ON SO(p, q) PURE SPINORS

By S. Giler, P. Kosiński, J. Rembieliński

Institute of Physics, University of Łódź*

and P. Maślanka

Institute of Mathematics, University of Łódź**

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Using the group-theoretical methods and the geometrical picture of pure spinors due to Cartan and Chevalley, we give the explicit construction of the manifold of such spinors for the group SO(p, q).

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1. *Introduction*

The nonlinear realization of groups arose in physics more than a decade ago in connection with current algebra and low-energy hadron physics [1–5]. This approach is known under the name of the method of effective Lagrangians. Some models, like for example the nonlinear sigma model, are interesting on its own right [6]. However, of all the investigated models, to our knowledge, the essentially nonlinear are bosonic fields. It was noticed recently [7] that a natural basis for nonlinear realization in the case of fermion fields is provided by E. Cartan theory of pure spinors [8, 9]. In this paper we discuss some topics concerning the mathematical structure of pure spinors for the pseudo-orthogonal groups SO(p, q). We emphasize the geometrical aspects of the problem. Similar results can be obtained by algebraic methods [10]. In Sect. 2 we remind some basic notions concerning the Chevalley construction of the representation of Clifford algebra and introduce the notion of pure spinors for the group SO(n, C) and its real forms SO(p, q), $p+q = n$. Section 3 is devoted to the general method of constructing of the manifold of pure spinors. In Sect. 4 we give a detailed discussion of the properties of SO(ν, ν) pure spinors. The general case is investigated in Sections 5 and 6.

* Address: Instytu Fizyki, Uniwersytet Łódzki, Nowotki 149/153, 90-236 Łódź, Poland.

** Address: Instytut Matematyki, Uniwersytet Łódzki, Banacha 22, 90-238 Łódź, Poland.

2. Preliminaries

In this section we remind some basic notions concerning the orthogonal groups and Clifford algebras.

A. The group SO(p, q) and the Witt decomposition

$SO(p, q)$ is the real form of the group $SO(p+q, C)$ or, equivalently, $SO(p+q, C)$ is the complex extension of $SO(p, q)$. Let M be a $p+q$-dimensional vector space over C, with $SO(p+q, C)$-invariant symmetric, nonsingular scalar product. Now, with a convenient change of basis in M, it is always possible to choose for our problem the metric tensor $\check{g}$ in the form

$$\check{g} \doteq [g_{\alpha\beta}] = \left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array}\right), \tag{2.1}$$

where I_p, resp. I_q, is $p\times p$, resp. $q\times q$, unit matrix. For definiteness we assume that $p \geqslant q$. All real transformations from $SO(p+q, C)$ leaving invariant the scalar product determined by $\check{g}$, form the group $SO(p, q)$. Namely, $\check{O} \in SO(p, q)$ if $\det \check{O} = 1$, $\check{O}^* = \check{O}$ and

$$\check{O}^{T}\check{g}\check{O} = \check{g}. \tag{2.2}$$

Consequently, if $\check{L}$ belongs to the Lie algebra of $SO(p, q)$, then (2.2) implies

$$\check{g}^{-1}\check{L}^{T}\check{g} = -\check{L}. \tag{2.3}$$

In the following we adopt the convention without imaginary unit in the definition of infinitesimal rotations.

According to the Witt theorem [9] in the case $p+q = \dim M =$ even, M is the direct sum of two maximal totally singular subspaces, say N and P:

$$M = N \oplus P,$$

with equal dimension $\dim N = \dim P = \dfrac{p+q}{2} \equiv \nu$. For odd $p+q \equiv 2\nu+1$

$$M = V_0 \oplus N \oplus P,$$

where V_0 is a one-dimensional subspace of M.

Note that the Witt decomposition is not unique. One possible choice is given by the following unitary transformation of coordinates

$$x^{\alpha} = \check{R}^{\alpha}_{\beta}\check{x}^{\beta}, \tag{2.4}$$

where $\alpha, \beta = 1, \ldots, 2\nu$ or $\alpha, \beta = 0, 1, \ldots, 2\nu$ for even and odd $p+q$ respectively, and

$$\check{R} = \frac{1}{\sqrt{2}}\left(\begin{array}{c|cc} I_\nu & iI_{\nu-q} & 0 \\ & 0 & -I_q \\ \hline I_\nu & -iI_{\nu-q} & 0 \\ & 0 & I_q \end{array}\right) \tag{2.4a}$$

for $p+q$ even, and

$$\check{R} = \frac{1}{\sqrt{2}}\left[\begin{array}{c|c|c|c} 1 & 0 & \multicolumn{2}{c}{0} \\ \hline 0 & I_\nu & iI_{\nu-q} & 0 \\ \cline{3-4} & & 0 & -I_q \\ \hline 0 & I_\nu & -iI_{\nu-q} & 0 \\ \cline{3-4} & & 0 & I_q \end{array}\right] \tag{2.4b}$$

for $p+q$ odd.

The metric tensor takes the form

$$g = \check{R}^{-1\mathrm{T}}\check{g}\check{R}^{-1} \tag{2.5}$$

i.e.

$$g = \left(\begin{array}{c|c} 0 & I_\nu \\ \hline I_\nu & 0 \end{array}\right) \tag{2.5a}$$

for $p+q$ even, and

$$g = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & I_\nu \\ \hline 0 & I_\nu & 0 \end{array}\right) \tag{2.5b}$$

for $p+q$ odd.

The subspaces N and P are spanned in both the cases by the coordinates

$$x_N^i = \frac{1}{\sqrt{2}}(\check{x}^i + i\check{x}^{i+\nu}), \quad x_N^k = \frac{1}{\sqrt{2}}(\check{x}^k - \check{x}^{k+\nu}),$$

$$x_P^i = \frac{1}{\sqrt{2}}(\check{x}^i - i\check{x}^{i+\nu}), \quad x_P^k = \frac{1}{\sqrt{2}}(\check{x}^k + \check{x}^{k+\nu}), \tag{2.6}$$

where $i = 1, 2, \ldots, (\nu-q)$, $k = (\nu-q+1), \ldots, \nu$. In the odd case, V_0 is spanned by $x^0 = \check{x}^0$. Note that

$$x_N^a = x_{Pa}, \quad x_P^a = x_{Na} \quad a = 1, \ldots \nu. \tag{2.7}$$

B. The Clifford algebra

In the sequel we consider the complex Clifford algebra C^n with $n = p+q$ and the anti-commutation rules for generating elements chosen appropriately to our problem (see Eq. (2.1)) as

$$\{\check{\gamma}_\alpha, \check{\gamma}_\beta\} = 2\check{g}_{\alpha\beta}\hat{I}. \tag{2.8}$$

The vector space spanned by the generating elements of C^n is naturally isomorphic to M by identification

$$[x^\alpha] \equiv \check{x} = \check{x}^\alpha\check{\gamma}_\alpha. \tag{2.9}$$

In the Witt basis defined by Eq. (2.6), we have

$$\{\gamma_N^a, \gamma_N^b\} = 0, \quad \{\gamma_P^a, \gamma_P^b\} = 0, \quad \{\gamma_N^a, \gamma_P^b\} = 2\delta^{ab}\hat{1} \tag{2.10}$$

for $a, b = 1, 2, \ldots, \nu$, and in addition

$$\gamma^{0^2} = \hat{1}, \quad \{\gamma^0, \gamma_N^a\} = \{\gamma^0, \gamma_P^a\} = 0$$

if $p+q$ is odd.
Furthermore,

$$x = x_{N_a}\gamma_N^a + x_{P_a}\gamma_P^a \quad (p+q \text{ even})$$

or

$$x = x_0\gamma^0 + x_{N_a}\gamma_N^a + x_{P_a}\gamma_P^a \quad (p+q \text{ odd}).$$

C. The Chevalley construction

In the case of Clifford algebra under consideration one can give the elegant construction of its representation resembling the construction of the adjoint representation for Lie algebra [9]. This is possible because of the existence of the Witt decomposition. In the following we restrict ourselves to the case $p+q$ even; for the extension to the odd $p+q$ case see Chevalley [9]. First, note that vectors belonging to N, resp. P, can be written as $x_N = x_N^a\gamma_{N_a}$, resp. $x_P = x_P^a\gamma_{P_a}$. The elements γ_{N_a} and γ_{P_a} generate two 2^ν-dimensional Grassman algebras C^N and C^P over N and P, respectively. Let f_P be the element of C^P of maximal order, namely

$$f_P = \gamma_{P_1}\gamma_{P_2} \cdots \gamma_{P_\nu} \tag{2.11}$$

It is obvious that f_P, up to a multiplicative factor, does not depend on the particular choice of basis in P. Furthermore, let us consider the left ideal $C^n f_P = C^N f_P$. This ideal spans the space of the representation ϱ of the Clifford algebra. We define this representation by the formula

$$\varrho(u)C^N f_P = uC^N f_P \subset C^N f_P \tag{2.12}$$

for each $u \in C^n$.

Note that the generators $\gamma_{N_a} = \gamma_P^a$ resp. $\gamma_{P_a} = \gamma_N^a$ act under ϱ as Grassman multiplication, resp. differentiation. Finally, the representation ϱ determines the representations of the Clifford group as well as the groups Pin and Spin [9].

D. Pure spinors

Let Z be a maximal totally singular subspace and let $M = Z \oplus Z'$ be the corresponding Witt decomposition (we will consider the even case). We put

$$f_z = \gamma_{z_1}\gamma_{z_2} \cdots \gamma_{z_\nu}.$$

As previously, f_z is determined by the choice of Z up to a multiplicative factor. Now $f_z C^n = f_z C^{Z'}$ is the minimal right ideal in C^n. It follows then that the intersection $C^N f_P$

$\cap f_Z C^{Z'}$ is the one-dimensional subspace of C^n [9]. Consequently, we can write

$$C^N f_P \cap f_Z C^{Z'} = \{s_Z f_P\}, \tag{2.13}$$

where $s_Z \in C^N$. We call this one-dimensional subspace, the space of SO(n, C) pure spinors associated with the maximal totally singular subspace Z. Note that for the particular case $Z = P$, this subspace has the form βf_P with $\beta \in C$. Equivalently, a pure spinor ψ associated with the subspace Z is determined up to a multiplicative factor by the equations:

$$z^\alpha \gamma_\alpha \psi = 0 \tag{2.14}$$

for each $z^\alpha \gamma_\alpha \in Z$.

It is obvious that SO(n, C) pure spinors form a nonlinear realization of SO(n, C). The manifold of pure spinors can be obtained by the group action on a fixed standard pure spinor, so it is simply the orbit of the group. The SO(p, q) pure spinors are identified with the points of the SO(p, q) orbit in the manifold of all SO(n, C) pure spinors.

3. The construction of pure spinors

From the discussion given above we conclude that SO(p, q) pure spinors form a nonlinear realization of this group. Having this in mind we can proceed in the standard fashion [2, 4] and construct the orbit of pure spinors as follows. We choose an arbitrary but fixed "standard" pure spinor and determine its stability group $G_0 \in$ SO(p, q). The group SO(p, q) acts transitively on the coset manifold SO(p, q)/G_0. It follows that we can identify the manifold of pure spinors with the above coset space. Every pure spinor can be obtained from the standard one by applying a suitable transformation from SO(p, q).

A. The stability group G_0

Let π denote the homomorphism, π: Spin (p, q) $\to$ SO(p, q), i.e. for $s \in$ Spin (p, q); $\pi(s) = O(s) \in$ SO(p, q). Putting

$$s_z f_p = f_z w$$

(compare with Eq (2.13)) we obtain

$$\varrho(s) s_z f_P = s f_z s^{-1}(sw)$$

but

$$f_z s^{-1} = \prod_{i=1}^{\nu} (\gamma_{z_i} s^{-1}).$$

Taking into account that

$$\gamma_\alpha s^{-1} = O^\beta_\alpha(s^{-1})\gamma_\beta$$

we conclude that

$$f_z s^{-1} = \beta(s) f_z, \quad \beta(s) \in R$$

if and only if the subspace Z is invariant under the action of $O(s)$. Then

$$\varrho(s)s_z f_P = f_z\beta(s)sw \in f_z C^{Z'} \cap C^N f_P.$$

It follows from the above consideration that the stability subgroup of the one-dimensional subspace of pure spinors associated with Z, consists of those elements of $SO(p, q)$ which leave Z invariant. Let us choose the pure spinor f_P related to the subspace P as the standard one, i.e. we put $Z = P$. It is easily seen from the above that the stability group of P should leave invariant the vectors of the form $\begin{pmatrix} (x_N^k) \\ 0 \end{pmatrix}$ in the even case or $\begin{pmatrix} 0 \\ (x_N^k) \\ 0 \end{pmatrix}$ the odd case. However, we need to know the stability group of f_P rather than the subspace P, namely the elements $s \in \mathrm{Spin}(p, q)$ such that

$$sf_P = f_P. \tag{3.1}$$

Let α be the main involution of the Clifford algebra [9]. Acting with α on both sides of Eq. (3.1), we get

$$f_P\alpha(s) = f_P. \tag{3.2}$$

On the other hand, multiplying Eq. (3.1) by $\alpha(s)$, we obtain

$$f_P = \pm\alpha(s)f_P. \tag{3.3}$$

From Eqs (3.2) and (3.3) we have

$$\alpha(s)f_P = \pm f_P\alpha(s).$$

Applying the operation α to the above equation we get

$$sf_P = \pm f_P s.$$

The continuity argument makes us choose the plus sign in the last equation, i.e.:

$$sf_P s^{-1} = f_P.$$

It follows then that $\pi(s)$ restricted to the subspace P should have the determinant equal to one.

Now, the condition that a global transformation of $SO(p, q)$ leaves the subspace P unchanged, can be rewritten in terms of elements of the Lie algebra as

$$(I - \Pi_P)L\Pi_P = 0, \tag{3.4}$$

where Π_P projects on the subspace P, and L is the general element of the Lie algebra $SO(p, q)$. In the following, Π_P and L will be used in the Witt basis.

The condition on the determinant mentioned above can be written in the form

$$\det_P(\Pi_P e^L \Pi_P) = 1. \tag{3.5}$$

Finally, the general pure spinor is obtained by action on the standard spinor, say f_P, the transformations from the part of $SO(p, q)$ corresponding to $SO(p, q)/G_0$, where $G_0 \subset SO(p, q)$ is a stability group of f_P.

4. *The case of* SO(ν, ν)

In this Section we restrict ourselves to the simplest case $SO(\nu, \nu)$ [12]. The general element of the $SO(\nu, \nu)$ Lie algebra has a form

$$\check{L} = \left(\begin{array}{c|c} \check{A}_+ & \check{S} \\ \hline \check{S}^{\mathrm{T}} & \check{A}_- \end{array}\right)$$

in the Cartesian basis, or

$$L = \left(\begin{array}{c|c} R & A \\ \hline B & -R^{\mathrm{T}} \end{array}\right) \tag{4.1}$$

in the Witt basis. Here, $\check{A}_\pm$, $\check{S}$, A, R and B are real matrices and $\check{A}^{\mathrm{T}}_\pm = -\check{A}_\pm$, $A^{\mathrm{T}} = -A$, $B^{\mathrm{T}} = -B$. In the Witt basis the projector Π_P has the simple form

$$\Pi_P = \left(\begin{array}{c|c} I_\nu & 0 \\ \hline 0 & 0 \end{array}\right). \tag{4.2}$$

From the conditions (3.4–3.5) and from the general form of L (Eq. (4.1)) in the Witt basis, we deduce that the general element of the Lie algebra of the stability group $G_0 \subset SO(\nu, \nu)$ of f_P is

$$L_0 = \left(\begin{array}{c|c} R_0 & A \\ \hline 0 & -R_0^{\mathrm{T}} \end{array}\right), \tag{4.3}$$

where $A^{\mathrm{T}} = -A$, $\mathrm{Tr}\, R_0 = 0$.

The Lie algebra of G_0 contains two subalgebras

$$\mathscr{R}_0 = \left\{\left(\begin{array}{c|c} R_0 & 0 \\ \hline 0 & -R_0^{\mathrm{T}} \end{array}\right)\right\} \quad \text{and} \quad \mathscr{A} = \left\{\left(\begin{array}{c|c} 0 & A \\ \hline 0 & 0 \end{array}\right)\right\}.$$

$\mathscr{R}_0$ is isomorphic to the $sl(\nu, \boldsymbol{R})$ Lie algebra, while $\mathscr{A}$ is the $\binom{\nu}{2}$-dimensional Abelian algebra.

Moreover, $\mathscr{A}$ is an ideal in the subalgebra under consideration. The general element of G_0 can be written in the form

$$\exp \mathscr{A} \cdot \exp \mathscr{R}_0 = \left(\begin{array}{c|c} I & A \\ \hline 0 & I \end{array}\right) \cdot \left(\begin{array}{c|c} \exp R_0 & 0 \\ \hline 0 & \exp(-R_0^{\mathrm{T}}) \end{array}\right) \equiv \{A, e^{R_0}\}_0, \tag{4.4}$$

with the composition law

$$\{A, e^{R_0}\}_0 \cdot \{A', e^{R_0'}\}_0 = \{A + e^{R_0} A' e^{R_0^{\mathrm{T}}}, e^{R_0} e^{R_0'}\}_0.$$

So, G_0 is isomorphic to the semidirect product of SL$(\nu, \boldsymbol{R})$ and the $\binom{\nu}{2}$-dimensional Abelian group N:

$$G_0 \simeq \mathrm{SL}\,(\nu, \boldsymbol{R}) \otimes_s N. \tag{4.5}$$

A. The manifold of SO(ν, ν) pure spinors

We can write the general element L of the Lie algebra of SO(ν, ν) (Eq. (4.1)) in the following form:

$$\left(\begin{array}{c|c} R & A \\ \hline B & -R^{\mathrm{T}} \end{array}\right) = \underbrace{\left(\begin{array}{c|c} R_0 & 0 \\ \hline 0 & -R_0^{\mathrm{T}} \end{array}\right) + \left(\begin{array}{c|c} 0 & A \\ \hline 0 & 0 \end{array}\right)}_{\text{the stability subalgebra } L_0} + \underbrace{\left(\begin{array}{c|c} 0 & 0 \\ \hline B & 0 \end{array}\right) + \left(\begin{array}{c|c} \alpha I & 0 \\ \hline 0 & -\alpha I \end{array}\right)}_{\text{the complement}},$$

with $\alpha = \frac{1}{\nu}\,\mathrm{Tr}\,R$ and $R_0 = R - \frac{1}{\nu}(\mathrm{Tr}\,R)I$. Note that the complements of L_0 to L do form the subalgebra by themselves. It consists of one-dimensional algebra $\left\{\left(\begin{array}{c|c} \alpha I & 0 \\ \hline 0 & -\alpha I \end{array}\right)\right\}$ and $\binom{\nu}{2}$-dimensional Abelian ideal $\left\{\left(\begin{array}{c|c} 0 & 0 \\ \hline B & 0 \end{array}\right)\right\}$. We parametrize the elements of the coset space W (which form the group) as follows:

$$W = \exp\left(\begin{array}{c|c} 0 & 0 \\ \hline B & 0 \end{array}\right) \cdot \exp\left(\begin{array}{c|c} \alpha I & 0 \\ \hline 0 & -\alpha I \end{array}\right) = \left(\begin{array}{c|c} I & 0 \\ \hline B & I \end{array}\right) \cdot \left(\begin{array}{c|c} e^{\alpha} I & 0 \\ \hline 0 & e^{-\alpha} I \end{array}\right) \equiv \{\alpha, B\} \tag{4.6}$$

so the composition law in W reads:

$$\{\alpha, B\} \cdot \{\alpha', B'\} = \{\alpha + \alpha', B + e^{-2\alpha} B'\}.$$

Consequently, W is the semidirect product of $\boldsymbol{R}$ and N':

$$W \simeq \boldsymbol{R} \otimes N', \tag{4.7}$$

with $\binom{\nu}{2}$-dimensional Abelian group N'. The group W is nilpotent so the exponential mapping gives the global map for the group manifold. In our case the group manifold of W is diffeomorphic to $\boldsymbol{R}^{\binom{\nu}{2}+1}$. We can parametrize the coset manifold $W \simeq \mathrm{SL}(\nu, \nu)/\mathrm{SL}(\nu, \boldsymbol{R}) \otimes_s N$, and consequently the manifold of SO(ν, ν) pure spinors, by the matrix elements $b_{ik} = -b_{kl}$ of B and by the parameter α. Now, the (nonlinear in general) action of SO(ν, ν) on the manifold W of pure spinors can be determined with the famous relation [2, 4]

$$gW = W' g_0, \tag{4.8}$$

$g \in \mathrm{SO}(\nu, \nu)$, $g_0 \in G_0$, $W, W' \in W$.

SO(ν,ν) acts on W according to the law $g: W \to W'$. From (4.8), (4.4) and (4.6) we obtain the following transformation law for coordinates (α, B) of W

$$\mathrm{SO}(\nu,\nu) \ni g: (\alpha, B) \to (\alpha', B').$$

Explicitly:

$$\text{if } g = \{0, e^{R_0}\}_0 \in \mathrm{SL}(\nu, R) \text{ then } \begin{cases} \alpha' = \alpha \\ B' = e^{-R_0^{\mathrm{T}}} B e^{-R_0}, \end{cases} \tag{4.9a}$$

$$\text{if } g = \{A, I\}_0 \in N \qquad \text{then } \begin{cases} \alpha' = \alpha + \dfrac{1}{\gamma} \ln \det (I + AB) \\ B' = B(I+AB)^{-1}, \end{cases} \tag{4.9b}$$

(the constraint $A \neq B^{-1}$ is connected with the fact that SO(ν,ν) cannot be covered by exponential map)

$$\text{if } g = \{\beta, C\} \in W \qquad \text{then } \begin{cases} \alpha' = \alpha + \beta \\ B' = e^{-2\beta} B + C. \end{cases} \tag{4.9c}$$

B. The explicit construction of SO(ν,ν) pure spinors in the spinor representation

First, we give the following general construction of the generators of SO(ν,ν) in the spinor representation. Let us introduce the following notation:

$$\gamma \equiv \begin{pmatrix} (\gamma_N^k) \\ (\gamma_P^k) \end{pmatrix}, \qquad \tilde{\gamma} \equiv (\gamma_N^1, \dots \gamma_N^\nu, \gamma_P^1, \dots \gamma_P^\nu) g, \tag{4.10}$$

where g is the metric tensor in the Witt basis (see Eq. (2.5)). Then the following relation holds:

$$\check{L} = -\tfrac{1}{4}\tilde{\gamma} L \gamma, \tag{4.11}$$

where $\hat{L}$ denotes the element of the Lie algebra of SO(ν,ν) in the spinor representation which corresponds to L. The formula (4.11) can be checked by considering the commutation rule

$$[\hat{L}, \gamma] = L\gamma \tag{4.12}$$

which is obtained with help of the relation (2.3) rewritten in the Witt basis.
Now, we can construct the general pure spinor corresponding to the point (α, B) of the manifold W. Putting

$$\mathscr{B} \equiv \left(\begin{array}{c|c} 0 & 0 \\ \hline B & 0 \end{array}\right), \qquad \mathcal{O} \equiv \left(\begin{array}{c|c} \alpha I & 0 \\ \hline 0 & -\alpha I \end{array}\right)$$

we obtain

$$f(\alpha, B) = \exp(-\tfrac{1}{4}\tilde{\gamma}\mathscr{B}\gamma) \cdot \exp(-\tfrac{1}{4}\tilde{\gamma}\mathcal{O}\gamma) \cdot f_P \tag{4.13a}$$

or expanding in the power series

$$f(\alpha, B) = e^{\frac{\alpha\nu}{2}} \prod_{\substack{(i,k)\\ i<k}} (1-b_{ik}\gamma_{N_i}\gamma_{N_k}) f_P \tag{4.13b}$$

(note that f_P can be determined in a matrix realization of C by the condition $\gamma_{P_k} f_P = 0$ for all $k = 1, \dots \nu$).

5. *The general case* $p+q = 2\nu$

A. Lie algebra of SO(p, q), $p+q = 2\nu$ in the Witt basis

The general element of the Lie algebra L of SO(p, q), $p+q = 2\nu$, in the Cartesian basis has the form

$$\check{L} = \begin{array}{c} \left(\begin{array}{c|c} \mathscr{A}_p & S \\ \hline S^{\mathrm{T}} & \mathscr{A}_q \end{array}\right) \begin{array}{l} \}p \\ \}q \end{array} \\ \begin{array}{cc} \underbrace{}_{p} & \underbrace{}_{q} \end{array} \end{array}$$

with $\mathscr{A}_p$ and $\mathscr{A}_q$ — real antisymmetric matrices, S — real matrix. After transformation to the Witt basis with help of $\check{R}$ given by Eq. (2.4a), we obtain

$$L = \begin{array}{c} \left(\begin{array}{cc|cc} A & B & C & D \\ -D^{\dagger} & R & -D^{\mathrm{T}} & E \\ \hline C^* & B^* & A^* & D^* \\ -B^{\dagger} & F & -B^{\mathrm{T}} & -R^{\mathrm{T}} \end{array}\right) \begin{array}{l} \nu-q \\ q \\ \nu-q \\ q \end{array} \\ \begin{array}{cccc} \underbrace{}_{\nu-q} & \underbrace{}_{q} & \underbrace{}_{\nu-q} & \underbrace{}_{q} \end{array} \end{array}, \tag{5.1}$$

where the matrices A, B, C and D are complex, while E, F and R are real. They satisfy

$$A^{\dagger} = -A, \quad C^{\mathrm{T}} = -C, \quad E^{\mathrm{T}} = -E, \quad F^{\mathrm{T}} = -F. \tag{5.2}$$

Let us denote the subspaces of the Lie algebra L of SO(p, q) generated by $A, B \dots$ as L_A, L_B etc., respectively. By L_A^0 and L_R^0 we denote the subspaces of L_A, and L_R under condition $\mathrm{Tr}\, A = 0$ and $\mathrm{Tr}\, R = 0$, respectively.

Then we see from (5.1) that:

a) L_A is the subalgebra of L generating direct sum of the basic representation of the group U($\nu-q$) and its conjugate. Consequently, L_A^0 is the Lie algebra of SU($\nu-q$).

b) L_R is the subalgebra of L generating direct sum of the selfrepresentation of GL(q, $\boldsymbol{R}$) and its contragradient selfrepresentation. Consequently, L_R^0 is the Lie algebra of SL(q, $\boldsymbol{R}$).

c) L_E is the subalgebra of L generating Abelian nilpotent $\binom{q}{2}$-dimensional subgroup of SO(p, q).

d) L_F is the subalgebra of L generating Abelian nilpotent $\binom{q}{2}$-dimensional subgroup of SO(p, q).

e) $L_A \oplus L_C$ is the subalgebra of L generating the group $SO(2(\nu-q))$. Note that $U(\nu-q) \subset SO(2(\nu-q))$ acts as automorphism of L_C ($[L_A, L_C] \subset L_C$).

f) $L_E \oplus L_D$ is the subalgebra of L generating nilpotent (non-Abelian) $\binom{q}{2}+2q(\nu-q)$-dimensional subgroup of $SO(p,q)$.

g) $L_F \oplus L_B$ is the subalgebra of L generating nilpotent (non-Abelian) $\binom{q}{2}+2q(\nu-q)$-dimensional subgroup of $SO(p,q)$.

h) $SO(2(\nu-q))$ and $GL(q,\boldsymbol{R})$ mentioned in (e) and (b) commute.

i) L_E and L_F are the irreducible antisymmetric tensors of $GL(q,\boldsymbol{R})$; namely, under $e^R \in GL(q,\boldsymbol{R})$ $E' = e^R E e^{R^T}$, $F' = e^{-R} F e^{-R^T}$.

j) L_E and L_F are singlets of $SO(2(\nu-q))$.

k) L_B and L_D are the representation spaces for $GL(q,\boldsymbol{R})$ and $U(\nu-q)$.

B. The stability subgroup G_0

Now, from the conditions (3.4), (3.5) and the general form of the element of Lie algebra of $SO(p,q)$ (see Eq. (5.1)), we obtain that elements of the stability algebra have the following form (in the Witt basis)

$$L_0 = \left(\begin{array}{cc|cc} A_0 & 0 & 0 & D \\ -D^\dagger & R_0 & -D^T & E \\ \hline 0 & 0 & A_0^* & D^* \\ 0 & 0 & 0 & -R_0^T \end{array}\right), \tag{5.3}$$

with $\operatorname{Tr} A_0 = \operatorname{Tr} R_0 = 0$, $A_0^\dagger = -A_0$, $E^T = -E$, E, R_0 are real, A, D are complex. Therefore, the stability algebra L^0 is the direct sum

$$L^0 = L_A^0 \oplus L_R^0 \oplus L_E \oplus L_D. \tag{5.4}$$

Consequently, as follows from the discussion in the first part of this section, the stability subgroup G_0 of a $SO(p,q)$ pure spinors associated with singular subspace P is the semi-direct product of $SU(\nu-q)\times SL(q,\boldsymbol{R})$ and the nilpotent non-Abelian subgroup N generated by $L_E L_D$ $\dim N = 2q(\nu-q)+\binom{q}{2}$.

$$G_0 = (SU(\nu-q)\times SL(q,\boldsymbol{R}))\otimes_s N. \tag{5.5}$$

If we use the exponential parametrization for N and denote by U and T the group elements of $SU(\nu-q)$ and $SL(q,\boldsymbol{R})$, respectively, then we obtain

$$G_0 \in g_0 \equiv \{U, T, D, E\}$$

$$\equiv \left(\begin{array}{cc|cc} I & 0 & 0 & D \\ -D^\dagger & I & -D^T & E-\frac{1}{2}(D^\dagger D + D^T D^*) \\ \hline 0 & 0 & I & D^* \\ 0 & 0 & 0 & I \end{array}\right)\cdot\left(\begin{array}{cc|cc} U & 0 & 0 & 0 \\ 0 & T & 0 & 0 \\ \hline 0 & 0 & U^* & 0 \\ 0 & 0 & 0 & T^{-1T} \end{array}\right). \tag{5.6}$$

Then the composition law for G_0 reads

$$\begin{aligned} g_0 g_0' &= \{U, T, D, E\} \cdot \{U', T', D', E'\} \\ &= \{UU', TT', D+UD'T^{\mathrm{T}}, E+TE'T^{\mathrm{T}} \\ &+\tfrac{1}{2}[TD'^{\dagger}U^{\dagger}D+TD'^{\mathrm{T}}U^{\mathrm{T}}D^{*}-D^{\dagger}UD'T^{\mathrm{T}}-D^{\mathrm{T}}U^{*}D'^{*}T^{\mathrm{T}}]\}. \end{aligned}$$

C. The coset space $SO(p, q)/G_0 \equiv L$

The elements of L corresponding to the coset space W have the form

$$L_W = L_F \oplus L_B \oplus L_C \oplus \overline{L}_A \oplus \overline{L}_R,$$

where $\overline{L}_A$ and $\overline{L}_R$ are the one-dimensional subalgebras — the orthogonal complement of L_A^0 and L_R^0 to L_A and L_R, respectively.

The dimension of L equals $\dim L_W = \nu^2-\nu+2-\binom{q}{2}-\delta_{\nu,0}-\delta_{q,0}$. The elements of the coset space W can be parametrized as follows

$$W \ni W \equiv \{\underbrace{X, Y,}_{L_F \oplus L_B} \quad \underbrace{\varphi, \lambda,}_{L_A \oplus L_R} \quad \underbrace{V}_{L_C}\}$$

$$\equiv \left(\begin{array}{cc|cc} I & X & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & X^{*} & I & 0 \\ -X^{\dagger} & Z & -X^{\mathrm{T}} & I \end{array}\right) \left(\begin{array}{cc|cc} e^{i\varphi}I & 0 & 0 & 0 \\ 0 & e^{\lambda}I & 0 & 0 \\ \hline 0 & 0 & e^{-i\varphi}I & 0 \\ 0 & 0 & 0 & e^{-\lambda}I \end{array}\right) V, \tag{5.7}$$

where $Z = Y-\frac{1}{2}(X^{\dagger}X+X^{\mathrm{T}}X^{*})$ and V generated by

$$L_C = \left(\begin{array}{cc|cc} 0 & 0 & \Omega & 0 \\ 0 & 0 & 0 & 0 \\ \hline \Omega^{*} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right);$$

here Ω is complex antisymmetric $(\nu-q)\times(\nu-q)$ matrix and parametrizes the coset space $SO(2(\nu-q))/U(\nu-q)$, and X is a complex $(\nu-q)\times q$ matrix, Y is real antisymmetric $q\times q$ matrix.

The nonlinear action of G on W can be determined with help of Eq. (4.8).

Summarizing, the manifold W of pure spinors

$$W \cong SO(2\nu-q, q)/(SU(\nu-q)\times SL(q, R))\otimes N$$

has the structure of the topological product:

$$R^{2q(\nu-q)+\binom{q}{2}+1} \times SO(2(\nu-q))/SU(\nu-q).$$

Two extreme situations: $\nu = q$ $(SO(\nu, \nu))$ and $q = 0$ $(SO(2\nu))$, correspond to $R^{\binom{\nu}{2}+1}$ (fully noncompact) and $SO(2\nu)/SU(\nu)$ (fully compact) cases, respectively.

D. SO(p, q), $p+q = 2\nu$, pure spinors in the spinor representation

Following the procedure from the SO(ν, ν) case, the general element $\hat{L}$ of the Lie algebra of SO(p, q) in the spinor representation has the form (4.11), namely

$$\hat{L} = -\tfrac{1}{4}\tilde{\gamma}L\gamma,$$

with γ and $\tilde{\gamma}$ defined as in Eq. (4.10) and L given by (5.1). Taking L_W in the form

$$L_W = \left(\begin{array}{cc|cc} i\varphi I & X & \Omega & 0 \\ 0 & \lambda I & 0 & 0 \\ \hline \Omega^* & X^* & -i\varphi I & 0 \\ -X^\dagger & Y & -X^{\mathrm{T}} & -\lambda I \end{array}\right) \tag{5.8}$$

(compare with Eq. (5.7)), we can express the general SO(p, q), $p+q = 2\nu$, pure spinor in the exponential parametrization as

$$\psi(X, Y, \varphi, \lambda, \Omega) = \exp(\hat{L}_W)\cdot f_P, \tag{5.9}$$

where $\hat{L}_W = -\frac{1}{4}\tilde{\gamma}L_W\gamma$.

6. *The general case of $p+q = 2\nu+1$*

In this case, the metric tensor g is given by (2.5b) and consequently the projector Π_P in the conditions (3.4), (3.5) should be of the form

$$\Pi_P = \begin{array}{c} \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \\ \begin{array}{ccc} \underbrace{}_{1} & \underbrace{}_{\nu} & \underbrace{}_{\nu} \end{array} \end{array} \begin{array}{l} 1 \\ \nu\,. \\ \nu \\ \end{array} \tag{6.1}$$

The general element of the Lie algebra of SO(p, q), $p+q = 2\nu+1$ in the Witt basis has the following form:

$$L = \begin{array}{c} \left[\begin{array}{c|c|c|c|c} 0 & b & f & b^* & e \\ \hline -b^\dagger & A & B & C & D \\ \hline -e^{\mathrm{T}} & -D^\dagger & R & -D^{\mathrm{T}} & E \\ \hline -b^{\mathrm{T}} & C^* & B^* & A^* & D^* \\ \hline -f^{\mathrm{T}} & -B^\dagger & F & -B^{\mathrm{T}} & -R^{\mathrm{T}} \end{array}\right] \\ \begin{array}{ccccc} 1 & \nu-q & q & \nu-q & q \end{array} \end{array} \begin{array}{l} 1 \\ \nu-q \\ q \\ \nu-q \\ q \\ \end{array} \tag{6.2}$$

where the matrices A, B, C, D, E, F and R have the same properties as in Eq. (5.1) while b is complex $1\times(\nu-q)$ matrix and e, f are real $1\times q$ matrices.

From Eqs (3.4), (3.5), we obtain the general element of the stability subalgebra

$$L_0 = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & e \\ \hline 0 & A_0 & 0 & 0 & D \\ \hline -e^{\mathrm{T}} & -D^{\dagger} & R_0 & -D^{\mathrm{T}} & E \\ \hline 0 & 0 & 0 & A_0^* & D^{*} \\ \hline 0 & 0 & 0 & 0 & -R_0^{\mathrm{T}} \end{array}\right] \tag{6.3}$$

i.e. $L^0 = L_A^0 \oplus L_R^0 \oplus L_E \oplus L_e \oplus L_D$.

Because the subalgebra $L_E \oplus L \oplus L_D$ is nilpotent (non-Abelian) ideal in L^0 (it is singlet of L_A^0 and a representation space for L_R^0) generating nilpotent $\left(\frac{q(q+1)}{2}+2q(v-q)\right)$-dimensional subgroup N, then G_0 is the semidirect product of $\mathrm{SU}(v-q)\times \mathrm{SL}(q, R)$ and N, namely

$$G_0 = (\mathrm{SU}(v-q)\times \mathrm{SL}(q, R))\otimes_s N, \tag{6.4}$$

with $\dim N = \frac{q(q+1)}{2}+2q(v-q)$.

Now the manifold W of pure spinors is the coset space

$$W \simeq \mathrm{SO}(2v+1-q, q)/(\mathrm{SU}(v-q)\times \mathrm{SL}(q, R))\otimes_s N \tag{6.5}$$

and $\dim W = v^2+v+2-\binom{q+1}{2}-\delta_{v,0}-\delta_{q,0}$.

Taking into account the fact that $L_A \oplus L_C \oplus L_b$ spans the Lie algebra of $\mathrm{SO}(2(v-q)+1)$, while $L_B \oplus L_F \oplus L_f$ is nilpotent subalgebra generating nilpotent subgroup isomorphic to N we see that W is topological product

$$R^{2q(v-q)+\frac{q(q+1)}{2}+1}\ \mathrm{SO}(2(v-q)+1)/\mathrm{SU}(v-q).$$

The manifold W can be locally parametrized by exponentiation of the complement of stability algebra L^0 in L, with general element

$$L_W = L - L^0. \tag{6.6}$$

A general pure spinor in the spinor representation is obtained analogously to Eq. (5.9) with L_W given by Eqs (6.6) and (6.3), (6.2). However, a modification is necessary in the definition of γ (and $\tilde{\gamma}$): one must add γ^0 in the first row:

$$\gamma = \begin{pmatrix} \gamma^0 \\ (\gamma_N^k) \\ (\gamma_P^k) \end{pmatrix}. \tag{6.7}$$

7. Conclusion

A formalism has been developed for describing the pure spinors. We have found explicitly the manifold of pure spinors for all pseudoorthogonal groups SO(p, q). It is osomorphic to the coset space

$$\mathrm{SO}(p, q)/(\mathrm{SU}(\nu-q)\times \mathrm{SL}(q, \boldsymbol{R}))\otimes_s N,$$

where $\nu = \left[\frac{p+q}{2}\right]$.

We have also given an explicit construction of the general pure spinor in the spinor representation. From the physical point of view the use of the pure spinors in field theory is motivated by the following fact. First, they reduce considerably the fermionic number of degrees of freedom. Moreover, at least part of the fermionic degrees of freedom can be expressed by the bosonic ones [11, 12]. Finally, it provides a way of introducing a dynamical principle like in the sigma model [11, 12].

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