THE NONABELIAN STOKES THEOREM AND BIANCHI IDENTITIES

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The path-ordered phase factors in nonabelian SU(N) gauge theory are analyzed and relation between the nonabelian Stokes theorem and the Bianchi identities is found.

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1. Introduction

Gauge invariant non-local operators play fundamental rôle in several programs formulated in recent years within the scope of the quantum chromodynamics (QCD) [1].

One of the methods employed in continuous QCD in the search for the nonperturbative (in the traditional sense) solutions is the expansion in 1/(number of colors). Within that approach the theory of QCD is reformulated in terms of objects having no color degrees of freedom. These are Wilson loop operators [2] constructed as traces of path-ordered phase factors:

$$U_x^y(C; A) = P\exp\left[-ig\int_{C_{yx}} A_\mu(x)dx^\mu\right], \tag{1.1}$$

where the field $A_\mu(x)$ takes values in the Lie algebra of $SU(N)$:

$$A_\mu(x) = \sum_{n=1}^{N^2-1} t^a A_\mu^a(x). \tag{1.2}$$

The path ordering operator P is defined for arbitrary curve C: if C_{xy} is a curve parametrized by $s \in \langle 0, 1\rangle : x_\mu = x_\mu(s)$ and $x = x(0)$, $x(1) = y$ then

$$U_x^y(C; A) = \sum_{n=0}^{\infty} (-ig)^n \int_0^1 ds_1 \int_0^{s_1} ds_2 \dots \int_0^{s_{n-1}} ds_n \frac{dx^{\mu_1}}{ds_1}(s_1) \dots \frac{dx^{\mu_n}}{ds_n}(s_n) A_{\mu_1}(s_1) \dots A_{\mu_n}(s_n). \tag{1.3}$$

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If the curve is closed, which is the case of Wilson loop operator, then $x = y$ and we have:

$$W(l) = \mathrm{tr}\, U_x^y(l).$$

The classical Yang-Mills theories can be formulated in terms of the path-ordered phase factors $U_x^y(C, A)$, where C is arbitrary curve with endpoints x and y. This version of classical Yang-Mills theory is a very natural and intuitive one [3, 4]. The aim of the present paper is to formulate the nonabelian Stokes theorem in a heuristic way (the formal proof of the theorem was given by Areféva [5]) and to point out the crucial role of the Bianchi identities for validity of this theorem.

2. *Heuristic formulation of nonabelian Stokes theorem*

We are now going to present a proof of the nonabelian Stokes theorem, which bears some resemblance to the one given by Menski [4]. The nonabelian Stokes theorem can be formulated in terms of path-ordered phase factors in a very natural way. As the ordered exponents of integrals, and not integrals themselves constitute here the fundamental objects, the addition of integrals is replaced by the multiplication of phase factors. We shall make use of the following properties of the ordered phase factors:

$$U_y^x(C^{-1}; A) \cdot U_x^y(C; A) = \mathbf{1}, \tag{2.1}$$

$$U_{y_1}^{y_2}(C_2; A) \cdot U_x^{y_1}(C_1; A) = U_x^{y_2}(C_2 \circ C_1; A), \tag{2.2}$$

where C is any curve connecting the point x and the point y, C^{-1} is the curve C with reversed orientation, C_1 is a curve originating in x and ending in y_1, C_2 is a curve originating in y_1 and ending in y_2, $C_2 \circ C_1$ is the conjunction of curves C_1 and C_2 and it connects the point x with the point y_2. With any loop attached to the point x we associate the corresponding operator $U_x^x(l)$. We are now going to transform $U_x^x(l)$ so as to convert its loop integral into the surface integral over S, with $\delta S = l$.

We begin by decomposing the loop l into a great number of infinitesimal loops Δl_k, each enclosing a small elements dS_k of the surface S respectively. dS_k should be small by diameter in addition. Fig. 1 illustrates the way this decomposition can be achieved.

The path-ordered phase factor $U_x^x(l)$ admits the following representation:

$$U_x^x(l) = \lim_{N \to \infty} P \prod_{k=1}^{N} U_{x_k}^x(C_k^{-1}; A) \cdot U_{x_k}^{x_k}(\Delta l_k) \cdot U_x^{x_k}(C_k; A), \tag{2.3}$$

where the term

$$U_{x_k}^x(C_k^{-1}; A) \cdot U_{y_k}^{x_k}(\Delta l_k) \cdot U_x^{x_k}(C_k; A)$$

is the integral contribution of elementary lasso (Fig. 2).

This is a consequence of properties (2.1) and (2.2) if the symbol P is taken to stand for the following two conditions:

a) every lasso begins and terminates in x,

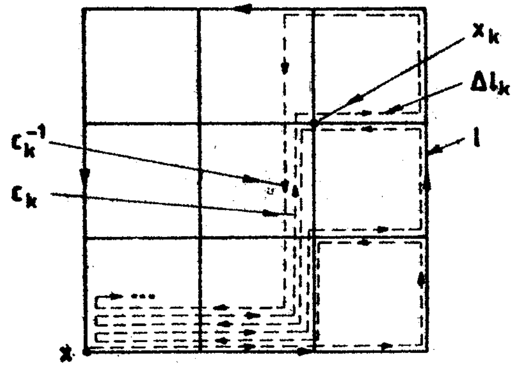

Fig. 1. The method of decomposition of the loop l into a great number of infinitesimal loops Δl_k

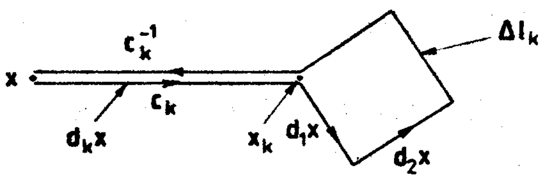

Fig. 2. The elementary lasso

b) each successive loop cancels in part with the preceding one, so as to reproduce the loop l when the tour is completed.

C_k is a curve connecting the point x with x_k, which is the point of attachment for Δl_k. The calculation of $U_{x_k}^{x_k}(\Delta l_k)$ gives

$$U_{x_k}^{x_k}(\Delta l_k) = 1 - \frac{ig}{2} F_{\mu\nu}(x_k) dS_k^{\mu\nu}, \tag{2.4}$$

where $F_{\mu\nu}$ is the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \tag{2.5}$$

$$dS^{\mu\nu} = d_1 x^\mu d_2 x^\nu - d_1 x^\nu d_2 x^\mu. \tag{2.6}$$

Hence we have

$$U_x^x(l) = \lim_{N\to\infty} P \prod_{k=1}^{N} \left\{ 1 - \frac{ig}{2} U_{x_k}^x(C_k^{-1}; A) F_{\mu\nu}(x_k) U_x^{x_k}(C_k; A) dS_k^{\mu\nu} \right\}. \tag{2.7}$$

On introducing the nonlocal counter part of the field strength tensor

$$\mathscr{F}_{\mu\nu}(C_k, x_k) = U_{x_k}^x(C_k^{-1}; A) F_{\mu\nu}(x_k) U_x^{x_k}(C_k; A) \tag{2.8}$$

one obtains:

$$U_x^x(l) = \lim_{N\to\infty} P \prod_{k=1}^{N} \exp\left[\frac{-ig}{2}\mathscr{F}_{\mu\nu}(C_k, x_k)dS_k^{\mu\nu}\right]$$

$$= P\exp\left[\frac{-ig}{2}\int_{S;\partial S=l} \mathscr{F}_{\mu\nu}(C_x, x)dS^{\mu\nu}\right]. \tag{2.9}$$

Writing the $U_x^x(l)$ out explicitly we finally have:

$$P\exp\left[-ig\oint_{\partial S} A_\mu(x)dx^\mu\right] = P\exp\left[\frac{-ig}{2}\int_S \mathscr{F}_{\mu\nu}(C_x, x)dS^{\mu\nu}\right] \tag{2.10}$$

The relation (2.10) is the nonabelian version of the Stokes theorem.

3. *The ordered-phase factor $U_x^x(l)$ independence of the surface S*

The consequence of (2.10) is, that all expressions

$$P\exp\left[\frac{-ig}{2}\int_S \mathscr{F}_{\mu\nu}(C_x; x)dS^{\mu\nu}\right]$$

(∂S fixed) must give the same quantity as the l.h.s. depends on ∂S only. Therefore, if $\partial S_1 = \partial S_2$ then we should have:

$$\left\{P\exp\left[\frac{-ig}{2}\int_{S_2} \mathscr{F}_{\mu\nu}(C_x, x)dS^{\mu\nu}\right]\right\}\cdot\left\{P\exp\left[\frac{-ig}{2}\int_{S_1} \mathscr{F}_{\mu\nu}(C_x, x)dS^{\mu\nu}\right]\right\}^{-1} = 1. \tag{3.1}$$

We shall show that the Bianchi identities:

$$D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0, \tag{3.2}$$

where

$$D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + ig[A_\lambda, F_{\mu\nu}] \tag{3.3}$$

form the necessary condition for this independence of S to hold. Our geometrical construction is shown in Fig. 3.

The two surfaces S_1 and S_2 have common edge $\partial S_1 = \partial S_2 = l$. With the use of nonabelian Stokes theorem the ordered exponent of surface integral:

$$P\exp\left[\frac{-ig}{2}\int_{S_1} \mathscr{F}_{\mu\nu}(C_x, x)dS^{\mu\nu}\right]$$

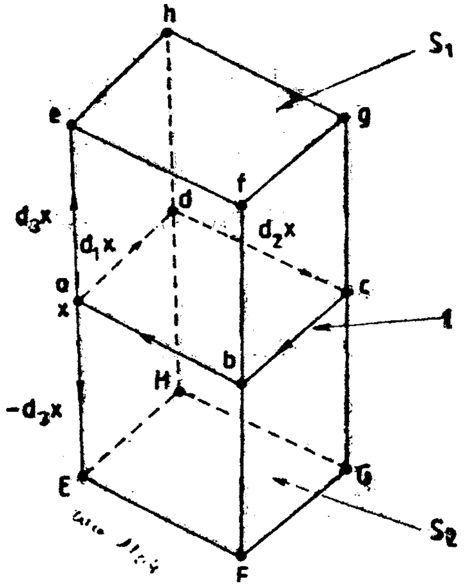

Fig. 3. Two surfaces having common edge $\partial S_1 = \partial S_2 = l$

can be expressed as the ordered of path integral, along the conjunction of five elementary lassos:

(aehgfeadcghdabfgcbaefbadhea).

For one elementary lasso one has (Fig. 2):

$$U_{x_k}^{x_k}(C_{x_k}; A) = 1 - \frac{ig}{2} F_{\mu\nu}(x_k) dS^{\mu\nu} - \frac{ig}{2} D_\lambda F_{\mu\nu}(x_k) d_k x^\lambda dS^{\mu\nu}. \tag{3.4}$$

For the whole indicated curve this yields:

$$P \exp\left[\frac{-ig}{2} \int_{S_1} \mathcal{F}_{\mu\nu}(C_x, x) dS^{\mu\nu}\right] = 1 - \frac{ig}{2} F_{\mu\nu}(x) dS^{\mu\nu} - \frac{ig}{2} dV^{\lambda\mu\nu} D_\lambda F_{\mu\nu}(x), \tag{3.5}$$

$$dV^{\lambda\mu\nu} = \begin{vmatrix} d_1x^\lambda & d_2x^\lambda & d_3x^\lambda \\ d_1x^\mu & d_2x^\mu & d_3x^\mu \\ d_1x & d_2x^\nu & d_3x^\nu \end{vmatrix}. \tag{3.6}$$

Considering the surface S and the path:

(aEHGFEadcGHdabFGcbaEFbadHEa)

one obtains respectively:

$$P \exp\left[\frac{-ig}{2} \int_{S_2} \mathcal{F}_{\mu\nu}(C_x, x) dS^{\mu\nu}\right] = 1 - \frac{ig}{2} dS^{\mu\nu} F_{\mu\nu}(x) + \frac{ig}{2} dV^{\lambda\mu\nu} D_\lambda F_{\mu\nu}. \tag{3.7}$$

Hence

$$\left\{P\exp\left[\frac{-ig}{2}\int\limits_{S_2}\mathscr{F}_{\mu\nu}(C_x,x)dS^{\mu\nu}\right]\right\}\cdot\left\{P\exp\left[\frac{-ig}{2}\int\limits_{S_1}\mathscr{F}_{\mu\nu}(C_x,x)dS^{\mu\nu}\right]\right\}^{-1}$$

$$=1+\frac{ig}{3}dV^{\lambda\mu\nu}(D_\lambda F_{\mu\nu}+D_\mu F_{\nu\lambda}+D_\nu F_{\lambda\mu}). \tag{3.8}$$

The validity of Bianchi identities

$$D_\lambda F_{\mu\nu}+D_\mu F_{\nu\lambda}+D_\nu F_{\lambda\mu}=0$$

is therefore the necessary condition for

$$P\exp\left[-\frac{ig}{2}\int\limits_{S}\mathscr{F}_{\mu\nu}(C_x,x)dS^{\mu\nu}\right]$$

to depend only on ∂S.

An example of the Yang-Mills field for which this condition does not hold at some points is the configuration of nonabelian monopole, which had been considered by Chan Hong-Mo and Tsou Sheung Tsun [6].

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