ANOMALOUS CHARGES OF FERMIONIC VACUUM IN CHIRAL BAGS*

By M. Jeżabek**

Max-Planck-Institut für Physik und Astrophysik, Werner-Heisenberg-Institut für Physik, P.O.Box 40 12 12, Munich, Fed. Rep. Germany

(Received April 6, 1987)

In the chiral bag the fermionic vacuum is polarized by an external meson field. A method is described for evaluation of physical effects due to the vacuum polarization in spherical chiral bags. We apply our method to the calculation of the vacuum baryon number. The possibility is discussed that the vacuum in chiral bags can be non-trivial with respect to other quantum numbers.

PACS numbers: 12.40.Aa

1. Introduction

During the last fifteen years quantum chromodynamics (QCD) has become established as the theory of strong interactions. Apart from aesthetical reasons like the beauty of gauge theories QCD is a very attractive theory because it is renormalizable and asymptotically free. The theory agrees with experiment for all processes for which we are able to work out its predictions, hard processes at the CERN Sp$\bar{\text{p}}$S collider being a newest example. There is little doubt that the new generation of experiments at SLC, LEP and HERA exploring the energy regions where the perturbative approach becomes a good approximation to QCD will strengthen our confidence in this theory. On the other hand attempts to describe the low- and intermediate-energy physics of hadrons are still not satisfactory. It is commonly believed that this is not a failure of QCD but only a consequence of our inability to deal with this theory when all its complexity becomes essential. Such a situation stimulates progress in theoretical physics and we believe that sooner or later the non-perturbative aspects of QCD will be understood quantitatively. A promising program is to use computers for solving QCD on a lattice. Another idea is to develop phenomenological models.

* Work supported in part by CPBP.01.09.

** On leave of absence from the Institute of Nuclear Physics, High Energy Physics Laboratory, Kawiory 26a, Cracow, Poland.

QCD, when already solved, will serve as a microscopic theory enabling calculation of phenomenological parameters from fundamental constants.

Two phenomenological models of this kind describe well the low energy properties of baryons. The first one is the MIT bag model [1] where quarks are confined inside a spherical cavity immersed in a non-perturbative medium. The second attempt describes baryons as topological solitons of a non-linear sigma model [2, 3]. Striking similarities between the results of these very different approaches suggest that some kind of duality exists relating these descriptions. The above idea is realized in the so-called hybrid (or chiral) bags [4]. In this model the bag is surrounded by a pion cloud, and the classical configuration of the pion field has a topologically non-trivial 'hedgehog' shape first considered by Skyrme [2]. The pion field polarizes the fermionic vacuum in the bag which becomes a complicated object carrying a non-zero baryon number [4] and probably other quantum numbers as well.

In the present paper we describe a method for systematic evaluation of the effects coming from the vacuum polarization in chiral bags [5–12]. We give a detailed presentation of this method (Chapter 3), and then we describe the results obtained so far (Chapter 4). In Chapter 5 we discuss an idea that the vacuum in chiral bags may have non-zero spin, isospin and other quantum numbers. Our results are summarized in Chapter 6.

2. *Skyrmions and chiral bags*

We start with a brief description of skyrmions and chiral bags. Our aim is to introduce some notions and results which we use in the following chapters, and not to review the vast literature on these subjects. For a comprehensive list of references and a much more complete account of recent developments, including those which are not closely related to our main topic and are not mentioned at all, the reader is referred to review articles on bags [13], skyrmions [14–16] and chiral bags [17–20], see also [21, 22].

In remarkable papers [2] Skyrme put forward the idea that baryons can be considered as solitons of a strongly interacting meson field. In [23–27] it has been shown that such solitons can carry half-integer spin and fulfill the Fermi-Dirac statistics, and in [3] the static properties of baryons in the Skyrme model have been calculated.

For two flavours the Skyrme lagrangian describes an isospin one pion field:

$$\mathscr{L}_{\mathrm{Sk}} = \frac{F_\pi^2}{16}\,\mathrm{Tr}\,(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32e^2}\,\mathrm{Tr}\,[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger]^2, \tag{2.1}$$

where U is an SU_2 matrix, $F_\pi = 186$ MeV is the pion decay constant, and e is a dimensionless parameter. Apart from standard harmonic excitations, i.e. pions, the spectrum of the model contains also solitons which we identify with baryons. (The second term in the lagrangian (2.1) was introduced by Skyrme to stabilize these solitons). We find soliton solutions looking for stationary configurations $U_0(\boldsymbol{r})$ of finite energy. Thus, for $|\boldsymbol{r}| \to \infty$ U_0 tends to a constant matrix, and we can choose

$$U_0(|\boldsymbol{r}| \to \infty) = 1 \tag{2.2}$$

because $\mathscr{L}_{Sk}$ is invariant under global $SU_2 \times SU_2$ transformations. Therefore U_0 is defined on $S^3 = R^3 + \{\infty\}$ and the set of maps

$$U_0(r) : S^3 \to SU_2$$

splits into homotopy classes not continuously deformable into each other and numbered by an integer number, the so-called winding number. The winding number of the configuration U_0 has been identified by Skyrme with the baryon number. In order to find a solution corresponding to a single baryon we substitute into (2.1) the Skyrme ansatz ('hedgehog' static configuration):

$$U_0(r) = \exp\{i\theta(X)\vec{\tau} \cdot \hat{n}\}, \tag{2.3}$$

where a dimensionless variable

$$X = eF_\pi r \tag{2.4}$$

is used. Then we solve the resulting variational equation [3]

$$\left(\frac{X^2}{4} + 2\sin^2\theta\right)\theta'' + \frac{X\theta'}{2} + \sin 2\theta, \theta'^2 - \frac{\sin 2\theta}{4} - \frac{\sin^2\theta \sin 2\theta}{X^2} = 0 \tag{2.5}$$

for the boundary conditions

$$\theta(0) = -\pi \tag{2.6a}$$

$$\theta(\infty) = 0. \tag{2.6b}$$

The radial density of baryon number is [3]

$$\varrho_{Sk}(r) = \frac{2\sin^2\theta}{\pi}\theta' \tag{2.7}$$

and Eqs. (2.6) imply that the baryon number of the solution (skyrmion) is equal to one.

The chiral bag is a bubble, within which quarks propagate freely, surrounded by a pion cloud in 'hedgehog' configuration. Chodos and Thorn [28] were the first who considered this object in their attempt to restore chiral symmetry in bag models. In bag models the surface of the bag is a source of explicit chiral symmetry breaking. However, it is possible to restore chiral symmetry when quarks are coupled chirally to the external pion field at bag surface. Thus, the action S of the system consists of three terms corresponding to the Dirac lagrangian for fermions in the bag, the Skyrme lagrangian for the pion cloud outside the bag, and the interaction between fermionic and bosonic fields at the surface of the bag:

$$S = \int_{\text{in}} d^4x(i\bar{\psi}\hat{\partial}\psi - \mathscr{B}) + \int_{\text{out}} d^4x\mathscr{L}_{Sk} - \int_{\text{surf}} d\Sigma\delta_{\text{surf}}(\bar{\psi}_R U\psi_R + \bar{\psi}_L U^\dagger\psi_L). \tag{2.8}$$

In the presence of a soliton the fermionic vacuum becomes non-trivial [29]. For the chiral bags the interaction with the pion field modifies the MIT boundary condition for quarks in the bag in such a way that the energy levels of quarks are not degenerate with the levels of antiquarks. Consequently, contributions from virtual quarks to some quantities are not canceled by contributions from virtual antiquarks, and the vacuum expectation values of these quantities are non-zero.

3. Spectral asymmetry for quarks in chiral bags

3.1. Massless quark in a spherical cavity [5, 9]

In the chiral bag model the quark wave function inside the bag satisfies the free particle Dirac equation and the boundary condition is

$$-i\vec{\gamma}\cdot\hat{\boldsymbol{n}}\Psi(\boldsymbol{r}) = \exp(i\Theta\vec{\tau}\cdot\hat{\boldsymbol{n}}\gamma_5)\Psi(\boldsymbol{r}). \tag{3.1}$$

For a spherical bag of radius R this condition applies at $|\boldsymbol{r}| = R$ and the unit vector $\hat{\boldsymbol{n}}$, which is the external normal to the bag surface, reduces to $\boldsymbol{r}/R$. Under isospin transformations the Dirac bispinors Ψ transform as a doublet. This corresponds to the inclusion of u and d quarks only. The components of the vector $\vec{\tau}$ are the Pauli matrices acting in isospin space. The parameter Θ is a real number describing the strength of the classical pion field at the surface of the bag.

For a Dirac isodoublet particle in a spherically symmetric and isospin independent potential the following (good) quantum numbers correspond to the operators commuting with the Hamiltonian: the sign of eigenenergy $\kappa = \pm 1$, total angular momentum J and its projection on the z-axis m, parity P, and the third component of isospin α. In order to define an energy level unambiguously it is necessary to add another quantum number, say n, which labels radial excitations, just like the principal quantum number in the hydrogen atom case. The above set of quantum numbers defines uniquely a given solution.

In our problem the boundary condition (3.1) involves the operator $\vec{\tau}\cdot\hat{\boldsymbol{n}}$ which is invariant neither with respect to rotations in ordinary space nor with respect to isospin rotations. Simultaneous rotations in space and isospace, however, leave $\vec{\tau}\cdot\hat{\boldsymbol{n}}$ and more generally the condition (3.1) unchanged. Thus, for $\Theta \neq 2\pi k$ $(k = 0, \pm 1, \ldots)$ the symmetry of our problem is reduced to SU_2, the diagonal subgroup of $SU_{2,\,\text{space}} \times SU_{2,\,\text{isospace}}$. Introducing the operator

$$\boldsymbol{K} = \boldsymbol{I}+\boldsymbol{J} = \boldsymbol{I}+\boldsymbol{L}+\boldsymbol{S}, \tag{3.2}$$

we can replace J, m and α by the good quantum numbers: K — such that the eigenvalue of $\boldsymbol{K}^2$ is $K(K+1)$, M, the eigenvalue of K_3, and ε. The quantum number (or label) ε is necessary in order to obtain a one to one correspondence between sets of indices and eigenstates. For the states of $K = 0$ we define $\varepsilon = 1$, whereas for $K > 0$ we have $\varepsilon = \pm 1$. Since the operators on both sides of the condition (3.1) are even under inversion in ordinary space parity P remains a good quantum number. Therefore, the quantum numbers which can be used to label the energy levels are: n, K, M, κ, P and ε. The energy levels are degenerate with respect to M as a consequence of the symmetry of the problem with respect to SU_2 rotations in K-space.

Let us derive explicit expressions for the wave functions and energy levels[1]. The building blocks for the quark wave function include: spherical harmonics $Y_{lm}(\Omega)$,[2] the eigenfunctions of the z-component of spin

$$\chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.3}$$

and λ_σ, the eigenfunctions of I_3, seemingly identical with χ_σ but defined in isospin space. Besides the usual spin-angular functions

$$\phi_{jlm} = \sum_{\sigma=\pm\frac{1}{2}} \langle l, m-\sigma; 1/2, \sigma | j, m\rangle Y_{l,m-\sigma}\chi_\sigma \tag{3.4}$$

where $\langle j_1, m_1; j_2, m_2 \mid j, m\rangle$ denote the Clebsch-Gordan coefficients, we shall need also the isospin-spin-angular functions

$$\Psi_{KjlM} = \sum_{\sigma=\pm\frac{1}{2}} \langle 1/2, \sigma; j, M-\sigma | KM\rangle \lambda_\sigma \phi_{jl,M-\sigma}. \tag{3.5}$$

We use the Dirac representation [33]

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and define eight families of free particle solutions of the massless Dirac equation [34]:

$$\Psi_1^\kappa = \gamma_5 \Psi_3^\kappa = \begin{pmatrix} j_K(|E|r)\Psi_{K,K+1/2,KM} \\ -\kappa j_{K+1}(|E|r)\Psi_{K,K+1/2,K+1,M} \end{pmatrix} \qquad K = 0, 1, \ldots \tag{3.6}$$

$$\Psi_2^\kappa = \gamma_5 \Psi_4^\kappa = \begin{pmatrix} j_K(|E|r)\Psi_{K,K-1/2,KM} \\ -\kappa j_{K-1}(|E|r)\Psi_{K,K-1/2,K-1,M} \end{pmatrix} \qquad K = 1, 2, \ldots \tag{3.7}$$

We choose the phase convention for the space parity in such a way that $P = (-1)^K$ for Ψ_1^κ and Ψ_2^κ whereas $P = -(-1)^K$ for Ψ_3^κ and Ψ_4^κ. Since P and κ are good quantum numbers the wave functions must be of the form:

$$\Psi_+^\pm = C_1\Psi_1^\pm + C_2\Psi_2^\pm, \tag{3.8a}$$

$$\Psi_-^\pm = C_3\Psi_3^\pm + C_4\Psi_4^\pm. \tag{3.8b}$$

In each of the four classes ($\kappa = \pm 1$, $P = \pm 1$) we expect two families of solutions corresponding to $\varepsilon = \pm 1$ and in each family an infinity of solutions ($n = 1, 2, \ldots$). $K = 0$ is an exception because according to (3.7)

$$C_2 = C_4 = 0 \quad \text{for} \quad K = 0 \tag{3.9}$$

---

[1] See also [30, 31].

[2] Our conventions for spherical harmonics, Clebsch-Gordan coefficients, $6j$ symbols etc. are taken from [32].

and only one family of solutions ($\varepsilon = +1$) exists. Our task is considerably simplified by relations between solutions for different combinations of P and κ. Let $\Psi^+_+(\Theta)$ be a solution of positive energy and parity $P = (-1)^K$. We can easily check that $\gamma_5 \Psi^+_+(\Theta+\pi)$ is also a solution of positive energy, but its parity is opposite. Thus,

$$\Psi^+_-(\Theta) = \gamma_5 \Psi^+_+(\Theta+\pi). \tag{3.10}$$

Similarly, using the identity

$$\exp(i\Theta\vec{\tau}\cdot\hat{\boldsymbol{n}}\gamma_5) = \cos\Theta + i\vec{\tau}\cdot\hat{\boldsymbol{n}}\gamma_5 \sin\Theta, \tag{3.11}$$

we find that, in self-explanatory notation,

$$\Psi^-_+(\Theta) = \gamma_0 \Psi^+_+(-\Theta+\pi), \tag{3.12}$$

$$\Psi^-_-(\Theta) = \gamma_0\gamma_5 \Psi^+_+(-\Theta). \tag{3.13}$$

The above relations imply relations between the energy levels and the coefficients C_i. For example it follows from (3.10) that

$$C_{i+2}(\Theta, \kappa) = C_i(\Theta+\pi, \kappa), \tag{3.14}$$

etc. As for the energy levels we have:

$$E^+_-(\Theta) = E^+_+(\Theta+\pi), \tag{3.15a}$$

$$E^-_+(\Theta) = -E^+_+(-\Theta+\pi), \tag{3.15b}$$

$$E^-_-(\Theta) = -E^+_+(-\Theta). \tag{3.15c}$$

Thus, we can limit our further study to one family of solutions, let them be Ψ^+_+, and then we can extend our results using the above symmetry relations.

In order to derive equations for energy levels we will need the identities:

$$i\vec{\sigma}\cdot\hat{\boldsymbol{n}}\Psi_{K,l\pm1/2,lM} = \mp\Psi_{K,l\pm1/2,l\pm1,M}, \tag{3.16}$$

$$i\vec{\tau}\cdot\hat{\boldsymbol{n}}\Psi_{KjlM} = \sum_{j'l'}(K;jl;j'l')\Psi_{Kj'l'M}, \tag{3.17}$$

where the only nonvanishing coefficients are

$$(K;K+1/2,K;K+1/2,K+1) = (K;K-1/2,K;K-1/2,K-1) = \frac{1}{2K+1} \tag{3.18a}$$

$$(K;K+1/2,K;K-1/2,K-1) = (K;K+1/2,K+1;K-1/2,K) = -\frac{2\sqrt{K(K+1)}}{2K+1} \tag{3.18b}$$

and those which can be obtained from them using the relation:

$$(K;jl;j'l') = -(K;j'l';jl). \tag{3.19}$$

A derivation of the formulae (3.16–3.19) is given in the Appendix A.

Using (3.8a, 3.11) and (3.16–3.19) we reduce the boundary condition for Ψ_+^+ to the following system of two linear equations for the coefficients C_1 and C_2:

$$C_1\left[j_K(x)\left(1+\frac{\sin\Theta}{2K+1}\right)-j_{K+1}(x)\cos\Theta\right]+C_2 j_K(x)\sin\Theta\,\frac{2\sqrt{K(K+1)}}{2K+1}=0, \tag{3.20a}$$

$$C_1 j_K(x)\sin\Theta\,\frac{2\sqrt{K(K+1)}}{2K+1}+C_2\left[j_K(x)\left(1-\frac{\sin\Theta}{2K+1}\right)+j_{K-1}(x)\cos\Theta\right]=0, \tag{3.20b}$$

where

$$x = |E|R. \tag{3.21}$$

For $K = 0$ the coefficient C_2 must be equal to zero, cf. (3.9), and Eq. (3.20b) is trivially fulfilled. Expressing the spherical Bessel functions j_0 and j_1 in terms of elementary functions [35] we derive from (3.20a) the following transcendental equation for the energy levels:

$$\sin x+\cos(\Theta-x)-\sin x\cos\Theta/x = 0. \tag{3.22}$$

For $K > 0$ the system (3.20a–b) has non-zero solutions for C_1 and C_2 only if the principal determinant vanishes, i.e.:

$$[j_{K-1}(x)j_{K+1}(x)-j_K^2(x)]\cos\Theta-j_K(x)\,[j_{K-1}(x)-j_{K+1}(x)]$$
$$-\frac{\sin\Theta}{2K+1}j_K(x)\,[j_{K-1}(x)+j_{K+1}(x)] = 0. \tag{3.23}$$

Using identities between Bessel functions [35] we can rewrite (3.23) in a simpler form:

$$(\mathscr{D}_\nu^2+1-\nu^2/x^2)\cos\Theta+2\mathscr{D}_\nu+\sin\Theta/x = 0, \tag{3.24}$$

where

$$\nu = K+1/2, \tag{3.25}$$

and

$$\mathscr{D}_\nu = J_\nu'(x)/J_\nu(x). \tag{3.26}$$

J_ν is the ordinary Bessel function of order ν and J_ν' its derivative. Another equivalent form is

$$\cos\Theta\mathscr{D}_\nu(x) = \varepsilon\mathscr{F}_\nu(\Theta, x)-1, \tag{3.27}$$

where

$$\mathscr{F}_\nu(\Theta, x) = \sqrt{1-(1-\nu^2/x^2)\cos^2\Theta-\sin 2\Theta/(2x)}. \tag{3.28}$$

Thus, we see once again that for $K > 0$ there are two families of solutions labelled by $\varepsilon = \pm 1$.

Let us consider now normalization of wave functions. The bispinors Ψ_i^κ defined by (3.5) and (3.6) are orthogonal. In particular

$$\int d\Omega \Psi_1^{\kappa\dagger} \Psi_2^\kappa = \int d\Omega \Psi_2^{\kappa\dagger} \Psi_1^\kappa = 0, \tag{3.29a}$$

whereas

$$\int d\Omega \Psi_1^{\kappa\dagger} \Psi_1^\kappa = j_K^2(ax) + j_{K+1}^2(ax) \tag{3.29b}$$

$$\int d\Omega \Psi_2^{\kappa\dagger} \Psi_2^\kappa = j_K^2(ax) + j_{K-1}^2(ax), \tag{3.29c}$$

where

$$a = \frac{r}{R}. \tag{3.30}$$

Let x_n ($n = 1, 2, \dots$) be n-th positive solution of (3.22). The corresponding (normalized) wave function is

$$\psi_n = \frac{x_n \Psi_1^1(x_n)}{\sqrt{4\pi R^3(1 - \sin^2 x_n / x_n^2)}}. \tag{3.31}$$

We shall need also expressions for the radial density

$$\varrho_n(r) = r^2 \int d\Omega \psi_n^\dagger \psi_n = \frac{1}{R\left(1 - \dfrac{\sin^2 x_n}{x_n^2}\right)} \left[1 - \frac{\sin 2ax_n}{ax_n} + \frac{\sin^2 ax_n}{a^2 x_n^2}\right] \tag{3.32}$$

and its second moment

$$M_n^{(2)} = \frac{1}{R^2} \int_0^R dr r^2 \varrho_n(r) = \tfrac{1}{3} + \frac{1}{x_n^2 - \sin^2 x_n} \left[\tfrac{2}{3} + \tfrac{1}{3}\cos 2x_n - \frac{\sin 2x_n}{2x_n}\right]. \tag{3.33}$$

The above formulae can be derived by elementary integrations, after j_0 and j_1 are expressed in terms of elementary functions. The formulae for the other moments are given in the Appendix B.

In the case $K > 0$ the formulae analogous to (3.32–3.33) are more complicated. Let $x_{n\nu\varepsilon}$ ($n = 1, 2, \dots$) be n-th positive solution of (3.27). The corresponding wave function is

$$\Psi_{n\nu\varepsilon} = C_1 \Psi_1^1(x_{n\nu\varepsilon}) + C_2 \Psi_2^1(x_{n\nu\varepsilon}). \tag{3.34}$$

Let

$$\varrho_{n\nu\varepsilon} = r^2 \int d\Omega \Psi_{n\nu\varepsilon}^\dagger \Psi_{n\nu\varepsilon} \tag{3.35}$$

and

$$M_{n\nu\varepsilon}^{(p)} = R^{-p} \int_0^R dr r^p \varrho_{n\nu\varepsilon}. \tag{3.36}$$

Using relations between Bessel functions and recurrence relations between integrals of Bessel functions we obtain the following expression for the radial density

$$\varrho_{n\nu\varepsilon}(r) = \frac{a}{RL_\nu^1(x_{n\nu\varepsilon})}\left\{J_\nu^2(\tilde{r})+\tfrac{1}{2}\left[J_{\nu+1}^2(\tilde{r})+J_{\nu-1}^2(\tilde{r})\right]+\frac{\alpha_\nu}{2}\left[J_{\nu+1}^2(\tilde{r})-J_{\nu-1}^2(\tilde{r})\right]\right\}, \quad (3.37)$$

where $\tilde{r} = r|E|$,

$$L_\nu^1(x) = \mathscr{A}_\nu + \frac{1}{x}J_\nu J_\nu' - \frac{\nu\alpha_\nu}{x^2}J_\nu^2 \quad (3.38)$$

$$\mathscr{A}_\nu = J_\nu'^2 + (1-\nu^2/x^2)J_\nu^2 \quad (3.39)$$

and

$$\alpha_\nu = \frac{C_1^2-C_2^2}{C_1^2+C_2^2} = \frac{\varepsilon}{\mathscr{F}_\nu(\Theta, x_{n\nu\varepsilon})}\left(\frac{\nu\cos\Theta}{x_{n\nu\varepsilon}} - \frac{\sin\Theta}{2\nu}\right). \quad (3.40)$$

For the second moment we obtain

$$M_{n\nu\varepsilon}^{(2)} = L_\nu^3(x_{n\nu\varepsilon})/L_\nu^1(x_{n\nu\varepsilon}), \quad (3.41)$$

where

$$L_\nu^3(x) = \left(1+\frac{2\nu^2}{x^2}\right)\frac{\mathscr{A}_\nu}{3} + \frac{J_\nu J_\nu'}{3x} + (\nu\alpha_\nu + \tfrac{1}{3})\frac{\mathscr{A}_\nu - J_\nu^2}{x^2}. \quad (3.42)$$

For a derivation of the above formulae see Appendix B where recurrence formulae for the even moments are also given. Eq. (3.40) can be easily derived from (3.20a–b) and (3.27).

3.2. Calculation of spectral asymmetries

The fermion number of the vacuum is defined by [4, 36]

$$F_{\text{vac}} = -\tfrac{1}{2}\lim_{t\to 0+}\Big(\sum_{E>0} e^{-tE} - \sum_{E<0} e^{tE}\Big), \quad (3.43)$$

where the summation extends over all single particle eigenstates. It is obvious from (3.43) that F_{vac} vanishes if the spectrum is CP symmetric because in this case each positive energy level is matched by a negative energy level with the same $|E|$ [3]. However, in some systems like an electron interacting with a point magnetic monopole [37–39] or chiral bags [4] F_{vac} can be different from zero. In the case of chiral bags CP symmetry is broken by the boundary condition (3.1) for $\Theta \neq \frac{k\pi}{2}$.

Let us remark that the definition (3.43) follows from a regulated version of the fermionic charge $\int d^3x\, \tfrac{1}{2}[\psi^\dagger(x), \psi(x)]$, where ψ is a quantum fermion field. More generally,

[3] We assume that there are no zero-energy levels in the spectrum.

we shall consider the vacuum expectation value of a symmetrized operator $\frac{1}{2}[\psi^{\dagger}, \hat{Q}\psi]$:

$$Q_{\text{vac}} = -\tfrac{1}{2} \sum_{\{E\}} \kappa Q(E).$$

The series in the above definition can be divergent, however, we will assume that it is summable by the Poisson method, i.e.

$$Q_{\text{vac}} = -\tfrac{1}{2} \lim_{t\to 0+} \sum_{\{E\}} \kappa Q(E) \exp(-t|E|) \tag{3.44}$$

is well defined. We shall assume also that

$$Q(E) = O(1) \quad \text{for} \quad E \to \infty. \tag{3.45}$$

In the present Section we describe a method which can be used in calculations of spectral asymmetries. The general idea is to divide the sum in the r.h.s. of (3.44) into two terms, the sums of two series $\mathscr{A}_E(\Theta, t)$ and $\mathscr{R}_E(\Theta, t)$. We call these terms the 'anomalous part' and the 'remainder', respectively. The first sum is not well defined at $t = 0$ and, consequently, it is discontinuous for $t \to 0^+$. However, $\mathscr{A}_E$ can be chosen in such a way that for $t > 0$ the sum of the series can be calculated analytically as well as its limit for $t \to 0^+$. In order to find $\mathscr{A}_E$ one can derive asymptotic expansions for the energy levels and $Q(E)$. Substituting into (3.44) the corresponding asymptotic expansions for these quantities one finds a finite number of terms which are not continuous for $t \to 0^+$. $\mathscr{A}_E$ can be defined as the sum of such terms. Both $\mathscr{A}_E$ and $\mathscr{R}_E$ are absolutely convergent for $t > 0$. It follows from the definition of $\mathscr{A}_E$ that for some ordering $\{E\}^*$ of energy levels the sum of $\mathscr{R}_E$ is well defined at $t = 0$ and continuous for $t \to 0^+$. Thus,

$$Q_{\text{vac}} = \lim_{t\to 0+} \sum_{\{E\}} \mathscr{A}_E(\Theta, t) + \sum_{\{E\}^*} \mathscr{R}_E(\Theta, 0) = \mathscr{A}(\Theta) + \mathscr{R}(\Theta). \tag{3.46}$$

The first term above is a well defined analytical expression, whereas the second term, the sum of a convergent series, can be calculated numerically.

We are interested in spectral asymmetries for quarks in chiral bags, so, we use the classification of energy levels given in the preceding Section. The dependence of the energies of single particle states on the quantum numbers P, κ and ε will play a crucial role in our further considerations. In order to simplify our formulae, and calculations as well, we introduce a special symbol for the average

$$\langle Q \rangle = \begin{cases} \frac{1}{8} \sum_{\kappa P \varepsilon} \kappa Q(\kappa, P, \varepsilon) & \text{if} \quad K > 0 \\ \frac{1}{4} \sum_{\kappa P} \kappa Q(\kappa, P) & \text{if} \quad K = 0 \end{cases} \tag{3.47}$$

for any expression Q. Let us note that according to (3.10, 3.12–3.13) the average over κ and P can be replaced by the average over $\pm\Theta$ and $\pi \pm \Theta$.

It is convenient to consider separately the contributions to Q_{vac} from the states of $K = 0$ and $K > 0$:

$$Q_{\text{vac}} = Q_{K=0} + Q_{K>0}. \tag{3.48}$$

Assuming that the expectation values of $\hat{Q}$ do not depend on M we obtain

$$Q_{K=0}(\Theta) = -2 \lim_{t\to 0+} \sum_{n=1}^{\infty} \langle Q_n(\Theta) \exp(-tx_n)\rangle \tag{3.49a}$$

and

$$Q_{K>0}(\Theta) = -8 \lim_{t\to 0+} \sum_{n,\nu} \langle \nu Q_{n\nu\varepsilon}(\Theta) \exp(-tx_{n\nu\varepsilon})\rangle. \tag{3.49b}$$

Let us assume now that

$$Q_n = q_0 + \frac{q_1}{n} + \tilde{Q}_n, \tag{3.50}$$

where

$$\tilde{Q}_n = O(n^{-2}) \quad \text{for} \quad n \to \infty \tag{3.51}$$

and

$$\langle q_0\rangle = \langle q_1\rangle = 0. \tag{3.52}$$

From (3.22) we can easily derive the following asymptotic expansion for energy levels:

$$x_n(\Theta) = n\pi + \frac{\Theta}{2} - \frac{\pi}{4} - \frac{1-\sin\Theta}{2\pi n} + O(n^{-2}). \tag{3.53}$$

We define

$$\bar{x}_n(\Theta) = n\pi + \frac{\Theta}{2} - \frac{\pi}{4} \tag{3.54}$$

and observe that replacing tx_n by $t\bar{x}_n$ in the r.h.s. of (3.49a) we make an error which vanishes when t tends to 0^+ because

$$\exp(-tx_n) - \exp(-t\bar{x}_n) = \exp(-t\bar{x}_n)O(tn^{-1}) \tag{3.55}$$

and

$$\sum_{n=1}^{\infty} \exp(-tn\pi)O(tn^{-1}) = O(t\ln t). \tag{3.56}$$

Thus, using (3.50) we obtain

$$Q_{K=0} = -2 \lim_{t\to 0+} \Bigg(\langle q_0 e^{t(\pi/4-\Theta/2)}\rangle \sum_{n=1}^{\infty} e^{-tn\pi}$$

$$+ \langle q_1 e^{t(\pi/4-\Theta/2)}\rangle \sum_{n=1}^{\infty} \frac{1}{n} e^{-tn\pi} + \sum_{n=1}^{\infty} \langle \tilde{Q}_n e^{-t\bar{x}_n}\rangle \Bigg). \tag{3.57}$$

The third term above is convergent for $t = 0$, so, the limit $t \to 0^+$ is trivial for it. The second term drops out because of (3.52) and (3.56). Expanding $e^{t(\pi/4-\Theta/2)}$ in powers of t, summing the geometrical series, and noting that $\langle \tilde{Q}_n \rangle = \langle Q_n \rangle$ we obtain:

$$Q_{K=0} = \frac{\langle \Theta q_0 \rangle}{\pi} - 2 \sum_{n=1}^{\infty} \langle Q_n \rangle. \tag{3.58}$$

Obviously, the first term in (3.58) gives a contribution to the anomalous part, whereas the second term to the remainder.

Before we calculate $Q_{K>0}$ let us describe some properties of the positive solutions of Eq. (3.27) for half-integer $\nu > 1/2$. There is no such solution in the region $x_{n\nu\varepsilon} \leqslant \nu$. Each solution can be uniquely related to $\beta_0(n, \nu)$, $0 < \beta_0 < \pi/2$, the only solution of the equation

$$\nu(\tan \beta_0 - \beta_0) = n\pi. \tag{3.59}$$

Using the Debye expansion [35] of Bessel functions[4] which is valid for $x_{n\nu\varepsilon} \gg 1$ and

$$0 < \frac{\nu}{x_{n\nu\varepsilon}} < 1 - \eta, \tag{3.60}$$

where $\eta = O(\nu^{-2/3})$, we can calculate the coefficients in the asymptotic expansions for energy levels $x_{n\nu\varepsilon}$:

$$x_{n\nu\varepsilon} \sim \frac{\nu}{\cos \beta_0} + \sum_{k=0}^{\infty} \frac{a_k(\beta_0, \varepsilon; \Theta)}{\nu^k} \tag{3.61}$$

or, equivalently, for

$$\beta = \arccos(\nu/x_{n\nu\varepsilon}) \sim \beta_0 + \sum_{k=1}^{\infty} \frac{\beta_k(\beta_0, \varepsilon; \Theta)}{\nu^k}. \tag{3.62}$$

Substituting in (3.24) the corresponding asymptotic series for $J_\nu(x_{n\nu\varepsilon})$ and $J'_\nu(x_{n\nu\varepsilon})$ we obtain, see Appendix C,

$$\beta_1 = \frac{\cot^2 \beta_0}{2} \left[(2l_\varepsilon + 1)\pi + \varepsilon \left(\arcsin(\sin \beta_0 \cos \Theta) - \frac{\pi}{2} \right) \right] \tag{3.63a}$$

$$\beta_2 = \frac{\varepsilon f_1}{2C} \cot^2 \beta_0 - \frac{2\beta_1^2}{\sin 2\beta_0} \tag{3.63b}$$

$$\beta_3 = \frac{\varepsilon f_2}{2C} \cot^2 \beta_0 + \frac{\varepsilon f_1^2 \cos^2 \beta_0}{4C^3 \sin \beta_0} \cos \Theta - \frac{4\beta_1 \beta_2}{\sin 2\beta_0} - \frac{4\beta_1^3(1 + 2\sin^2 \beta_0)}{3 \sin^2 2\beta_0} \tag{3.63c}$$

---

[4] See also Appendix C.

etc. In Eqs. (3.63)

$$C = \sqrt{1-\sin^2 \beta_0 \cos^2 \Theta} \tag{3.64}$$

and l_ε is an integer,

$$f_1 = \frac{1+\varepsilon C}{2} \cot \beta_0 \sin \Theta - C^2 \phi_+ - \varepsilon C \phi_- + \beta_1 \cos \beta_0 \cos \Theta \tag{3.65a}$$

$$f_2 = \beta_2 \cos \beta_0 \cos \Theta - \frac{1+\varepsilon C}{2 \sin^2 \beta_0} \beta_1 \sin \Theta - \frac{\varepsilon f_1}{4C} \cos \beta_0 \sin 2\Theta$$

$$+ \frac{\varepsilon f_1}{C} \phi_- \sin \beta_0 \cos \Theta + \tfrac{1}{4} (2\phi_+ - \phi_- + \varepsilon C \phi_+) \cos \beta_0 \sin 2\Theta + \delta. \tag{3.65b}$$

In the above formulae

$$\phi_+ = \frac{\cot \beta_0}{2 \sin^2 \beta_0} \tag{3.66a}$$

$$\phi_- = \frac{\cot \beta_0 (1+2 \sin^2 \beta_0)}{12 \sin^2 \beta_0} \tag{3.66b}$$

and

$$\langle \delta \rangle = \langle \varepsilon \delta \rangle = 0. \tag{3.66c}$$

An apparent ambiguity in β_1 can be removed in a natural way. Solving Eq. (3.24) numerically for given v, ε and Θ, and using as a starting point

$$\tilde{x}_{nv\varepsilon} = \frac{v}{\cos \beta_0} + \beta_1 \frac{\tan \beta_0}{\cos \beta_0}, \tag{3.67}$$

we can fix l_ε demanding that for large n

$$|x_{nv\varepsilon} - \tilde{x}_{nv\varepsilon}| \ll 1.$$

The above condition implies

$$l_\varepsilon = -1. \tag{3.68}$$

However, cf. [9], other choices of l_ε can be also useful, so, below we treat l_ε as a free integer parameter.

Let us now derive a sufficient condition for the finiteness of $Q_{K>0}$. We rewrite (3.49b):

$$Q_{K>0} = \bar{\mathscr{A}}_{K>0} + \bar{\mathscr{R}}_{K>0} = -8 \lim_{t \to 0^+} \Big[\sum_{nv} \langle v Q_{nv\varepsilon} (e^{-txnv\varepsilon} - e^{-tv/\cos \beta_0}) \rangle$$

$$+ \sum_{nv} \langle v Q_{nv\varepsilon} \rangle e^{-tv/\cos \beta_0} \Big] \tag{3.69}$$

and assume that for both series the limit $t \to 0^+$ exists. The first series in the above formula contributes to the anomalous part and the second term contributes to the remainder and to the anomalous part. Let

$$Q_{n\nu\varepsilon} \sim \sum_{j=0}^{\infty} \frac{m_j(\beta_0, \varepsilon; \Theta)}{\nu^j} \tag{3.70}$$

and

$$\langle m_0 \rangle = \langle m_1 \rangle = \langle m_2 \rangle = 0 \tag{3.71a}$$

$$\int_0^{\pi/2} d\beta_0 \tan^2 \beta_0 \langle m_3 \rangle = 0. \tag{3.71b}$$

If these conditions are fulfilled the remainder is well defined at $t = 0$:

$$\mathcal{R}_{K>0} = -8 \sum_{n\nu} \nu \langle Q_{n\nu\varepsilon} \rangle. \tag{3.72}$$

In order to calculate the anomalous part we expand $\exp\{-t(x_{n\nu\varepsilon} - \nu/\cos\beta_0)\}$ in powers of t, then we replace $x_{n\nu\varepsilon}$ and $Q_{n\nu\varepsilon}$ by the corresponding asymptotic expansions (3.61) and (3.70). In this way we obtain

$$\overline{\mathscr{A}}_{K>0} = -8 \lim_{t\to 0^+} \sum_{n\nu} e^{-t\nu/\cos\beta_0} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} c_{jk}(\beta_0, \varepsilon; \Theta) \nu^{1-j} t^k. \tag{3.73}$$

A sufficient condition for the finiteness of $\overline{\mathscr{A}}_{K>0}$ is

$$c_{jk} = 0 \quad \text{for} \quad k + j \leqslant 2. \tag{3.74}$$

Using the Euler-MacLaurin formula we replace the summation over n by integration, cf. (3.59):

$$\sum_n \ldots \to \frac{\nu}{\pi} \int_0^{\pi/2} d\beta_0 \tan^2 \beta_0 \ldots, \tag{3.75}$$

then we calculate the sums over ν using

$$t^k \sum_\nu \nu^p e^{-t\nu/\cos\beta_0} = p! \cos^{p+1}\beta_0 t^{k-p-1} + O(t^{k-p}), \qquad p \geqslant 0, \tag{3.76}$$

and, finally, we obtain

$$\mathscr{A}_{K>0} = \frac{8}{\pi} \int_0^{\pi/2} d\beta_0 \sin^2 \beta_0 [(\langle m_2 a_0 \rangle + \langle m_1 a_1 \rangle + \langle m_0 a_2 \rangle)/\cos\beta_0 - \langle m_0 a_0 a_1 \rangle$$
$$- \langle m_3 \rangle \ln \cos\beta_0 / \cos^2 \beta_0] \tag{3.77}$$

The last term in the above formula is the contribution coming from $\overline{\mathscr{R}}_{K>0}$. The lower limit of the integral in (3.77) corresponds to β well into the transition region $|x_{n\nu\varepsilon}-\nu| = O(\nu^{1/3})$. In the transition region the asymptotic expansion which we are using breaks down. Indeed, one can easily check that

$$\beta_{\min} = O(\nu^{-1/3}), \tag{3.78}$$

where $\beta_{\min}$ is the solution of (3.59) for $n = 1$. So, from (3.61) we deduce that

$$x_{1\nu\varepsilon}-\nu = O(\nu^{1/3}). \tag{3.79}$$

The same leading ν-dependence can be obtained from the asymptotic expansions of Bessel functions in the series of Airy functions which is valid in the transition region. More generally, we expect that the ν-dependence of the leading contributions to spectral asymmetries, but not their magnitude, can be obtained from the Debye expansion. This conjecture has been confirmed by explicit calculations for the baryon number of the vacuum and for the second moment of its distribution. Thus, if our conjecture is true, the convergence of the integral in (3.77) implies that the contribution of the transition region to the anomalous parts of spectral asymmetries vanishes in the limit $t \to 0^+$, whereas (3.71a–b) imply that the remainder is convergent for $t = 0$.

We have seen that the summability of the series in (3.14) is of vital importance for our method based on the Debye expansion. We can ignore the contribution to spectral asymmetries from the transition regions only if series are summable. In this case it does not matter that the information about the transition region is rather poor, because the contribution from this region is negligible anyway. However, when we consider series which are not summable the contribution of the transition region may be important and, consequently, results derived from the Debye expansion need not be true. Calculation of the Casimir energy for fermions in the chiral bag is an important example of this phenomenon [40–42].

Let us now review briefly other calculational methods. Our aim is to illustrate and compare different approaches rather than to give a complete list of relevant papers. For some quantities like the total baryon number of the vacuum [36], the Casimir energy, or the flux of the axial current through the surface of the bag [40, 31] one can express the corresponding spectral asymmetry in terms of fermionic propagators or their derivatives. Then, one can use either the multiple reflection method [43] or the Debye expansion of the confined propagator [44, 40] in order to calculate the leading behaviour of the quantity of interest for $t \to 0^+$. For the total baryon number of the vacuum one can obtain the exact result in this way [36]. However, these methods are applicable for a very limited class of problems. Perhaps the most radical idea has been developed by the Stony Brook group [41, 42, 45]. They abandon analytical methods, evaluate spectral asymmetries numerically for a few positive t, and from these data they extract the limit when t tends to 0^+. Obviously, this method has a few advantages. It is simple and applicable to any finite quantity. Moreover, it is free from all possible errors or misprints of analytical approaches. Unfortunately, this method has also some problems. It is difficult to estimate the accuracy of the method, and, moreover, the accuracy of numerical manipulations may seriously reduce the precision of the results for rapidly oscillating quantities. The above-mentioned difficulties are much less serious for the 'analytically improved' numerical calculations described in our paper.

We close this section with a remark about the regularization-dependence of the results for spectral asymmetries. In [12] it has been shown that the result for the baryon number of the vacuum remains unchanged when the regulator of the Poisson method $\exp(-t|E|)$ is replaced by another one from a broad class of regulators. The considerations of [12] can be trivially extended to the case of the summable series discussed in our paper.

4. Applications

4.1. Baryon numbers of empty chiral bags [5]

Calculation of the baryon number of the vacuum in a spherical chiral bag is the simplest application of the method described in the preceding chapter. We start from the observation that the baryon number of a quark is equal to its fermion number divided by N_c, the number of colours. On the other hand for N_c colours we have an N_c-fold degeneracy of energy levels. Thus, the baryon number of the vacuum is equal to the fermion number F_{vac} calculated for $N_c = 1$:

$$B_{vac} = -\tfrac{1}{2} \lim_{t\to 0^+} \sum_E \kappa \exp(-t|E|). \tag{4.1}$$

Consequently,

$$Q(E) = 1 \tag{4.2}$$

and

$$q_0 = m_0 = 1, \tag{4.3}$$

whereas all other coefficients q_k and m_k are equal to zero. From (3.58) and (3.77) we obtain

$$B_{vac} = \frac{\Theta}{\pi} + \frac{8}{\pi} \int_0^{\pi/2} d\beta_0 \sin^2\beta_0 \left(\frac{\langle a_2 \rangle}{\cos\beta_0} - \langle a_0 a_1 \rangle \right). \tag{4.4}$$

After some algebra, using (3.61–3.63) and the definition (3.47), we derive the following expressions for the averages:

$$\begin{aligned} \langle a_2 \rangle = {} & \sin\Theta\cos\Theta \left(\frac{\cos^2\beta_0}{4C^2\sin^4\beta_0} - \frac{(1+\cos^2\Theta)\cos^2\beta_0}{8C^2\sin^2\beta_0} \right) \\ & + \operatorname{sac}(\Theta,\beta_0) \left(-\frac{\cos^2\beta_0}{4C\sin^4\beta_0} + \frac{\cos^2\beta_0}{8C\sin^2\beta_0} + \frac{\cos^2\Theta\cos^4\beta_0}{8C^3\sin^2\beta_0} \right) \end{aligned} \tag{4.5}$$

and

$$\langle a_0 a_1 \rangle = \operatorname{sac}(\Theta,\beta_0) \frac{\cos\beta_0}{8C\sin^2\beta_0}, \tag{4.6}$$

where

$$\mathrm{sac}\,(\Theta, \beta_0) = \sin \Theta \frac{\arcsin (\sin \beta_0 \cos \Theta)}{\sin \beta_0} \tag{4.7}$$

and C is defined by (3.64). Substituting these expressions in (4.4) and integrating by parts the difference of two singular terms ($\sim \sin^{-2} \beta_0$) we can express the resulting integral in terms of elementary functions:

$$B_{vac}(\Theta) = \frac{\Theta - \sin \Theta \cos \Theta}{\pi}, \qquad |\Theta| < \frac{\pi}{2}. \tag{4.8}$$

The above result can be extended to other values of $\Theta \neq (k+\frac{1}{2})\pi$. The symmetry relations (3.15a–c) imply that

$$B_{vac}(\Theta) = -B_{vac}(-\Theta) = B_{vac}(\Theta+\pi). \tag{4.9}$$

For the 'magic' angles $(k+\frac{1}{2})\pi$ there exists a zero energy level in the spectrum of fermions. In spite of the fact that in these cases the spectrum is CP symmetric, B_{vac} is non-zero and fractional ($\pm\frac{1}{2}$). A comprehensive discussion of this phenomenon is given in the review article [46].

It is noteworthy that the sum of the baryon number of the vacuum and the topological charge of the bosonic field outside the bag is an integer number. This number changes when the strength of the bosonic field at the surface of the bag crosses any of the 'magic' values. The change is exactly compensated by the change of the baryon numbers of valence quarks and antiquarks in the bag. Thus, the total baryon number of the system is a homotopy invariant as in the case of the pure skyrmion. Let us remark that none of the contributions enjoys this property when considered separately from the others. For example, the outside of the bag is topologically equivalent to a non-compact three-dimensional Euclidean space which admits only the trivial homotopy class for the configurations of the bosonic field.

4.2. Massive quarks [6]

In the preceding section we demonstrated that the baryon numbers of the Dirac vacuum in the bag and the Skyrme soliton outside the bag add to an integer number. The baryon number of the soliton is of topological origin, and, therefore, it does not depend on mass parameters explicitly. The question naturally arises whether the baryon number of the Dirac vacuum in the bag depends on quark masses. More generally, one can ask about the dependence of B_{vac} on such 'dynamical details' of the model like the gauge interactions between quarks or deformations of the surface of the bag. In our opinion the consistency of the model requires that there is no such dependence, because the bosonic contribution does not depend on these 'details'. It is known [36] that continuous deformations of the surface of the bag do not change B_{vac} for massless quarks. However, the above-mentioned conjecture is still to be proven. Such a general statement is out of reach for our formalism. However, we can show that for spherical bags there is indeed no dependence of B_{vac} on quark masses[5].

[5] See also [47].

We consider solutions of mass m of the Dirac equation satisfying the boundary condition (3.1) at the surface of the bag. These solutions can be labelled in the same way as those in the massless case. Let us introduce dimensionless quantities $x = |E|R$, $\mu = mR$ and $y = \sqrt{x^2-\mu^2}$. For $\kappa = 1$ and $P = (-1)^K$ the energy levels can be obtained by solving the transcendental equations:

$$\sqrt{1+\frac{\mu}{x}}(1+\sin\Theta)j_0(y) = \sqrt{1-\frac{\mu}{x}}\cos\Theta j_1(y) \tag{4.10}$$

for $K = 0$, and

$$\left[\left(1-\frac{\mu}{x}\right)\mathscr{D}_\nu^2(y)+1-\frac{\nu^2}{y^2}+\frac{\mu}{x}\left(1+\frac{\nu^2}{y^2}\right)\right]\cos\Theta$$

$$+\sqrt{1-\frac{\mu^2}{x^2}}\left(2\mathscr{D}_\nu(y)+\frac{\sin\Theta}{y}\right) = 0 \tag{4.11}$$

for $K > 0$; cf. (3.20a) and (3.24), respectively. Replacing a solution Ψ by $\gamma_5\Psi$, $\gamma_0\Psi$ and $\gamma_0\gamma_5\Psi$ one easily derives the following relations, cf. (3.15a–c)

$$E_-^+(\Theta, m) = E_+^+(\Theta+\pi, -m) \tag{4.12a}$$

$$E_+^-(\Theta, m) = -E_+^+(-\Theta+\pi, -m) \tag{4.12b}$$

$$E_-^-(\Theta, m) = -E_+^+(-\Theta, m). \tag{4.12c}$$

Thus, for the baryon number of the vacuum we obtain the following generalization of (4.9):

$$B_{\text{vac}}(\Theta, m) = -B_{\text{vac}}(-\Theta, m) = B_{\text{vac}}(\Theta+\pi, -m). \tag{4.13}$$

It is easy to see that there is no mass dependent contribution to B_{vac} from the levels for $K = 0$. As is evident from (4.10) the mass-dependent shifts of energy levels are of order μ/n. However, such shifts do not contribute to B_{vac}, cf. (3.56–3.58). For $K > 0$ the non--zero contribution to B_{vac} comes from the region $x_{n\nu\varepsilon} > \nu$ and $x_{n\nu\varepsilon} \gg 1$ where we can use the Debye expansion of Bessel functions. We define β and β_0 such that

$$\cos\beta = \frac{\nu}{y} \tag{4.14}$$

and

$$\nu(\tan\beta_0-\beta_0) = n\pi. \tag{4.15}$$

Let us introduce also the following decomposition:

$$A(\beta_0, \Theta, \mu) = \bar{A}(\beta_0, \Theta)+\delta A(\beta_0, \Theta, \mu) \tag{4.16a}$$

$$\bar{A}(\beta_0, \Theta) \equiv A(\beta_0, \Theta, 0) \tag{4.16b}$$

for any quantity of interest A.

As already explained, only a few coefficients in the asymptotic expansion

$$x_{n\nu\varepsilon} \sim \frac{\nu}{\cos \beta_0} + \sum_{k=0}^{\infty} \frac{a_k(\beta_0, \varepsilon; \Theta, \mu)}{\nu^k} \tag{4.17}$$

for the energy levels contribute to B_{vac}, cf. (3.77) or (4.4). Repeating calculations described in Sect. 3.2 we obtain

$$\delta B_{\text{vac}}(\Theta, m) = \frac{8}{\pi} \int_0^{\pi/2} d\beta_0 \sin^2 \beta_0 \left(\frac{\langle \delta a_2 \rangle}{\cos \beta_0} - \langle \delta(a_0 a_1) \rangle \right). \tag{4.18}$$

Calculation of the averages in (4.18) greatly simplifies when one notices that:

1. each factor μ 'costs' the factor ν^{-1}, so, only terms linear and quadratic in μ can contribute
2. the averages must be odd functions of Θ.

These two conditions limit very strongly the form of the averages in (4.18): they must be proportional to $\mu \sin \Theta f(\cos^2 \Theta)$. An explicit calculation, see Appendix D, shows that

$$\langle \delta a_2 \rangle = \langle \delta(a_0 a_1) \rangle = 0. \tag{4.19}$$

Therefore,

$$B_{\text{vac}}(\Theta, m) = \frac{\Theta - \sin \Theta \cos \Theta}{\pi}, \quad |\Theta| < \frac{\pi}{2}. \tag{4.20}$$

4.3. Distribution of baryon number [7]

The method described in Section 3.2 is not directly applicable in the calculation of the distribution of baryon number in chiral bags. When calculating the corresponding spectral asymmetry we have to consider contributions from various asymptotics of Bessel functions. In particular we have to cross the transition region

$$x = \nu + z\nu^{1/3}, \tag{4.21}$$

where the Debye expansion used in Section 3.2 is not valid. From the calculational point of view the problem is more complicated because the distribution of the baryon number is given by a spectral asymmetry apparently 'less convergent' than that for the baryon number. Consequently, one has to calculate the asymptotic expansion for the energy levels up to the order ν^{-3}, i.e., one order higher than for the baryon number of the vacuum.

We can avoid the above-mentioned difficulties by considering the normalized moments of the baryon number distribution:

$$M_p^{\text{vac}} = -\frac{1}{2R^p} \lim_{t \to 0^+} \sum_E \kappa \exp(-t|E|) \int_0^R dr r^{p+2} \int d\Omega \Psi_E^\dagger \Psi_E \tag{4.22}$$

cf. (3.32), (3.35)–(3.37) and the Appendix B. We write the radial density of the vacuum baryon number in the following form;

$$\varrho_{\rm vac}(r) = R^{-1}B_{\rm vac}(\Theta)\tilde{\varrho}_{\rm vac}(a), \tag{4.23}$$

where $a = r/R$ and

$$\int_0^1 da\,\tilde{\varrho}_{\rm vac}(a) = 1. \tag{4.24}$$

We can assume that $\tilde{\varrho}_{\rm vac}(a)$ is an even function of $a \in (-1, 1)$ and

$$\tilde{\varrho}_{\rm vac}(a) = (1-a^2)^{\alpha-1/2}\sum_{n=0}^{\infty} d_{2n}^{(\alpha)}C_{2n}^{(\alpha)}(a), \tag{4.25}$$

where $C_{2n}^{(\alpha)}$ are the ultraspherical (Gegenbauer) polynomials [35]. Of course, the coefficients $d_{2n}^{(\alpha)}$ can be expressed as known linear combinations of the ratios μ_{2p} $(p \leqslant n)$, where

$$\mu_p = \frac{M_p^{\rm vac}}{M_0^{\rm vac}}. \tag{4.26}$$

The parameter $\alpha > -1/2$ is to be chosen in such a way that the coefficients $d_{2n}^{(\alpha)}$ tend to zero for large n as fast as possible. If this goal is achieved we can gain a fairly precise information about $\varrho_{\rm vac}$ calculating a finite number of the even moments.

It is convenient to write $M_p^{\rm vac}$ as a sum of two terms

$$M_p^{\rm vac}(\Theta) = M_{K=0}^{(p)} + M_{K>0}^{(p)}, \tag{4.27}$$

where $M_{K=0}^{(p)}$ denotes the contribution of $K = 0$ states and $M_{K>0}^{(p)}$ is the contribution of the other states. It follows that

$$M_{K=0}^{(p)} = -2\lim_{t\to 0^+}\sum_{n=1}^{\infty}\langle M_n^{(p)}e^{-tx_n}\rangle, \tag{4.28}$$

cf. (B.1) and (3.49a). Using (B.2) and (3.58) we obtain

$$M_{K=0}^{(p)} = \mathscr{A}_{K=0}^{(p)} + \mathscr{R}_{K=0}^{(p)}, \tag{4.29}$$

where

$$\mathscr{A}_{K=0}^{(p)} = \frac{\Theta}{\pi(p+1)} \tag{4.30}$$

and

$$\mathscr{R}_{K=0}^{(p)} = -2\sum_{n=1}^{\infty}\langle M_n^{(p)}\rangle. \tag{4.31}$$

Similarly, cf. (3.69), (3.77) and the Appendix B,

$$M_{K>0}^{(2p)} = \overline{\mathscr{A}}_{K>0}^{(2p)} + \overline{\mathscr{R}}_{K>0}^{(2p)}, \tag{4.32}$$

where

$$\overline{\mathscr{A}}^{(2p)}_{K>0} = \frac{8}{\pi} \int_0^{\pi/2} d\beta_0 \sin^2 \beta_0 [(\langle m_2^{(2p)} a_0 \rangle + \langle m_1^{(2p)} a_1 \rangle$$

$$+ \langle m_0^{(2p)} a_2 \rangle) \cos^{-1} \beta_0 - m_0^{(2p)} \langle a_0 a_1 \rangle] \tag{4.33}$$

and

$$\overline{\mathscr{R}}^{(2p)}_{K>0} = -8 \lim_{t \to 0^+} \sum_{nv} \langle v M^{(2p)}_{nv\varepsilon} \rangle e^{-tv/\cos\beta_0}. \tag{4.34}$$

The coefficients $m_k^{(2p)}$ are defined by the asymptotic expansion

$$M^{(2p)}_{nv\varepsilon} \sim \sum_{k=0}^{\infty} \frac{m_k^{(2p)}(\beta_0, \varepsilon; \Theta)}{v^k}. \tag{4.35}$$

From the recurrence formulae (B.14–B.16) we obtain, see Appendix E,

$$m_0^{(2p)} = \frac{1}{2p+1} + \frac{2p}{2p+1} W_{2p}(\cos\beta_0) \tag{4.36a}$$

$$\langle m_1^{(2p)} a_1 \rangle = -\frac{p}{4(2p+1)C} \frac{\cos^3\beta_0}{\sin^2\beta_0} \frac{dW_{2p}(\cos\beta_0)}{d(\cos\beta_0)} \operatorname{sac}(\Theta, \beta_0)$$

$$+ \frac{p}{4(2p+1)} \Big\{ \cot^4 \beta_0 (W_{2p}(\cos\beta_0) - 1)(1 + C^{-2})$$

$$+ \frac{2p+1}{C^2} \cot^2 \beta_0 W_{2p}(\cos\beta_0) + \cot^2 \beta_0 V_{2p}(\cos\beta_0) \Big\} \sin\Theta \cos\Theta, \tag{4.36b}$$

$$\langle m_2^{(2p)} a_0 \rangle = \frac{p}{4(2p+1)C} \Big\{ -\frac{\cos^3\beta_0}{\sin^2\beta_0} \frac{dW_{2p}(\cos\beta_0)}{d(\cos\beta_0)}$$

$$- (W_{2p}(\cos\beta_0) - 1) \left(\frac{\sin^2\Theta}{C^2} + \cos^2\beta_0 \cos^2\Theta \right) \cot^2 \beta_0$$

$$- \frac{2p+1}{C^2} W_{2p}(\cos\beta_0) \sin^2\Theta + C^2 V_{2p}(\cos\beta_p) \cot^2\beta_0 \Big\} \operatorname{sac}(\Theta, \beta_0), \tag{4.36c}$$

where C is defined by (3.64) and $\operatorname{sac}(\Theta, \beta_0)$ by (4.7),

$$W_{2p}(x) = \frac{(2p-2)!!}{(2p-1)!!} \sum_{k=0}^{p-1} \frac{(2k+1)!!}{(2k+1)(2k)!!} x^{2(p-k)} \tag{4.37}$$

$$V_{2p}(x) = \frac{(2p-2)!!}{(2p-1)!!} \sum_{k=0}^{p-1} \frac{(2k+1)!!}{(2k)!!} x^{2(p-k)} \tag{4.38}$$

and $\langle a_2\rangle$, $\langle a_0 a_1\rangle$ are given by (4.5)–(4.6). Thus, calculating the integral in (4.33) numerically we obtain the 'anomalous parts' of the even moments. The 'remainders' can be computed numerically from (4.34).

The results of this work as well as further details will be published elsewhere.

4.4. Isoscalar charge radius

The moment M_2^{vac}, cf. (4.22), gives the contribution of the fermionic vacuum to the isoscalar charge radius of the baryon. For $p = 1$ the formulae (4.36a–c) simplify considerably. We obtain

$$m_0^{(2)} = 1-\tfrac{2}{3}\sin^2\beta_0 \tag{4.39a}$$

$$\langle m_1^{(2)}a_1\rangle = \frac{\cos^4\beta_0}{6C^2\sin^2\beta_0}(\sin\Theta\cos\Theta - C\,\mathrm{sac}(\Theta,\beta_0)) \tag{4.39b}$$

$$\langle m_2^{(2)}a_0\rangle = -\frac{\cos^2\beta_0}{12C^3}(C^2\cot^2\beta_0+2\sin^2\Theta)\,\mathrm{sac}(\Theta,\beta_0). \tag{4.39c}$$

From these expressions and (4.33) performing integration we derive

$$\overline{\mathscr{A}}{}^{(2)}_{K>0} = \frac{1}{\pi}\{2\tan\Theta(1-|\gamma\tan\Theta|)-\tfrac{5}{6}\sin 2\Theta-\tfrac{2}{3}\sin\Theta\,\Phi(\gamma)\}, \tag{4.40}$$

where $\gamma = \arcsin(\cos\Theta)$ and

$$\Phi(\gamma) = \int_0^\gamma \frac{\gamma du u}{\sin u}. \tag{4.41}$$

We can also calculate the average

$$\langle m_3^{(2)}\rangle = \left\{\frac{\sin^2\Theta\cos^3\beta_0(1+\sin^2\beta_0)}{12C^2\sin^2\beta_0}\right.$$
$$\left. -\frac{\cos^5\beta_0}{6C^2\sin^4\beta_0}-\frac{\sin^2\Theta\cos^5\beta_0}{2C^4\sin^2\beta_0}\right\}\sin\Theta\cos\Theta$$
$$+\left\{\frac{\cos^5\beta_0}{6C\sin^4\beta_0}+\frac{\sin^2\Theta\cos^3\beta_0(1+\sin^2\beta_0)}{6C^3\sin^2\beta_0}\right.$$
$$\left.+\frac{\cos^2\Theta\cos^7\beta_0}{12C^3\sin^2\beta_0}-\frac{\sin^2 2\Theta\cos^5\beta_0}{8C^5}\right\}\mathrm{sac}(\Theta,\beta_0) \tag{4.42}$$

and the integral

$$\mathscr{G}(\beta,\Theta) = \int_0^\beta d\beta_0\tan^2\beta_0\langle m_3^{(2)}(\beta_0,\varepsilon;\Theta)\rangle$$

$$= \frac{1-u}{6} \sin\Theta \left\{ \cos\Theta \left(\frac{1-u^2-u\sin^2\Theta}{1-u^2\cos^2\Theta} + \frac{1-u}{2u} \right) - \frac{(1+u)\arcsin(u\cos\Theta)}{\sqrt{1-u^2\cos^2\Theta}} \left(\frac{\sin^2\Theta}{1-u^2\cos^2\Theta} + \frac{1-u^2}{2u^2} \right) \right\}, \tag{4.43}$$

where $u = \sin\beta$. Derivation of the above formulae is rather lengthy and we will not reproduce it here. Let us remark, however, that the formulae (4.42)–(4.43) match very well the results of numerical calculations [9] int he regions where the Debye expansion used in Section 3.2 is valid. Moreover, it follows from (4.43) that

$$\mathscr{G}(\pi/2, \Theta) = 0, \tag{4.44}$$

i.e. the condition (3.71b) for the finiteness of $\mathscr{R}^{(2)}_{K>0}$ is fulfilled. Since $\langle m_3^{(2)} \rangle$ is a rather complicated expression this condition provides also a fairly non-trivial cross check of (4.42)–(4.43).

Let us show now that $\mathscr{R}^{(2)}_{K>0}$ is finite. Let $\beta_{\min}$ be the zeroth order approximation of $x_{1\nu\varepsilon}$, the first energy level for given ν and ε, and let ν be large. When we are using the Debye expansion $\beta_{\min}$ is the solution of (3.59) for $n = 1$, and, cf. (3.78),

$$\langle m_3^{(2)}(\beta_0, \varepsilon; \Theta) \rangle = O(\beta_{\min}^{-2}) = O(\nu^{2/3}). \tag{4.45}$$

Using the Euler-MacLaurin formula we estimate

$$\sum_n \langle m_3^{(2)} \rangle = \frac{\nu}{\pi} \int_{\beta_{\min}}^{\pi/2} d\beta_0 \tan^2\beta_0 \langle m_3^{(2)} \rangle + \tfrac{1}{2} \langle m_3^{(2)}(\beta_0, \varepsilon; \Theta) \rangle + \ldots = O(\nu^{2/3}) \tag{4.46}$$

and

$$\sum_n \nu \langle M^{(2)}_{n\nu\varepsilon} \rangle = O(\nu^{-4/3}). \tag{4.47}$$

Thus, the remainder $\mathscr{R}^{(2)}_{K>0}$ is finite, and it can be rewritten in the following simple form, cf. (3.72),

$$\mathscr{R}^{(2)}_{K>0}(\Theta) = -\sum_{\nu>1/2} 8\nu \sum_n \langle M^{(2)}_{n\nu\varepsilon} \rangle. \tag{4.48}$$

Moreover, we can simplify the numerical task writing

$$-8\nu \sum_n \langle M^{(2)}_{n\nu\varepsilon} \rangle \approx -8\nu \sum_{n<N} \langle M^{(2)}_{n\nu\varepsilon} \rangle + d^{an}_{KN}(\Theta), \tag{4.49}$$

where

$$d^{an}_{KN} = -\frac{8}{\nu^2} \left\{ \tfrac{1}{2} \langle m_3^{(2)}(\beta_{N\nu}, \varepsilon; \Theta) \rangle - \frac{\nu}{\pi} \mathscr{G}(\beta_{N\nu}, \Theta) \right\} \tag{4.50}$$

and $\beta_{N\nu}$ is the solution of (3.59) for $n = N$, $N \gg 1$. In Fig. 1 we show the result for $\mathscr{R}^{(2)}_{K>0}$ given in [9].

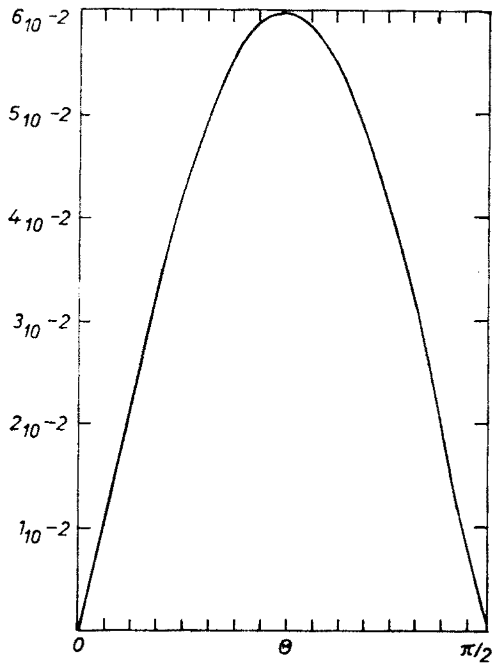

Fig. 1. $\mathscr{R}^{(2)}_{K>0}$ as a function of Θ

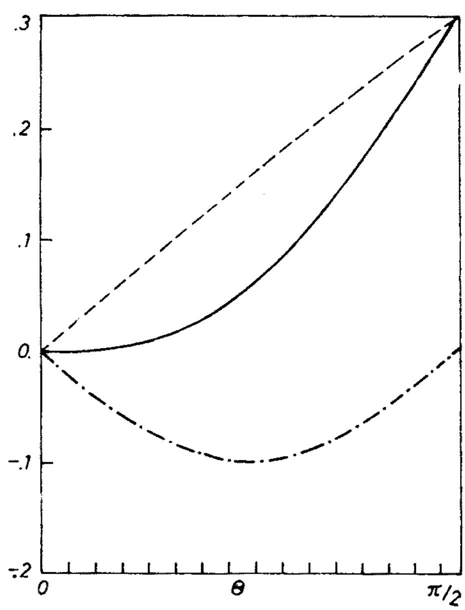

Fig. 2. Second moment of the baryon number radial distribution in the vacuum: $M_2^{\text{vac}}(\Theta)$ (solid line, $M^{(2)}_{K=0}$ (dashed line) and $M^{(2)}_{K>0}$ (dashed-dotted line)

Numerical evaluation of the sum in Eq. (4.31) leads to the result which can be well approximated by the following formula:

$$\mathscr{R}^{(2)}_{K=0} = 0.298\,\frac{\Theta}{\pi} - 0.128\left(\frac{\Theta}{\pi}\right)^3 + \delta, \tag{4.51}$$

where $|\delta/\mathscr{R}^{(2)}_{K=0}| < 0.01$.

In Fig. 2 M_2^{vac} is plotted as a function of the chiral angle Θ for $0 \leqslant \Theta \leqslant \frac{\pi}{2}$. The terms $M^{(2)}_{K=0}$ and $M^{(2)}_{K>0}$ are also shown. Symmetry relations:

$$M_p^{\text{vac}}(-\Theta) = -M_p^{\text{vac}}(\Theta) \tag{4.52a}$$

$$M_p^{\text{vac}}(\Theta+\pi) = M_p^{\text{vac}}(\Theta) \tag{4.52b}$$

provide an extension of these results for all values of $\Theta \neq (k+\frac{1}{2})\pi$.

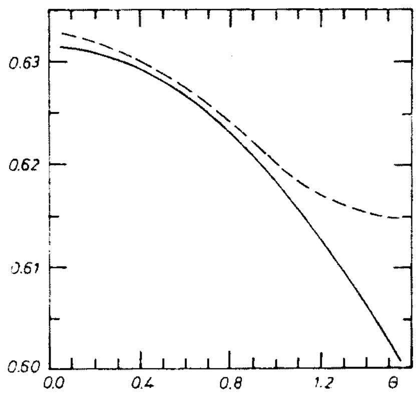

Fig. 3. The ratios $M^{(2)}_{K=0}/M^{(0)}_{K=0}$ (solid line) and $M^{(2)}_{K>0}/M^{(0)}_{K>0}$ (dashed line) as functions of Θ

In Fig. 3 the ratios $M^{(2)}_{K=0}/M^{(0)}_{K=0}$ and $M^{(2)}_{K>0}/M^{(0)}_{K>0}$ are shown. For the uniform distribution such a ratio would be equal to $\frac{3}{5}$. Calculating higher moments and inverting them as described in the preceding section we can check [7] that the contribution of the $K = 0$ states is nearly uniform. The curves plotted in Fig. 3 suggest that the same may be true for the contribution of the other states.

5. *Other charges*

5.1. Motivation: Cheshire cat hypothesis

The more sophisticated bag models of hadrons become, the less precisely they seem to determine the bag radius. This observation put forward in [48] has led the authors of this paper to the so-called Cheshire cat models (CCM). In CCM the bag in an unphysical concept, having mainly to do with the fact that the effective mesonic theory, say, given by the Skyrme lagrangian, may be not accurate enough. According to this idea for exact bag models the physics should be completely independent of changes in the size or shape of the bag. Thus, fitting the bag radius in a particular model one minimizes the effects of the errors of this model. A model without errors should lead to an undefined bag radius. In 1+1 space-time dimensions this remarkable idea can be realized in models for which bosonization is known to be exact[6]. In 3+1 dimensions the situation is much less clear. However, one may still hope that CCM is a good approximation of reality.

Coming back from the heavens to our harsh world we have to choose a model with tractable bosonic and fermionic sectors and check whether the results are indeed insensitive to the bag radius, at least for some range of this parameter. A natural idea is to consider non-interacting quarks in the bag surrounded by the hedgehog configuration of the pion field. Then, the requirement of chiral invariance can be fulfilled if an interaction between pions and quarks is added at the surface of the bag [28]. The form of this interaction is fixed by the chiral symmetry and the boundary condition (3.1) follows from the lagrangian obtained in this way. According to CCM the physical quantities should not depend on the bag radius R, so outside the bag the strength of the pion field, or in other words the chiral angle $\theta(X)$, should be the same as in the pure soliton case. Thus, the function $\theta(X)$ is the solution of the Euler-Lagrange equations originating from the Skyrme lagrangian for the boundary conditions $\theta(0) = -\pi$, $\theta(\infty) = 0$, cf. Chapter 2. The CCM implies that

$$\Theta = \theta(\tilde{R}), \tag{5.1}$$

where

$$d = eF_\pi \tag{5.2}$$

$$\tilde{R} = dR. \tag{5.3}$$

In Fig. 4 we show the result for the second moment of the baryon number distribution in the proton which follows from the model described above [8]. In this figure the contributions from the soliton, the vacuum, and the valence quarks to

$$d^2\langle r^2\rangle_{I=0} = M_2^S + \tilde{R}^2(M_2^{\mathrm{vac}} + M_2^{\mathrm{val}}) \tag{5.4}$$

are shown. In (5.4) $\langle r^2\rangle_{I=0}$ denotes the square of the isoscalar charge radius. The two scales drawn for the figure (i.e. the bottom scale Θ and the top scale $\tilde{R}$) are related by the condition (5.1).

In the region $-\pi \leqslant \Theta \leqslant -\pi/2$, where there are no valence quarks in the baryons, M_2^{val} (dashed line) is equal to zero. For $\Theta > -\pi/2$ the valence quarks occupy the lowest energy level in the bag. Their contribution to $\langle r^2\rangle_{I=0}$ increases like R^2 for large R and becomes dominant for large bags. M_2^{val} does not vanish when Θ tends to $-\pi/2$ from above. Thus, M_2^{val} is discontinuous at $\Theta = -\pi/2$. However, the total contribution of the fermions in the bag, i.e. the sum of M_2^{val} and M_2^{vac}, is continuous.

As we see, for our simple chiral bag the Cheshire cat hyphotesis works reasonably

[6] See e.g. [49] and the references quoted therein.

well for $\tilde{R} \leqslant 2.5$. In particular for $1 \leqslant \tilde{R} \leqslant 2.5$ we can see some indications for the CCM because M_2^S falls while the fermionic contribution rises, and they roughly compensate. In the region $\tilde{R} > 2.5$ the valence quarks take over. The contribution of the pion cloud is already small in this region and its variation cannot compensate the increase coming from the valence quarks. This is not surprising because we have switched off the gauge interactions between quarks, and this approximation is rather poor when the bag radius

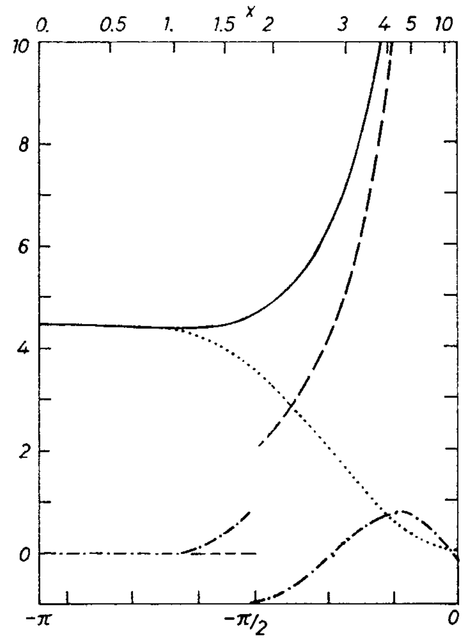

Fig. 4. Contributions to $d^2\langle r^2\rangle_{I=0}$ (solid line) from: soliton — M_2^S (dotted line), vacuum — $\tilde{R}^2 M_2^{vac}$ (dashed-dotted line), and valence quarks — $R^2 M_2^{val}$ (dashed line)

becomes comparable with the confinement scale. Therefore, it is plausible that the Cheshire cat principle can be applied to the chiral bags discussed in our paper and the approximation obtained in this way is reasonably accurate. Taking this seriously we are immediately led to the conclusion that the vacuum in the chiral bag carries not only baryon number but also other quantum numbers like spin[7], isospin and strangeness in the $SU_{3\text{flavour}}$ case. In particular for $-\pi \leqslant \Theta \leqslant -\pi/2$ the variations with R of the skyrmion contributions to these quantities must be compensated by the variations of the vacuum contributions if CCM is to work.

In order to prove the conjecture that it is possible to obtain non-zero contributions from the vacuum polarization not only to baryon numbers but to other quantities as well one has to calculate appropriate averages over single particle energy levels. In principle such calculation is similar to the calculation of spectral asymmetries for the baryon number,

[7] Induced fractional angular momentum has been found in (2+1)-dimensional QED [50].

as described in the preceding sections. However, such work is still to be done. In what follows we show that there exists a coupling between the modes of collective motion and the single particle energy levels for quarks in chiral bags. Spin, isospin, and other internal quantum numbers are related to the collective motion of the bag. Therefore, such a coupling is necessary if the vacuum is to contribute to these quantities.

5.2. Spin and isospin [10]

Let Ψ_{KM} be a solution of the Dirac equation for massless quarks and the chiral boundary condition (3.1) for given K and M, cf. 3.1. The action S for the chiral bag, see (2.8) remains unchanged by the following global rotations:

$$U \to U = AU_0A^{-1} \tag{5.5}$$

$$\Psi_{KM} \to \Psi = AD^{(K)}\Psi_{KM}, \tag{5.6}$$

where A is an SU_2 matrix acting in isospin space, and $D^{(K)}$ is a matrix from $(2K+1)$-dimensional representation of rotations in K-space. Thus, the system posses zero modes which have to be treated by the method of collective coordinates.

Following [3] we consider A and $D^{(K)}$ as time dependent operators and write:

$$A = a_0 + ia_k\tau_k \equiv a_0 + i\vec{a}\cdot\vec{\tau}, \tag{5.7}$$

where

$$a_\alpha a_\alpha \equiv a_0^2 + \vec{a}\cdot\vec{a} = 1 \tag{5.8}$$

and τ_k are the Pauli matrices acting in isospace. Rotations in K-space are parametrized in a similar way: a transformation of Ψ_{KM} is specified by the transformation of the fundamental representation, i.e. by some SU_2 matrix B:

$$B = b_0 + i\vec{b}\cdot\vec{\sigma}, \tag{5.9}$$

where

$$b_\alpha b_\alpha = 1. \tag{5.10}$$

The lagrangian of the collective motion reads:

$$\mathscr{L}_{\text{coll}} = 2\lambda \dot{a}_\alpha \dot{a}_\alpha + \mathscr{L}_{\text{cf}}, \tag{5.11}$$

where

$$\mathscr{L}_{\text{cf}} = i\int d^3r \Psi^\dagger_{KM} D^{(K)\dagger}(\dot{D}^{(K)} + \dot{A}^\dagger A D^{(K)})\Psi_{KM} \tag{5.12}$$

and [3]

$$\lambda = \frac{4\pi}{6e^3F_\pi}\int\limits_{\tilde{R}}^{\infty} dX X^2 \sin^2\theta(X)\left\{1+4\left(\theta'^2 + \frac{\sin^2\theta}{X^2}\right)\right\}. \tag{5.13}$$

Calculating the matrix elements in (5.12) we obtain:

$$\mathscr{L}_{\mathrm{cf}} = -2M\mathscr{W}(b_\alpha, \dot{b}_\alpha) - \frac{1-\alpha_v}{2K}\mathscr{V}_j(a_\alpha, \dot{a}_\alpha)\mathscr{T}_j(b_\alpha), \tag{5.14}$$

where

$$\mathscr{W}(b_\alpha, \dot{b}_\alpha) = b_0\dot{b}_3 - b_3\dot{b}_0 + b_1\dot{b}_2 - b_2\dot{b}_1 \tag{5.15}$$

$$\mathscr{V}_j(a_\alpha, \dot{a}_\alpha) = a_0\dot{a}_j - a_j\dot{a}_0 + \varepsilon_{jpq}a_p\dot{a}_q \tag{5.16}$$

$$\mathscr{T}_j(b_\alpha) = M\{(b_0^2 - b_kb_k)\delta_{j3} + 2b_jb_3 - \varepsilon_{jr3}b_0b_r\}. \tag{5.17}$$

In order to derive the formula for the rotational shifts of energy levels we perform canonical quantization of the system [51][8]. We calculate the canonical momenta

$$p_\alpha = \frac{\partial \mathscr{L}_{\mathrm{coll}}}{\partial \dot{a}_\alpha} \tag{5.18a}$$

$$\pi_\alpha = \frac{\partial \mathscr{L}_{\mathrm{coll}}}{\partial \dot{b}_\alpha} \tag{5.18b}$$

and the hamiltonian of the collective motion

$$\mathscr{H}_{\mathrm{coll}} = p_\alpha\dot{a}_\alpha + \pi_\alpha\dot{b}_\alpha - \mathscr{L}_{\mathrm{coll}} = \frac{1}{8\lambda}\sum_{\alpha=0}^{3}\left(p_\alpha + \frac{1-\alpha_v}{2K}\mathscr{T}_j\frac{\partial\mathscr{V}_j}{\partial\dot{a}_\alpha}\right)^2. \tag{5.19}$$

We see that the π's do not depend on the generalized velocities. Thus, Eqs. (5.18b) should be considered as the following set of primary constraints:

$$\phi_\alpha = \pi_\alpha + 2M\frac{\partial\mathscr{W}}{\partial\dot{b}_\alpha} = 0. \tag{5.20}$$

These constraints are all second class because the matrix of Poisson brackets for constraints has a non-vanishing determinant. Following the standard procedure we calculate the canonical Dirac brackets and obtain the following canonical commutation relations:

$$[\hat{a}_\alpha, \hat{p}_\beta] = i\delta_{\alpha\beta} \tag{5.21a}$$

$$[\hat{b}_1, \hat{b}_2] = [\hat{b}_0, \hat{b}_3] = \frac{i}{4M} \tag{5.21b}$$

$$[\hat{\pi}_1, \hat{\pi}_2] = [\hat{\pi}_0, \hat{\pi}_3] = iM \tag{5.21c}$$

$$[\hat{b}_\alpha, \hat{\pi}_\beta] = \frac{i}{2}\delta_{\alpha\beta}, \tag{5.21d}$$

whereas all other independent commutators vanish.

[8] For a recent pedagogical review see [52].

In order to write the hamiltonian in a form in which its symmetries become explicit we define three sets of operators:

$$\hat{\mathscr{I}}_j = \tfrac{1}{2}(\hat{a}_j\hat{p}_0 - \hat{a}_0\hat{p}_j + \varepsilon_{jlm}\hat{a}_l\hat{p}_m) \tag{5.22}$$

$$\hat{\mathscr{S}}_j = \tfrac{1}{2}(\hat{a}_0\hat{p}_j - \hat{a}_j\hat{p}_0 + \varepsilon_{jlm}\hat{a}_l\hat{p}_m) \tag{5.23}$$

and $\hat{\mathscr{T}}_j$ define by (5.17). It follows from the commutation relations (5.21) that the commutators of the operators from the different sets vanish and each set forms an SU_2 algebra. In our method of quantization the constraints (5.8), (5.10) limit the Hilbert space of states to the subspace of physical states H_{phys}. For a physical state

$$\hat{p}_\alpha\hat{p}_\alpha|\phi_{\text{phys}}\rangle = 2(\hat{\mathscr{I}}^2 + \hat{\mathscr{S}}^2)\,|\phi_{\text{phys}}\rangle \tag{5.24}$$

and

$$\hat{\mathscr{T}}^2|\phi_{\text{phys}}\rangle = (M^2 - \tfrac{1}{4})\,|\phi_{\text{phys}}\rangle, \tag{5.25}$$

where $\hat{\mathscr{I}}^2 = \hat{\mathscr{I}}_j\,\hat{\mathscr{I}}_j$ et c. The quantum hamiltonian acting in H_{phys} can be written as follows:

$$\mathscr{H}_{\text{coll}} = \frac{\hat{\mathscr{I}}^2 + \hat{\mathscr{S}}^2}{4\lambda} = \frac{1-\alpha_v}{4\lambda K}\hat{\mathscr{T}}_j\hat{\mathscr{S}}_j + \frac{(1-\alpha_v)^2}{32\lambda K^2}\hat{\mathscr{T}}^2. \tag{5.26}$$

The shifts of quark energy levels depend on both Θ and the modes of the collective motion of the whole system. Thus, the spectral asymmetries corresponding to the vacuum expectation values of spin and isospin may be non-zero.

5.3. Strangeness

A lot of studies have been devoted to the problem of other flavours; for a review see [53]. In spite of interesting results concerning quantization of skyrmions in this case [27, 54], the outcome of the studies for pure skyrmions is unsuccessful from the phenomenological point of view. Ass hown in [55] the pattern of energy levels for the lowest octet and decuplet of baryons resembles the experimental data but the overall scale of energy splittings is too small; however, see also [56]. Since this scale factor is related to well establish parameters like F_π and the masses of pseudoscalar mesons we have to conclude that the Skyrme model is unable to describe the spectrum of baryons for more than two flavours. It is possible that for chiral bags the situation may look better but calculations are still to be done.

Callan and Klebanov [57] have argued that baryons carrying heavy flavours can be described by bound states of the corresponding heavy measons in the background field of the basic SU_2 skyrmion. In particular hyperons can be considered as bound kaon--skyrmion systems. They have shown that the bound states exist in the channels needed to reproduce the quark model quantum numbers of strange baryons. Moreover, model--independent mass relations derived within this approach work very well. From a more theoretical point of view one can show that the baryonic current is conserved, despite a defect of the bound state wave function at the center of the soliton [58].

In [59] it has been shown that the distribution of the baryon number in hyperons resulting from the model of [57] is not spherically symmetric and reads:

$$\varrho(r,\vartheta,\varphi)=\frac{1}{2\pi^2r^2}\left\{\Theta'\left(\sin^2\Theta\cos^4\frac{\gamma}{2}+\tfrac{1}{2}\sin^2\gamma\cos^2\frac{\Theta}{2}+4\sin^4\frac{\gamma}{2}\cos^4\frac{\Theta}{2}\cos^2\vartheta\right)\right.$$

$$\left.+\gamma'\sin\Theta\cos^2\frac{\Theta}{2}\sin\gamma\left(\cos^2\frac{\gamma}{2}+\sin^2\frac{\gamma}{2}\cos^2\vartheta\right)\right\} \tag{5.27a}$$

for the ground states and

$$\varrho(r,\vartheta,\varphi)=\frac{\sin^2\frac{\Theta}{2}}{2\pi^2r^2}\left\{\Theta'\left(1+(1+2\cos\Theta)\cos\gamma+4\sin^4\frac{\gamma}{2}\sin^2\frac{\Theta}{2}\cos^2\vartheta\right.\right.$$

$$\left.\left.+2\cos\Theta\sin^4\frac{\gamma}{2}\right)-\gamma'\sin\Theta\sin\gamma\left(\cos^2\frac{\gamma}{2}+\sin^2\frac{\gamma}{2}\cos^2\vartheta\right)\right\} \tag{5.27b}$$

for the odd-parity excited states. The function γ gives the strength of the kaon field for the bound state, cf. [59]. If follows from (5.27a) that the ground states of hyperons are larger than the ground states of non-strange baryons [60]. However, (5.27b) implies the opposite, i.e. the odd-parity excited states are smaller than the ground states of non-strange baryons. This result is indeed unexpected.

The lack of spherical symmetry for hyperons in the Callan-Klebanov approach should be manifest for the corresponding chiral bags as well. Consequently, the wave functions of quarks in the bag are not K-symmetric and degeneracy of the energy levels with respect to M disappears. Thus, the calculation of spectral asymmetries becomes nearly impossible in this case.

6. *Summary*

The fermionic vacuum in the chiral bag carries a non-zero baryon number. The sum of the baryon numbers of the vacuum, valence quarks, and the pion field outside the bag is an integer number which does not depend on quark masses and at least in the massless case on the shape of the bag as well. We give arguments showing that the baryon number of the whole system is a homotopy invariant which should not depend on 'dynamical details' like for example gauge interactions between quarks.

The effects due to the vacuum polarization by a topologically non-trivial configuration of pion field can be described in terms of spectral asymmetries for corresponding operators. A convenient method for calculation of these spectral asymmetries for spherical chiral bags is to use the Poisson method, or equivalently the point splitting regularization technique, for summation of the corresponding summable series. Then, using the Debye expansion of Bessel functions we can express the spectral asymmetry under study as a sum of two pieces: an 'anomalous' one which can be calculated analytically and a 'remainder' which is convergent without any regularization and can be evaluated numerically. The method allows calculation of the vacuum baryon number as well as of its distribution in chiral bags. The calculations performed so far support the Cheshire cat hypothesis for chiral angles Θ near the 'magic' value $-\pi/2$. It follows from the Cheshire cat principle that the vacuum in chiral bags should carry other quantum numbers like spin and isospin. Considering the collective motion of chiral bags we have derived the quantum hamiltonian of the collective motion which may result in non-zero vacuum expectation values for these quantities. Extension of the chiral bag model to more than two flavours turns out to be difficult from the calculational point of view. In particular a recent proposal to consider strange baryons as bound states of SU_2 skyrmions and kaons results in a non-spherical distribution of hadronic matter in hyperons. A corresponding chiral bag would be also non-spherical, if the Cheshire cat principle is applicable in SU_3 case. Thus, the calculation of spectral asymmetries for more than two flavours becomes very difficult.

I would like to thank Professor Kacper Zalewski for a lot of discussions during our work on problems discussed in this report, as well as for an encouragement afterwards. I have benefited from discussions and collaboration with Krzysztof Heller, Paweł Mazur, Michał Praszałowicz, Maciek A. Nowak and last but not least Mannque Rho. I thank also many other colleagues from the Institute of Nuclear Physics and the Jagellonian University in Cracow, the Max-Planck-Institut in Munich and the Theory Division in CEN Saclay. In particular I am indebted to Hans Kühn and William Thacker for helpful remarks about the manuscript and to the Authors of Ref. [61] for their work which was of much help in the course of preparation of the present paper.

APPENDIX A

In this Appendix we specify our conventions for the spherical harmonics, $6j$ symbols etc., and we give a derivation of Eqs. (3.16)–(3.19).

Our conventions for spherical harmonics and addition of angular momenta are taken from [32, 34]. The spherical harmonics $Y_{l,m}$ are defined as follows

$$Y_{l,m}(\vartheta, \varphi) = (-1)^{(m+|m|)/2} i^l \left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P_l^{|m|}(\cos\vartheta) e^{im\vartheta}, \tag{A.1}$$

where the associated Legendre polynomials are defined by

$$P_l^m(\cos\vartheta) = \frac{1}{2^l l!} \sin^m \vartheta \frac{d^{l+m}}{(d\cos\vartheta)^{l+m}} (\cos^2\vartheta - 1)^l. \tag{A.2}$$

In particular for $\vartheta = 0$ we have

$$Y_{l,m}(\hat{n}_z) = i^l \sqrt{\frac{2l+1}{4\pi}}\, \delta_{m0}. \tag{A.3}$$

The spin-angular functions ϕ_{jlm}, see (3.4), read:

$$\phi_{l+1/2,lm} = \sqrt{\frac{j+m}{2j}}\,\chi_{1/2}Y_{l,m-1/2}+\sqrt{\frac{j-m}{2j}}\,\chi_{-1/2}Y_{l,m+1/2} \tag{A.4a}$$

$$\phi_{l-1/2,lm} = -\sqrt{\frac{j-m+1}{2j+2}}\,\chi_{1/2}Y_{l,m-1/2}+\sqrt{\frac{j+m+1}{2j+2}}\,\chi_{-1/2}Y_{l,m+1/2}. \tag{A.4b}$$

The formula (3.16) follows immediately from

$$i\vec{\sigma}\cdot\hat{\boldsymbol{n}}\phi_{l\pm 1/2,lm} = \mp\phi_{l\pm 1/2,l\pm 1,m} \tag{A.5}$$

because the operator $\vec{\sigma}\cdot\hat{\boldsymbol{n}}$ acts trivially on the isospin eigenfunctions λ_σ. In order to prove (A.5) we note that $\vec{\sigma}\cdot\hat{\boldsymbol{n}}$ does not change the total angular momentum

$$\boldsymbol{J} = \boldsymbol{L}+\boldsymbol{S} \tag{A.6}$$

of a state and changes its spatial parity. So, the l.h.s. and r.h.s. of (A.5) must be proportional. Due to rotational symmetry the proportionality constant cannot depend on m, and its modulus is equal to one because $(\vec{\sigma}\cdot\hat{\boldsymbol{n}})^2 = 1$. Then, comparing the phases of $\phi_{l\pm 1/2,lm}$ and $\phi_{l\pm 1/2,l\pm 1,m}$ at $\vartheta = 0$ (i.e. for $\hat{\boldsymbol{n}} = \hat{\boldsymbol{n}}_z$) and using (A.3)–(A.4), we fix the phase in (A.5).

In order to prove (3.17) we introduce an operator

$$\boldsymbol{T} = \boldsymbol{I}+\boldsymbol{L}, \tag{A.7}$$

its eigenfunctions

$$\zeta_{tlm} = \sum_{\sigma=\pm\frac{1}{2}}\langle 1/2,\sigma;l,m-\sigma|t,m\rangle\lambda_\sigma Y_{l,m-\sigma} \tag{A.8}$$

and define

$$\xi_{KtlM} = \sum_{\sigma=\pm\frac{1}{2}}\langle t,M-\sigma;1/2,\sigma|KM\rangle\zeta_{tl,M-\sigma}\chi_\sigma \tag{A.9}$$

which are the eigenfunctions of

$$\boldsymbol{K} = \boldsymbol{T}+\boldsymbol{S}, \tag{A.10}$$

cf. (3.2). Using a relation between the Clebsch-Gordan coefficients

$$\langle j_1,m_1;j_2,m_2|j,m\rangle = (-1)^{j-j_1-j_2}\langle j_2,m_2;j_1,m_1|j,m\rangle, \tag{A.11}$$

we deduce from (A.5) that

$$i\vec{\tau}\cdot\hat{\boldsymbol{n}}\zeta_{l\pm 1/2,lm} = \pm\zeta_{l\pm 1/2,l\pm 1,m} \tag{A.12}$$

and, consequently,

$$i\vec{\tau}\cdot\hat{\boldsymbol{n}}\xi_{K,l\pm 1/2,lM} = \pm\xi_{K,l\pm 1/2,l\pm 1,M}. \tag{A.13}$$

The two sets of eigenfunctions of $\boldsymbol{K}$, Ψ_{KjlM} and ξ_{KtlM}, are related by an orthogonal transformation, see [32]:

$$\Psi_{KjlM} = \sum_t (-1)^{K+l+1} \sqrt{(2j+1)(2t+1)} \begin{Bmatrix} \frac{1}{2} & l & t \\ \frac{1}{2} & K & j \end{Bmatrix} \xi_{KtlM}, \tag{A.14}$$

where { } denote $6j$ symbols. The relation inverse to (A.14) reads:

$$\xi_{KtlM} = \sum_j (-1)^{K+l+1} \sqrt{(2j+1)(2t+1)} \begin{Bmatrix} \frac{1}{2} & l & t \\ \frac{1}{2} & K & j \end{Bmatrix} \Psi_{KjlM}. \tag{A.15}$$

Now, we rewrite the l.h.s. of (3.17): we use (A.14), then (A.13), and then (A.15). In this way we obtain the following identity:

$$i\vec{\tau}\cdot\hat{\boldsymbol{n}}\Psi_{K,l\pm 1/2,lM} = -\sum_{\beta=\pm 1}\sum_{\gamma=\pm 1} \beta(2l+\beta+1)\sqrt{(2l\pm 1+1)(2l+2\beta+\gamma+1)}$$

$$\begin{Bmatrix} \frac{1}{2} & l & l+\dfrac{\beta}{2} \\ \frac{1}{2} & K & l\pm\frac{1}{2} \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & l+\beta & l+\dfrac{\beta}{2} \\ \frac{1}{2} & K & l+\beta+\dfrac{\gamma}{2} \end{Bmatrix} \Psi_{K,l+\beta+\gamma/2,l+\beta,M}. \tag{A.16}$$

All $6j$ symbols which appear in (A.16) can be evaluated from the formulae [32]:

$$\begin{Bmatrix} a & b & c \\ \frac{1}{2} & c-\frac{1}{2} & b+\frac{1}{2} \end{Bmatrix} = (-1)^s \left[\frac{(s-2b)(s-2c+1)}{(2b+1)(2b+2)2c(2c+1)}\right]^{1/2} \tag{A.17a}$$

$$\begin{Bmatrix} a & b & c \\ \frac{1}{2} & c-\frac{1}{2} & b-\frac{1}{2} \end{Bmatrix} = (-1)^s \left[\frac{(s+1)(s-2a)}{2b(2b+1)2c(2c+1)}\right]^{1/2} \tag{A.17b}$$

and the symmetries of $6j$ symbols: permutations of columns and simultaneous transpositions of upper and lower elements for any pair of columns do not change their values. In (A.17) $s = a+b+c$. Substituting the numerical values for $6j$ symbols in (A.16) we obtain (3.18–3.19).

APPENDIX B

Moments of the radial distribution of fermion number

For the states of $K = 0$ the p-th moment is defined by

$$M_n^{(p)} = R^{-p}\int_0^R dr r^p \varrho_n(r), \tag{B.1}$$

where ϱ_n is given by (3.32). Thus,

$$M_n^{(0)} = 1 \tag{B.2a}$$

$$M_n^{(1)} = \tfrac{1}{2} + \frac{\mathrm{Cin}\,(2x_n) - \sin^2 x_n}{2(x_n^2 - \sin^2 x_n)} \tag{B.2b}$$

$$M_n^{(p)} = \frac{1}{p+1} + \frac{1}{(x_n^2 - \sin^2 x_n)}\left[\frac{1}{2(p-1)} - \frac{p(p-2)!}{2^p x_n^{p-1}} \sin\frac{p\pi}{2}\right.$$

$$\left. + \frac{\sin^2 x_n}{p+1} + \frac{\cos 2x_n}{2} + \frac{p}{2}\sum_{k=1}^{p-1} \frac{(p-2)!}{(p-k-1)!}\,\frac{\cos(2x_n + k\pi/2)}{(2x_n)^k}\right]. \tag{B.2c}$$

In the case $K > 0$ we start from Eqs. (3.29) and express the spherical Bessel functions in terms of the ordinary Bessel functions:

$$j_K(z) = \sqrt{\frac{\pi}{2z}}\, J_\nu(z). \tag{B.3}$$

In this way we obtain (3.37), where

$$L_\nu^1(x) = \frac{1}{x^2}\int_0^x du\,u\left\{J_\nu^2(u) + \tfrac{1}{2}\,[J_{\nu+1}^2(u) + J_{\nu-1}^2(u)] + \frac{\alpha_\nu}{2}\,[J_{\nu+1}^2(u) - J_{\nu-1}^2(u)]\right\}. \tag{B.4}$$

Let us consider the following expressions, $\lambda \geqslant 1$:

$$L_\nu^\lambda(x) = I_\nu^\lambda(x) + \tfrac{1}{2}\,[I_{\nu+1}^\lambda(x) + I_{\nu-1}^\lambda(x)] + \frac{\alpha_\nu}{2}\,[I_{\nu+1}^\lambda(x) - I_{\nu-1}^\lambda(x)], \tag{B.5}$$

where

$$I_\nu^\lambda(x) = x^{-\lambda-1}\int_0^x du\,u^\lambda J_\nu^2(u). \tag{B.6}$$

Integrating by parts and using the Bessel equation we derive the formulae:

$$I_\nu^\lambda = \frac{\lambda-1}{\lambda}\left[\frac{\nu^2}{x^2} - \frac{(\lambda-1)^2}{4x^2}\right] I_\nu^{\lambda-2} + \frac{\mathscr{A}_\nu}{2\lambda} - \frac{\lambda-1}{2\lambda x}\, J_\nu J'_\nu + \frac{(\lambda-1)^2}{4\lambda x^2}\, J_\nu^2, \tag{B.7}$$

$$I_{\nu\pm1}^\lambda = U_\nu^\lambda + \frac{\nu^2}{x^2}\, I_\nu^{\lambda-2} \mp \frac{2\nu}{x^2}\, W_\nu^{\lambda-1}, \tag{B.8}$$

where $\mathscr{A}_\nu$ is defined by (3.39), and

$$U_\nu^\lambda = \frac{1}{x^{\lambda+1}}\int_0^x du\,u^\lambda J_\nu'^2(u)$$

$$= I_\nu^\lambda + \left[\frac{(\lambda-1)^2}{2x^2} - \frac{\nu^2}{x^2}\right] I_\nu^{\lambda-2} + \frac{J_\nu J'_\nu}{x} - \frac{\lambda-1}{2x^2}\, J_\nu^2 \tag{B.9}$$

$$W_\nu^\lambda = \frac{1}{x^\lambda}\int_0^x du\,u^\lambda J_\nu(u) J'_\nu(u) = \tfrac{1}{2}\, J_\nu^2 - \frac{\lambda}{2}\, I_\nu^{\lambda-1}. \tag{B.10}$$

Putting everything together we obtain:

$$L_\nu^\lambda = \frac{2(\lambda-1)\nu^2}{\lambda x^2} I_\nu^{\lambda-2} + \frac{\mathscr{A}_\nu}{\lambda} + \frac{J_\nu J_\nu'}{\lambda x} + \left(\frac{\nu\alpha_\nu}{x^2} + \frac{\lambda-1}{2\lambda x^2}\right)[(\lambda-1)I_\nu^{\lambda-2} - J_\nu^2]. \tag{B.11}$$

In particular, for $\lambda = 1$ we obtain (3.38), and, as it follows from (B.7),

$$I_\nu^1 = \frac{\mathscr{A}_\nu}{2}, \tag{B.12}$$

so, from (B.11)–(B.12) we derive (3.42). Obviously, the p-th moment defined by (3.36) can be written as

$$M_{n\nu c}^{(p)} = \frac{L_\nu^{p+1}}{L_\nu^1} \tag{B.13}$$

so, using (B.7), (B.11), we can calculate an arbitrary even moment. Thus,

$$M_{n\nu c}^{(2p)} = \frac{1}{2p+1} + \frac{1}{L_\nu^1(x)}\left\{-\frac{2p}{2p+1}\frac{\nu\alpha_\nu+\frac{1}{2}}{x^2}J_\nu^2\right.$$

$$\left. + I_\nu^{2p-1}\left[\frac{4p}{2p+1}\frac{\nu^2}{x^2} + \frac{2p}{x^2}\left(\nu\alpha_\nu + \frac{p}{2p+1}\right)\right]\right\}, \tag{B.14}$$

where

$$I_\nu^{2p+1} = w_p(\nu^2/x^2, x^{-2})\frac{\mathscr{A}_\nu}{2} - v_p(\nu^2/x^2, x^{-2})\frac{J_\nu J_\nu'}{2x} + u_p(\nu^2/x^2, x^{-2})\frac{J_\nu^2}{x^2} \tag{B.15}$$

and

$$w_p(y, z) = \frac{2p}{2p+1}(y - p^2 z)w_{p-1}(y, z) + \frac{1}{2p+1}, \tag{B.16a}$$

$$v_p(y, z) = \frac{2p}{2p+1}(y - p^2 z)v_{p-1}(y, z) + \frac{2p}{2p+1}, \tag{B.16b}$$

$$u_p(y, z) = \frac{2p}{2p+1}(y - p^2 z)u_{p-1}(y, z) + \frac{p^2}{2p+1}. \tag{B.16c}$$

APPENDIX C

Asymptotic expansions for energy levels

In this Appendix we give a derivation of Eqs. (3.63a–c). Then, we discuss relations between asymptotic expansions in various regions in the plane (ν, $x_{n\nu\varepsilon}$). From the Debye expansion of Bessel functions [35]

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x \sin\beta}}\left\{\cos\psi \sum_{k=0}^{\infty} \frac{u_{2k}(i\cot\beta)}{\nu^{2k}} - i\sin\psi \sum_{k=0}^{\infty} \frac{u_{2k+1}(i\cot\beta)}{\nu^{2k+1}}\right\}, \tag{C.1}$$

$$J'_\nu(x) \sim \sqrt{\frac{2\sin\beta}{\pi x}}\left\{-\sin\psi \sum_{k=0}^{\infty} \frac{v_{2k}(i\cot\beta)}{\nu^{2k}} - i\cos\psi \sum_{k=0}^{\infty} \frac{v_{2k+1}(i\cot\beta)}{\nu^{2k+1}}\right\}, \tag{C.2}$$

where

$$\cos\beta = \frac{\nu}{x} \tag{C.3}$$

$$\psi = \nu(\tan\beta - \beta) - \frac{\pi}{4}, \tag{C.4}$$

and the equation (3.24) for the energy levels we obtain:

$$\begin{aligned}\sin 2\psi = {} & \cos\Theta \sin\beta + \frac{1}{\nu}\left\{\frac{\sin\Theta}{2}(1+\cos 2\psi)\cot\beta\right. \\ & \left. -(1-\cos\Theta\sin 2\psi\sin\beta)\frac{\cos\beta}{2\sin^3\beta} - \cos 2\psi\frac{\cos\beta(1+2\sin^2\beta)}{12\sin^3\beta}\right\} \\ & + \frac{1}{\nu^2}\left\{\sin\Theta\sin 2\psi\,\frac{3\cot^2\beta+5\cos^4\beta}{24} + D_1(\beta)\cos\Theta\right. \\ & \left. + D_2(\beta)\cos\Theta\cos 2\psi + D_3(\beta)\sin 2\psi\right\} + O(\nu^{-3}).\end{aligned} \tag{C.5}$$

The functions D_i in the above equation do not contribute to the 'anomalous parts' of the quantities discussed in the present paper, so we do not give explicit expressions for them. Writing

$$\sin 2\psi = \sin\beta_0\cos\Theta + \frac{f_1(\beta_0, \beta_1)}{\nu} + \frac{f_2(\beta_0, \beta_1, \beta_2)}{\nu^2} + \dots \tag{C.6a}$$

$$\cos 2\psi = \varepsilon\left\{C - \frac{\sin\beta_0\cos\Theta}{\nu C} f_1(\beta_0, \beta_1) + \dots\right\}, \tag{C.6b}$$

where C is defined by (3.64), and putting (3.62), (C.6) into the r.h.s. of (C.5) we derive an asymptotic expansion for $\sin 2\psi$. Then, we calculate $\sin 2\psi$ from the definition (C.4) replacing β by the asymptotic expansion (3.62) and using (3.59):

$$\sin 2\psi = \sin(2\beta_1 \tan^2\beta_0 - \pi/2) + \frac{2\varepsilon C}{\nu}\left\{\beta_2 \tan^2\beta_0 + \beta_1^2 \frac{\sin\beta_0}{\cos^3\beta_0}\right\}$$

$$+\frac{1}{\nu^2}\left\{2\varepsilon C\left[\beta_3 \tan^2\beta_0 + 2\beta_1\beta_2 \frac{\sin\beta_0}{\cos^2\beta_0} + \beta_1^3 \frac{1+2\sin^2\beta_0}{3\cos^4\beta_0}\right]\right.$$

$$\left. -2\cos\Theta \sin\beta_0\left[\beta_2 \tan^2\beta_0 + \beta_1^2 \frac{\sin\beta_0}{\cos^3\beta_0}\right]^2\right\} + O(\nu^{-3}). \tag{C.7}$$

Comparing coefficients in these two expansions we derive (3.63a–c).

In the transition region

$$x_{n\nu\varepsilon} = \nu + z\nu^{1/3} \tag{C.8}$$

the expansions (C.1)–(C.2) break down. We can use another Debye's expansion in this region [35] and express the Bessel functions in terms of the Airy function and its derivative. Let

$$z \sim \sum_{k=0}^{\infty} \frac{z_k}{\nu^{k/3}}. \tag{C.9}$$

From this expansion and (3.28) we obtain

$$\mathrm{Ai}(-2^{1/3} z_0) = 0 \quad \text{for} \quad \varepsilon = 1, \tag{C.10a}$$

$$\mathrm{Ai}'(-2^{1/3} z_0) = 0 \quad \text{for} \quad \varepsilon = -1. \tag{C.10b}$$

Thus, for the solutions of the equation for the energy levels

$$z_0 > 0. \tag{C.11}$$

Using the asymptotic formula for the n-th zero of the Airy function (or the zero of its derivative), cf. [35], we obtain

$$z_0 \approx 2^{-1/3}\left[\frac{3\pi(4n-1)}{8}\right]^{2/3}, \tag{C.12a}$$

to be compared with

$$x_{n\nu\varepsilon} = \nu + \frac{(3n\pi)^{2/3}}{2}\nu^{1/3} + \dots, \tag{C.12b}$$

the result obtained from (3.59) for n fixed and $\nu \to \infty$. We see that the ν-dependence of $|x_{n\nu\varepsilon} - \nu|$ obtained from the expansions (C.1–C.2) is the same as those derived from the expansion valid in the transition region (C.8). It is noteworthy that this observation seems to be more general. For example, from (4.5) we obtain for fixed n and $\nu \gg 1$:

$$\langle x_{n\nu\varepsilon} \rangle = \frac{\langle a_2 \rangle}{\nu^2} + \dots = O(\nu^{-2} \sin^{-2} \beta_0) = O(\nu^{-4/3}), \tag{C.13}$$

whereas the first non-vanishing average of the coefficients z_k is $\langle z_5 \rangle$.

Let us consider the region $x_{n\nu\varepsilon} < \nu$, $\nu \gg 1$. In this region

$$J_\nu(\nu/\cosh\alpha) \sim \frac{\exp\{\nu(\tanh\alpha - \alpha)\}}{\sqrt{2\pi\nu\tanh\alpha}} + \dots \tag{C.14a}$$

$$J'_\nu(\nu/\cosh\alpha) \sim \sqrt{\frac{\sinh 2\alpha}{4\pi\nu}} \exp\{\nu(\tanh\alpha - \alpha)\} + \dots . \tag{C.14b}$$

Substituting these expansions for J_ν and J'_ν into (3.24) we find that there is no energy level for large $\nu > x_{n\nu\varepsilon}$ outside the transition region (C.8). Thus, taking into account (C.11) we conclude that there is no solution of Eq. (3.24) for $x_{n\nu\varepsilon} \leqslant \nu$ and $\nu \gg 1$. Solving (3.24) numerically we check that there is no solution in the region $x_{n\nu\varepsilon} \leqslant \nu$ for $\nu \geqslant 3/2$.

APPENDIX D

In this Appendix we give a derivation of Eqs. (4.19). Let us introduce the asymptotic expansion

$$y \sim \frac{\nu}{\cos\beta_0} + a_0 + \frac{b_1}{\nu} + \frac{b_2}{\nu^2} + \dots \tag{D.1}$$

$$\beta = \arccos(\nu/y) \sim \sum_{k=0}^{\infty} \frac{\beta_k}{\nu^k}, \tag{D.2}$$

where β_0 is defined by (3.59). One can easily check that

$$a_0 = \frac{\sin\beta_0}{\cos^2\beta_0} \beta_1 \tag{D.3a}$$

$$a_1 = b_1 + \frac{\mu^2 \cos^2\beta_0}{2} \tag{D.3b}$$

$$a_2 = b_2 - \frac{\mu^2 \cos^2\beta_0}{2} a_0 \tag{D.3c}$$

$$b_1 = \frac{\sin\beta_0}{\cos^2\beta_0}\beta_2 + \frac{1+\sin^2\beta_0}{2\cos^3\beta_0}\beta_1^2 \tag{D.3d}$$

$$b_2 = \frac{\sin\beta_0}{\cos^2\beta_0}\beta_3 + \frac{1+\sin^2\beta_0}{\cos^3\beta_0}\beta_1\beta_2 + \frac{\sin\beta_0(5+\sin^2\beta_0)}{6\cos^4\beta_0}\beta_1^3 \tag{D.3e}$$

and, cf. (4.16a–b),

$$\bar{\beta}_0 = \beta_0, \tag{D.4a}$$

$$\bar{\beta}_1 = \beta_1, \tag{D.4b}$$

$$\bar{a}_0 = a_0. \tag{D.4c}$$

It follows from (D.3b–c) that

$$\langle a_2 \rangle = \langle b_2 \rangle, \tag{D.5a}$$

$$\langle a_0 a_1 \rangle = \langle a_0 b_1 \rangle. \tag{D.5b}$$

Let us rewrite (4.11) as follows:

$$\sin 2\psi = F_1(\beta, \Theta) + F_2(\beta, \Theta, \mu), \tag{D.6}$$

where

$$F_1(\beta, \Theta) = \frac{\cos^2\psi}{\sin\beta}\left\{(\mathscr{D}_\nu^2 + \sin^2\beta)\cos\Theta + 2(\mathscr{D}_\nu + \sin\beta\tan\psi) + \frac{\sin\Theta\cos\beta}{\nu}\right\} \tag{D.7}$$

$$F_2(\beta, \Theta, \mu) = \frac{\cos^2\psi}{\sin\beta}\left\{\frac{\mu}{x}\cos\Theta(1+\cos^2\beta - \mathscr{D}_\nu^2) + \left(\sqrt{1-\frac{\mu^2}{x^2}} - 1\right)\left(2\mathscr{D}_\nu + \frac{\sin\Theta\cos\beta}{\nu}\right)\right\} \tag{D.8}$$

and $\mathscr{D}_\nu$ is defined by (3.26). We introduce asymptotic expansions

$$F_1(\beta, \Theta) \sim \tilde{F}_1(\{\beta_k\}, \Theta) = \tilde{F}_1(\{\bar{\beta}_k\}, \Theta) + \sum_{k=2}^{\infty} \frac{g_k(\beta_0, \varepsilon; \Theta, \mu)}{\nu^k} \tag{D.9}$$

and

$$F_2(\beta, \Theta, \mu) \sim \sum_{k=1}^{\infty} \frac{h_k(\beta_0, \varepsilon; \Theta, \mu)}{\nu^k}. \tag{D.10}$$

Comparing the coefficients in (C.7) and (D.9, D.10) we see that

$$\delta\beta_2 = \frac{\varepsilon h_1}{2C}\cot^2\beta_0 \tag{D.11}$$

is an even function of Θ. Thus,

$$\langle\delta(a_0 b_1)\rangle = \frac{\sin\beta_0}{\cos^2\beta_0}\langle a_0\delta\beta_2\rangle = 0. \tag{D.12}$$

As for β_3 we have:

$$\langle\delta\beta_3\rangle = \frac{\cot^2\beta_0}{2C}\langle\varepsilon(g_2+h_2)\rangle + \frac{\sin\beta_0}{C^2}\langle\cos\Theta\bar{\beta}_2 h_1\rangle. \tag{D.13}$$

From (D.8), (3.63b) we easily obtain:

$$\langle\cos\Theta\bar{\beta}_2 h_1\rangle = \frac{\mu}{4}\sin\Theta\cos^2\Theta\cot^4\beta_0(1+\cos^2\beta_0) \tag{D.14a}$$

$$\langle\varepsilon h_2\rangle = -\frac{\mu}{2C}\sin\Theta\cos^2\Theta\frac{\cos^2\beta_0}{\sin\beta_0}. \tag{D14.b}$$

When calculating $\langle\varepsilon g_2\rangle$ we notice that the coefficient g_2 can be obtained from the r.h.s. of (C.5). Moreover, only the term

$$\frac{1}{2v}\sin\Theta\cos 2\psi\cot\beta$$

contributes to the average of interest, and

$$\langle\varepsilon g_2\rangle = -\frac{\mu}{2C}\sin\Theta\cos^2\Theta\frac{\cos^4\beta_0}{\sin\beta_0}. \tag{D.14c}$$

From (D.11), (D.14) and (D.13) we obtain

$$\langle\delta\beta_3\rangle = 0 \tag{D.15}$$

and, consequently,

$$\langle\delta b_2\rangle = 0. \tag{D.16}$$

Equations (D.12), (D.16) and (D.5) imply (4.19).

APPENDIX E

In this Appendix we give a derivation of Eqs. (4.36). In order to calculate the averages which appear in these equations we have to calculate the first three coefficients in the asymptotic expansion (4.35). Our task is even simpler because we know that the averages in (4.36) must be odd and periodic functions of Θ with period π. Moreover, the results must be proportional to $\sin\Theta$. These observations reduce seriously the number of those terms in the expressions for the coefficients $m_k^{(2p)}$ which can contribute to the averages in (4.36)[9]. For example, when calculating $\langle m_2^{(2p)} a_0\rangle$ we have to know only the part of $m_2^{(2p)}$ antisymmetric in Θ because a_0 is an even function of Θ, cf. (D.3a). Then, for the terms which depend only on β the leading order in $\frac{1}{\nu}$ does not depend on Θ because β_0 does not depend on Θ. The next-to-leading order is proportional to β_1 and, consequently, it is an even function of Θ. A non-zero antisymmetric part appears in the second-to-leading order. However, it is periodic with period 2π. Therefore, when calculating $m_0^{(2p)}$, $m_1^{(2p)}$ and $m_2^{(2p)}$ from (B.14) we can replace I_ν^{2p-1} by

$$\tilde{I}_\nu^{2p-1} = \bar{w}_{p-1}(\cos^2\beta)\frac{\mathscr{A}_\nu}{2} - \bar{v}_{p-1}(\cos^2\beta)\frac{J_\nu J'_\nu \cos\beta}{2\nu}, \tag{E.1}$$

where

$$\bar{w}_p(y) = w_p(y, 0) \tag{E.2}$$

etc., cf. (B.15), (B.16). Then, we use Eq. (3.24) for the energy levels in order to eliminate $J_\nu J'_\nu$ in (3.38) and (B.14), cf. (3.39),

$$J_\nu J'_\nu = -\frac{\mathscr{A}_\nu}{2}\left(\cos\Theta + \frac{\sin\Theta\cos\beta}{2\nu\sin^2\beta}\mathscr{B}_\nu\right), \tag{E.3}$$

where

$$\mathscr{B}_\nu = \frac{\sin^2\beta J_\nu^2}{\mathscr{A}_\nu}. \tag{E.4}$$

In this way we obtain:

$$L_\nu^1 = \mathscr{A}_\nu\left\{1 - \frac{1}{\nu}\left(\frac{\cos\Theta\cos\beta}{2} + \cot^2\beta\,\alpha_\nu\mathscr{B}_\nu\right) - \frac{1}{2\nu^2}\sin\Theta\cot^2\beta\,\mathscr{B}_\nu\right\} \tag{E.5}$$

and

$$M_{n\nu\varepsilon}^{(2p)} = \frac{1}{2p+1} + \frac{2p}{2p+1}W_{2p}(\cos\beta)$$
$$+\frac{1}{\nu}\left\{\frac{2p}{2p+1}\cot^2\beta(W_{2p}-1)\alpha_\nu\mathscr{B}_\nu + pW_{2p}\alpha_\nu + \frac{p}{2p+1}\cos\beta V_{2p}\cos\Theta\right\}$$
$$+\frac{1}{\nu^2}\left\{\frac{p}{2p+1}\cot^2\beta V_{2p}\mathscr{B}_\nu\sin\Theta + \text{even}(\Theta)\right\} + \dots, \tag{E.6}$$

[9] This reduction is even more important when one calculates $\langle m_2^{(2p)}\rangle$.

where

$$W_{2p}(y) = y^2\overline{w}_{p-1}(y^2) \tag{E.7a}$$

$$V_{2p}(y) = y^2[\overline{w}_{p-1}(y^2)+\overline{v}_{p-1}(y^2)]. \tag{E.7b}$$

The asymptotic expansions for $\mathscr{B}_v$ and α_v read[10]

$$\mathscr{B}_v = \frac{1+\varepsilon C}{2} - \frac{\sin\beta_0\cos\Theta\tilde{g}}{v}$$

$$+\frac{1}{v^2}\{-h\sin\beta_0\cos\Theta-\varepsilon C\tilde{g}^2+\varphi_+\tilde{g}\sin^2\beta_0\cos^2\Theta+\text{even}\,(\Theta)\}+\dots \tag{E.8}$$

$$\alpha_v = \frac{\varepsilon}{C}\cos\beta_0\cos\Theta - \frac{\varepsilon\sin^2\Theta}{2vC^3}(\sin\Theta+2\sin\beta_0\cos\Theta\beta_1)$$

$$-\frac{\varepsilon\sin^2\Theta}{v^2C^3}\left\{\beta_2\sin\beta_0\cos\Theta+\frac{3\sin 2\beta_0}{4C^2}\beta_1\sin\Theta\cos^2\Theta+\delta\right\}+\dots, \tag{E.9}$$

where $\langle\delta\rangle = \langle\varepsilon\delta\rangle = 0$, and

$$\tilde{g} = \frac{\varepsilon+C}{4C}\cot\beta_0\sin\Theta+\frac{\varepsilon\beta_1}{2C}\cos\beta_0\cos\Theta \tag{E.10}$$

$$h = \beta_3\tan^2\beta_0+\frac{2\sin\beta_0}{\cos^3\beta_0}\beta_1\beta_2+\frac{1+2\sin^2\beta_0}{3\cos^4\beta_0}\beta_1^3. \tag{E.11}$$

As we see, in the leading order $\mathscr{B}_v$ is an even function of Θ. This is why in (E.1) we can neglect the term proportional to u_{p-1}. From (E.6)–(E.10) and (3.63) we can easily derive (4.36).

REFERENCES

[1] A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, V. F. Weisskopf, *Phys. Rev.* **D9**, 3471 (1974); A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, *Phys. Rev.* **D10**, 2599 (1974); T. De Grand, R. L. Jaffe, K. Johnson, J. Kiskis, *Phys. Rev.* **D12**, 2060 (1975).

[2] T. H. R. Skyrme, *Proc. Roy. Soc.* (London) **A260**, 127 (1961); *Nucl. Phys.* **31**, 556 (1962).

[3] G. S. Adkins, C. Nappi, E. Witten, *Nucl. Phys.* **B228**, 552(1983); G. S. Adkins, C. Nappi, *Nucl. Phys.* **B233**, 109 (1984).

[4] M. Rho, A. S. Goldhaber, G. E. Brown, *Phys. Rev. Lett.* **51**, 747 (1983).

[5] M. Jeżabek, K. Zalewski, *Z. Phys. C — Particles and Fields* **26**, 385 (1984).

[6] M. Jeżabek, K. Zalewski, *Phys. Lett.* **148B**, 334 (1984).

[7] K. J. Heller, M. Jeżabek, preprint MPI-PAE/PTh 89/87, submitted to *Nucl. Phys.* A.

[8] M. Jeżabek, K. J. Heller, M. A. Nowak, *Phys. Lett.* **166B**, 101 (1986).

[9] K. J. Heller, M. Jeżabek, M. A. Nowak, *Z. Phys. C — Particles and Fields* **30**, 483 (1986).

[10] These expansions can be used in calculation of $\langle m_3^{(2p)}\rangle$. The terms $O(v^{-2})$ in Eqs. (E8),(E9) do not contribute to the averages in (4.36).

[10] M. Ježabek, *Phys. Lett.* **174B**, 429 (1986).
[11] K. Zalewski, *Acta Phys. Pol.* **B17**, 83 (1986).
[12] K. Zalewski, *Acta Phys. Pol.* **B17**, 305 (1986).
[13] P. Hasenfratz, J. Kuti, *Phys. Rep.* **40**, 75 (1978).
[14] I. Zahed, G. E. Brown, *Phys. Rep.* **142**, 1 (1986).
[15] A. P. Balachandran, *Skyrmions*, Lectures at Theor. Adv. Study Inst. in Elem. Part. Phys., Yale 1985.
[16] G. Holzwarth, B. Schwesinger, *Rep. Prog. Phys.* **49**, 825 (1986).
[17] A. W. Thomas, *Adv. Nucl. Phys.* **13**, 1 (1984).
[18] M. Rho, *Chiral Bags, Skyrmions and Quarks in Nuclei*, Lecture at Int. School of Physics, Varenna 1984; M. Rho, *Cheshire Cat Phenomena and Quarks in Nuclei*, Saclay preprint SPhT/86-159.
[19] G. E. Brown, M. Rho, *The Chiral Bag*, Comments on Nucl. and Part. Phys., to appear.
[20] R. Jaffe, in *Pointlike Structures Inside and Outside Hadrons*, A. Zichichi, ed., Plenum, New York 1982.
[21] *Solitons in Nuclear and Elementary Particle Physics*, A. Chodos, E. Hadjimichael, C. Tze, eds., World Scientific, Singapore 1984.
[22] R. F. Alvarez-Estrada, F. Fernández, J. L. Sánchez-Gómez, V. Vento, *Models of Hadron Structure Based on Quantum Chromodynamics*, Lecture Notes in Physics 259, Springer-Verlag, Berlin--Heidelberg 1986.
[23] D. Finkelstein, J. Rubinstein, *J. Math. Phys.* **9**, 1762 (1968).
[24] L. D. Faddeev, *Lett. Math. Phys.* **1**, 289 (1976).
[25] N. K. Pak, H. Ch. Tze, *Ann. Phys.* **117**, 164 (1979).
[26] A. P. Balachandran, V. P. Nair, S. G. Rajeev, A. Stern, *Phys. Rev. Lett.* **49**, 1124 (1982); *Phys. Rev. Lett.* **50**, 1630 (E) (1983); *Phys. Rev.* **D27**, 1153 (1983); *Phys. Rev.* **D27**, 2722(E) (1983).
[27] E. Witten, *Nucl. Phys.* **B223**, 422 and 433 (1983).
[28] A. Chodos, C. B. Thorn, *Phys. Rev.* **D12**, 2733 (1975).
[29] J. Goldstone, F. Wilczek, *Phys. Rev. Lett.* **47**, 986 (1981).
[30] V. Vento, M. Rho, *Nucl. Phys.* **A412**, 413 (1984).
[31] P. J. Mulders, *Phys. Rev.* **D30**, 1073 (1984).
[32] L. D. Landau, E. M. Lifshitz, *Quantum Mechanics, Nonrelativistic Theory*, Addison & Wesley, London-Amsterdam-Paris 1965.
[33] J. D. Bjorken, S. D. Drell, *Relativistic Quantum Mechanics*, Mc Graw — Hill, New York 1964.
[34] V. B. Berestetskiĭ, E. M. Lifshitz, L. P. Pitaevskiĭ, *Relativistic Quantum Theory*, Pergamon Press, Oxford 1971.
[35] *Handbook of Mathematical Functions*, M. Abramovitz, I. A. Stegun, eds., Dover Publications, New York 1972.
[36] J. Goldstone, R. L. Jaffe, *Phys. Rev. Lett.* **51**, 1518 (1983).
[37] H. Yamagishi, *Phys. Rev.* **D27**, 2383 (1983).
[38] B. Grossman, *Phys. Rev. Lett.* **50**, 464 (1983).
[39] K. J. Heller, K. Zalewski, *Acta Phys. Pol.* **B17**, 839 (1986).
[40] I. Zahed, A. Wirzba, U.-G. Meissner, *Ann. Phys.* **165**, 406 (1985).
[41] L. Vepstas, A. D. Jackson, A. S. Goldhaber, *Phys. Lett.* **140B**, 280 (1984).
[42] L. Vepstas, thesis (unpublished).
[43] T. H. Hansson, R. L. Jaffe, *Phys. Rev.* **D28**, 882 (1983).
[44] K. A. Milton, *Phys. Rev.* **D22**, 1441 and 1444 (1980); *Phys. Rev.* **D25**, 3441(E) (1982); *Phys. Rev.* **D27**, 439 (1983).
[45] E. Wüst, L. Vepstas, A. D. Jackson, *Phys. Lett.* **173B**, 217 (1986).
[46] A. J. Niemi, G. W. Semenoff, *Phys. Rep.* **135**, 99 (1986).
[47] T. Jaroszewicz, *Phys. Lett.* **143B**, 217 (1984).
[48] S. Nadkarni, H. B. Nielsen, I. Zahed, *Nucl. Phys.* **B253**, 308 (1985).
[49] Y. K. Ha, *Phys. Rev.* **D29**, 1744 (1984).

[50] M. B. Paranjape, *Phys. Rev. Lett.* **55**, 2390 (1985).
[51] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University, New York 1964.
[52] K. Sundermeyer, *Constrained Dynamics*, Lecture Notes in Physics 169, Springer-Verlag 1985.
[53] G. E. Brown, M. Rho, W. Weise, *Phenomenological Sizes of Confinement Regions in Baryons*, Stony Brook preprint 1986.
[54] P. Mazur, M. A. Nowak, M. Praszałowicz, *Phys. Lett.* **147B**, 137 (1984); E. Guadagnini, *Nucl. Phys.* **B236**, 35 (1984); S. Jain, S. Wadia, *Nucl. Phys.* **B258**, 713 (1985).
[55] M. Praszałowicz, *Phys. Lett.* **158B**, 264 (1985); M. Chemtob, *Nucl. Phys.* **B256**, 600 (1985).
[56] H. Yabu, K. Ando, *A New Approach to the SU_3 Skyrme Model*, Kyoto University preprint KUNS 851.
[57] C. G. Callan, I. Klebanov, *Nucl. Phys.* **B262**, 365 (1985).
[58] M. Jeżabek, talk at XXVI Cracow School of Theoretical Physics, Zakopane 1986.
[59] M. Jeżabek, M. A. Nowak, M. Rho, *Phys. Lett.* **182B**, 139 (1986).
[60] H. Nadeau, M. A. Nowak, M. Rho, V. Vento, *Phys. Rev. Lett.* **57**, 2127 (1986).
[61] P. Breitenlohner, A. Kabelschacht, R. J. Drofnats, *The PHYSTEX Manual*, preprint MPI/PAE/PTh 1986.