THE LIE ALGEBRA OF THE POST-GALILEIAN RELATIVITY GROUP

BY P. MULTARZYŃSKI

Institute of Nuclear Physics, Cracow*

(Received January 31, 1986; revised version received August 4, 1986)

The Lie algebra of the post-Galileian relativity group is constructed and investigated.

PACS numbers: 02.20.+b

1. *Introduction*

A general Poincare transformation changes the coordinates of points of the Minkowski space-time in the following way:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + \alpha^{\mu}. \tag{1.1}$$

We can also present a general Poincare transformation of coordinates in the form:

$$\vec{x}' = \mathscr{R}\vec{x} + \vec{v} \cdot t + \vec{\alpha}, \tag{1.2}$$

$$t = \gamma t + c^{-2} \vec{v}^{\mathrm{T}} \mathscr{R} x + b, \tag{1.3}$$

where $\vec{x}^{\mathrm{T}} = (x^1, x^2, x^3)$, $ct = x^0$, $\vec{v}^{\mathrm{T}} = (v^1, v^2, v^3)$ and $\gamma = (1 - c^{-2}\vec{v}^{\mathrm{T}}\vec{v})^{-\frac{1}{2}}$. When $\vec{\alpha}$ and b vanish we obtain a Lorentz transformation which preserves the scalar product $(ct)^2 - \vec{x}^{\mathrm{T}}\vec{x}$ and, therefore, the 3×3 matrix $\mathscr{R}$ must satisfy the condition

$$\mathscr{R}^{\mathrm{T}}\mathscr{R} = 1 + \frac{\mathscr{R}^{\mathrm{T}}\vec{v}\vec{v}^{\mathrm{T}}\mathscr{R}}{c^2}. \tag{1.4}$$

One can easily check that the solution of the condition (1.4) may be written in the form

$$\mathscr{R} = \left(1 + \frac{\gamma^2}{\gamma+1} \cdot \frac{\vec{v}\vec{v}^{\mathrm{T}}}{c^2}\right) R, \tag{1.5}$$

where R denotes an arbitrary 3×3 orthogonal matrix.

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* Address: Instytut Fizyki Jądrowej, Radzikowskiego 152, 31-342 Kraków, Poland.

The aim of the present paper is to investigate the structure of the Lie algebra generated by the low-energy approximation to the relativity group described above. Executing this program we shall follow the way described in Ref. [1] and represent the corresponding quantities as the sums:

$$\vec{x} = \sum_{n=0}^{\infty} x_{2n+1}, \tag{1.6}$$

$$t = \sum_{n=0}^{\infty} t_{2n}, \tag{1.7}$$

$$\vec{v} = \sum_{n=0}^{\infty} \vec{v}_{2n+1}, \tag{1.8}$$

$$\mathscr{R} = \sum_{n=0}^{\infty} \mathscr{R}_{2n}, \tag{1.9}$$

$$R = \sum_{n=0}^{\infty} R_{2n}, \tag{1.10}$$

$$\vec{\alpha} = \sum_{n=0}^{\infty} \vec{\alpha}_{2n+1}, \tag{1.11}$$

$$b = \sum_{n=0}^{\infty} b_{2n}, \tag{1.12}$$

where the terms on the right-hand sides are labelled by a corresponding order of smallness.

Now we may substitute all the representations into (1.2) and (1.3). Comparing the terms of the same order of smallness on both sides we obtain an infinite series of transformation properties for $\vec{x}_{2n+1}$ and t_{2n}. When $n = 0$ we have the usual Galileian relativity group. The first-order corrections coming from relativistic physics appear in our considerations when $n \in \{0, 1\}$. In this case the transformations also form a group, called in this paper the post-Galileian group of the first. In the present paper we restrict our investigations to relativistic corrections of the first order. Therefore it will be quite enough for us to write explicitly the series of transformation properties only up to quantities of the third order of smallness:

$$t_0' = t_0 + b, \tag{1.13}$$

$$\vec{x}_1 = R_0\vec{x}_1 + \vec{v}_1 t_0 + \vec{\alpha}_1, \tag{1.14}$$

$$t_2' = t_2 + \frac{\vec{v}_1^{\mathrm{T}}\vec{v}_1}{2c^2} t_0 + \frac{\vec{v}_1^{\mathrm{T}} R_0 \vec{x}_1}{c^2} + b_2, \tag{1.15}$$

$$\vec{x}_3 = R_0\vec{x}_3 + Q\vec{x}_1 + \frac{\vec{v}_1^{\mathrm{T}} R_0 \vec{x}_1}{2c^2} v_1 + \vec{v}_1 t_2 + \left(\frac{\vec{v}_1\vec{v}_1^{\mathrm{T}}\vec{v}_1}{2c^2} + \vec{v}_3\right) t_0 + \vec{\alpha}_3, \tag{1.16}$$

where Q denotes R_2 and since R is an orthogonal matrix Q must fulfil the following condition:

$$R_0^{\mathrm{T}} Q + Q^{\mathrm{T}} R_0 = 0. \tag{1.17}$$

2. *The Lie algebra of the post-Galileian group*

For further investigations of the group of transformations given by the rules (1.13)–(1.16) it is necessary, however, to know the exact form of the matrix Q. The formula (1.17) is the only source of our information about Q. The left side of the formula (1.17) is a symmetrical matrix and, therefore, there are only three independent parameters in Q.

R_0 describes the pure rotation and, therefore, we may assume that it is a differentiable function of its parameters

$$R_0 \equiv R_0(\varphi^1, \varphi^2, \varphi^3). \tag{2.1}$$

One can easily check that

$$Q = \sum_{i=1}^{3} q^i \frac{\partial R_0}{\partial \varphi^i} (\varphi^1, \varphi^2, \varphi^3) \tag{2.2}$$

is a good solution of the condition (1.17).

Looking at the form of the matrix Q we see that it will not be easy to make use of it. Denoting by R_1, R_2, R_3 the generators of rotations we may write R_0 in the form

$$R_0 = \exp\left(\sum_{i=1}^{3} \varphi^i R_i\right). \tag{2.3}$$

In general, the rotations form a non-Abelian group and, therefore, it would be very difficult to find the composition law for the parameters $\{q_i\}$. Hence, instead of direct investigations of the post-Galileian group we present in this paper corresponding investigations of its Lie algebra.

The whole transformation given by (1.13)–(1.16) can be presented in a compact form

$$\vec{\xi}' = L\vec{\xi} + \vec{\xi}_0, \tag{2.4}$$

where $\vec{\xi}^{\mathrm{T}} = (t_0, t_2, \vec{x}_1^{\mathrm{T}}, \vec{x}_3^{\mathrm{T}})$ is a vector of eight-dimensional vector space, $\vec{\xi}_0$ is a constant vector corresponding to the translations and L is an automorphism (8×8 matrix) of the eight-dimensional vector space. The group of transformations L is parametrized by twelve parameters $(p_1, \ldots, p_{12}) \equiv (\vec{\varphi}, \vec{v}_1, \vec{q}, \vec{v}_3)$

$$L \equiv L(\vec{\varphi}, \vec{v}_1, \vec{q}, \vec{v}_3). \tag{2.5}$$

One can easily check that the mappings

$$p_i \to L(0, 0, \ldots, p_i, \ldots, 0) \tag{2.6}$$

are homomorphisms except the case when $i \in \{10, 11, 12\}$. But if we change $\vec{v}_3$ into $\vec{u}$ in the following way:

$$\vec{v}_3 \to \vec{u} := \vec{v}_3 + \frac{\vec{v}_1^{\mathrm{T}}\vec{v}_1}{3c^2}\vec{v}_1 \quad (2.7$$

we see that the mappings (2.6) are homomorphisms for every $i \in \{1, 2, \ldots, 12\}$ and therefore $L(\vec{\varphi}, \vec{v}, \vec{q}, \vec{u})$ — where for the sake of simplicity we have denoted $\vec{v}_1$ by $\vec{v}$ — has the exponential form

$$L(\vec{\varphi}, \vec{v}, \vec{q}, \vec{u}) = \exp\left(\sum_{i=1}^{3} \varphi^i G_{R_i} + v^i G_{V_i} + q^i G_{Q_i} + u^i G_{U_i}\right). \tag{2.8}$$

From the transformation rules (1.13)–(1.16) we see that the generators G_{R_i}, G_{V_i}, G_{Q_i}, G_{U_i} have the following form:

$$G_{R_i} \equiv \left.\frac{\partial L}{\partial \varphi^i}\right|_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_i & 0 \\ 0 & 0 & A_i \end{bmatrix}, \tag{2.9}$$

$$G_{V_i} \equiv \left.\frac{\partial L}{\partial v^i}\right|_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & c^{-2}\hat{n}_i^{\mathrm{T}} & 0 \\ \hat{n}_i & 0 & 0 & 0 \\ 0 & n_i & 0 & 0 \end{bmatrix}, \tag{2.10}$$

$$G_{Q_i} \equiv \left.\frac{\partial L}{\partial q^i}\right|_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_i & 0 \end{bmatrix}, \tag{2.11}$$

$$G_{U_i} \equiv \left.\frac{\partial L}{\partial u^i}\right|_0 = \begin{bmatrix} 0 & 0 \\ n_i & 0 \end{bmatrix}, \tag{2.12}$$

where the right-hand sides of the formulas denote, of course, 8×8 matrices, $\hat{n}_i$ is a unit 3-vector $((\hat{n}_i)_j = \delta_{ij})$ and $A_i = \varepsilon_{ikm}(\hat{n}_m \hat{n}_k^{\mathrm{T}} - \hat{n}_k \hat{n}_m^{\mathrm{T}})$.

The generators fulfil the following commutation relations:

$$[G_{R_i}, G_{R_k}] = \varepsilon_{ikm} G_{R_m}, \tag{2.13}$$

$$[G_{R_i}, G_{V_k}] = \varepsilon_{ikm} G_{V_m}, \tag{2.14}$$

$$[G_{R_i}, G_{Q_k}] = \varepsilon_{ikm} G_{Q_m}, \tag{2.15}$$

$$[G_{R_i}, G_{U_k}] = \varepsilon_{ikm} G_{U_m}, \tag{2.16}$$

$$[G_{V_i}, G_{V_k}] = -c^{-2} \varepsilon_{ikm} G_{Q_m}, \tag{2.17}$$

$$[G_{V_i}, G_{Q_k}] = \varepsilon_{ikm} G_{U_m}, \tag{2.18}$$

$$[G_{V_i}, G_{U_k}] = 0, \tag{2.19}$$

$$[G_{Q_i}, G_{Q_k}] = 0, \tag{2.20}$$

$$[G_{Q_i}, G_{U_k}] = 0, \tag{2.21}$$

$$[G_{U_i}, G_{U_k}] = 0. \tag{2.22}$$

The commutation relation between a generator G_L of the homogeneous part of the transformation (2.4) and a generator $G_{\hat{\xi}_0}$ of a translation has the form

$$[G_L, G_{\hat{\xi}_0}] = G_{G_L \cdot \hat{\xi}_0}. \tag{2.23}$$

Using the formula (2.23) we can now complete the list of commutation relations (2.13)–(2.22):

$$[G_{R_i}, G_{\hat{t}_0}] = 0, \tag{2.24}$$

$$[G_{R_i}, G_{\hat{t}_2}] = 0, \tag{2.25}$$

$$[G_{R_i}, G_{\hat{x}_{1k}}] = 2\varepsilon_{ikm} G_{\hat{x}_{1m}}, \tag{2.26}$$

$$[G_{R_i}, G_{\hat{x}_{3k}}] = 2\varepsilon_{ikm} G_{\hat{x}_{3m}}, \tag{2.27}$$

$$[G_{V_i}, G_{\hat{t}_0}] = G_{\hat{x}_{1i}}, \tag{2.28}$$

$$[G_{V_i}, G_{\hat{t}_2}] = G_{\hat{x}_{3i}}, \tag{2.29}$$

$$[G_{V_i}, G_{\hat{x}_{1k}}] = c^{-2} \delta_{ik} G_{\hat{t}_2}, \tag{2.30}$$

$$[G_{Vi}, G_{\hat{x}_{3k}}] = 0, \tag{2.31}$$

$$[G_{Q_i}, G_{\hat{t}_0}] = 0, \tag{2.32}$$

$$[G_{Q_i}, G_{\hat{t}_2}] = 0, \tag{2.33}$$

$$[G_{Q_i}, G_{\hat{x}_{1k}}] = 2\varepsilon_{ikm} G_{\hat{x}_{3m}}, \tag{2.34}$$

$$[G_{Q_i}, G_{\hat{x}_{3k}}] = 0, \tag{2.35}$$

$$[G_{U_i}, G_{\hat{t}_0}] = G_{\hat{x}_{3i}}, \tag{2.36}$$

$$[G_{U_i}, G_{\hat{t}_2}] = 0, \tag{2.37}$$

$$[G_{U_i}, G_{\hat{x}_{1k}}] = 0, \tag{2.38}$$

$$[G_{U_i}, G_{\hat{x}_{3k}}] = 0. \tag{2.39}$$

The pure translation commutators obviously vanish.

3. Classification of the post-Galileian Lie algebra

In order to investigate the physical consequences of the post-Galileian relativity principle, one should know the representations of the Lie algebra constructed in Sect. 2.

The representations of Lie groups and Lie algebras are vastly studied in mathematical literature [3, 6]. All Lie algebras are well classified and up to isomorphism there are only some types of them. Every type of Lie algebra has its own representations and, therefore, also the physical consequences are different for different types of algebras. One must then first determine the type of a Lie algebra in order to read out its representations and then its physical aspects. In particular, we have in mind the quantum numbers of particles described by the representations of our group.

Let us denote with $\mathscr{L}$ the post-Galileian algebra given by the commutation relations (2.13)–(2.22) and (2.24)–(2.39). The algebra $\mathscr{L}$ is a real twenty-dimensional Lie algebra.

In order to recognize what type of algebra we have, two sequences are introduced:

$$\mathscr{L}^{(0)} := \mathscr{L}, \quad \mathscr{L}^{(n+1)} := [\mathscr{L}^{(n)}, \mathscr{L}^{(n)}], \tag{3.1}$$

$$\mathscr{L}_{(0)} := \mathscr{L}, \quad \mathscr{L}_{(n+1)} := [\mathscr{L}_{(n)}, \mathscr{L}], \tag{3.2}$$

where $n = 0, 1, 2, \ldots$. One can easily see that for the algebra $\mathscr{L}$ the following fact comes true:

$$\mathscr{L}^{(1)} = \mathscr{L}_{(1)} = \mathscr{L} \tag{3.3}$$

and, therefore, the algebra $\mathscr{L}$ is not solvable as well as it is not nilpotent. It is also not semisimple because it has non-zero commutative ideal which is a subalgebra generated by G_{Q_i} and G_{U_k}. There is also another reason why $\mathscr{L}$ cannot be semisimple: the Killing form of algebra $\mathscr{L}$, Cartan metric tensor of which is:

$$g_{A_iB_j} = -24\delta_{AR} \cdot \delta_{BR} \cdot \delta_{ij}, \tag{3.4}$$

is strongly degenerated. As we see from (3.4), $g_{A_iB_j}$ is diagonal and it has only three non--zero elements which come from rotations G_{R_i}. Moreover, the algebra $\mathscr{L}$ is not simple because it has many ideals which are different than $\{0\}$ and $\mathscr{L}$.

The subalgebra of $\mathscr{L}$ generated by G_V, G_Q, G_U and by generators of translations is an ideal of $\mathscr{L}$. Denoting that ideal by I, we obtain that

$$I^{(3)} = \{0\} \tag{3.5}$$

and

$$I_{(4)} = \{0\}. \tag{3.6}$$

From (3.6) (and directly from (3.5)) we see that the ideal I is solvable. It is easy to see that I is the maximal solvable ideal of $\mathscr{L}$ (the so-called radical) and, therefore, the quotient algebra $\mathscr{L}/I$ is semisimple (and isomorphic with the subalgebra generated by rotations G_R).

From the Levi-Malcev theorem we know that there exists a semisimple subalgebra S of $\mathscr{L}$ such that $\mathscr{L}$ is given by

$$\mathscr{L} = I \oplus_s S, \tag{3.7}$$

where the symbol $\oplus_s$ denotes the semidirect sum of two Lie algebras. S is obviously equal to the subalgebra generated by G_R because there are no more subalgebras in $\mathscr{L}$ which would be semisimple and larger than the one generated by G_R. In this way we have obtained the well-known fact that the subalgebra generated by G_R (the rotation Lie algebra) is semisimple. Denoting the subalgebra generated by G_R with $\mathscr{L}_R$ we have that:

$$\mathscr{L} = I \oplus_s \mathscr{L}_R. \tag{3.8}$$

Because the algebra $\mathscr{L}_R$ is very well known it will be quite sufficient for further investigations of $\mathscr{L}$ to consider the radical I only.

From (3.6) we see that I has a nontrivial center $I_{(3)}$ which is the subalgebra of $\mathscr{L}$ generated by $G_{\hat{x}_{3k}}$.

According to the theorem of Ado every Lie algebra is isomorphic to some linear subalgebra of the full linear algebra gl (n, C). Therefore we can represent I by certain algebra of matrices.

Let us denote with $T^{(m)}$ the vector space of all $m \times m$ upper triangular matrices with equal diagonal elements and let $T^{(m_1, m_2, \ldots, m_k)}$ denote the set of all linear transformations acting in the space

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_k \tag{3.9}$$

being a direct sum of vector spaces $V_1, V_2, \ldots, V_k$ in such a way that:

1) the subspaces V_i, $i = 1, 2, \ldots, k$, are invariant with respect to transformations $A \in T^{(m_1, m_2, \ldots, m_k)}$,
2) in each subspace V_i with the basis $\{u_j^{(i)}\}_{j=1}^{m_i}$ every $A \in T^{(m_i)}$ has the form

$$\begin{bmatrix} \lambda_i & & A_{kl}^{(i)} \\ & \lambda_i & \\ 0 & & \ddots \\ & & & \lambda_i \end{bmatrix}. \tag{3.10}$$

Because the matrices $A \in T^{(m_1, m_2, \ldots, m_k)}$ are triangular they form a Lie algebra. An arbitrary nilpotent linear Lie algebra is isomorphic to a subalgebra of some Lie algebra $T^{(m_1, m_2, \ldots, m_k)}$ (for the proof see [2]).

According to that fact and to the theorem of Ado, the radical I which is nilpotent because of (3.6), can be represented by triangular matrices.

A connected Lie group is called solvable, nilpotent, Abelian, simple or semisimple, if its Lie algebra has one of those properties.

Now, when we already know the type of the post-Galileian Lie algebra we can use the known mathematical methods to construct its representations. The next and the most important question would be the physical interpretation of the parameters which determine the given representation. This problem, however, will be discussed in a future paper.

The author is indebted to Professor E. Kapuścik for suggesting these investigations, for valuable discussions and encouragement. I would also like to express my gratitude to J. Szczęsny for many critical remarks.

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