been found in Sects. 5 and 9, respectively. The separation vector seems to be well suited to the definition of conjugated points along a Lorentzian world line as well as to a study of singularities of spacetime, since in the limiting case of geodesics it is just this vector field which plays an important role in the proofs of the singularity theorems (cf. [6]).

In principle, the whole scheme discussed here applies to both spacelike as well as timelike Lorentzian world lines. The sign differences between the time- and spacelike character have been taken care of by the operation of the absolute value and by appropriately chosen indicator k. The null case is left as a challenge for the future.

Another possibility of a generalization of the approach discussed here is connected with a tacit assumption made in this paper that the charge-to-mass ratio is always the same, along all the world lines adjacent to the basic one. In a discussion of some more realistic physical models, one would rather expect that the charge-to-mass ratio changes as one passes from one of the world lines to the others.

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