ON THE THEORY OF FIELDS IN FINSLER SPACE — IV

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Some physical aspects of the previously introduced Finslerian structure based on the (generalized) Finsler metric $g_{\lambda\kappa}(x, y) = \gamma_{\lambda\kappa}(x)+h_{\lambda\kappa}(x, y)$ are considered.

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1. Introduction

In the previous paper [1], we have introduced the following new (generalized) Finsler metric:

$$g_{\lambda\kappa}(x, y) = \gamma_{\lambda\kappa}(x)+h_{\lambda\kappa}(x, y),$$

where $\gamma_{\lambda\kappa}(x)$ ($\kappa, \lambda = 1, 2, 3, 4$) denotes the Riemann metric of the external (x)-field, while $h_{\lambda\kappa}(x, y)$ is the (generalized) Finsler metric induced from the Riemann metric $h_{ij}(y)$ ($i, j = 1, 2, 3, 4$) of the internal (y)-field by means of the mapping process of the (y)-field on the (x)L-field (see Section 1 of [1]). This mapping process has been called the N-mapping. The word "generalized" means that the Finsler metrics $h_{\lambda\kappa}$ and $g_{\lambda\kappa}$ do not satisfy the ordinary homogeneity conditions with respect to y (cf. [2, 3].

Hitherto, we have completely determined the metrical Finsler connection $Dg_{\lambda\kappa} = 0$ by taking account of the intrinsic behaviour of y (i.e., δy), where Kawaguchi's theorem [4] plays the most important role in regard to the relation $Dg_{\lambda\kappa} = 0$ and $\delta g_{\lambda\kappa} \neq 0$ (see Section 4 of [1]).

In this paper, which is a continuation of [1], we shall consider some physical aspects of this Finslerian structure.

2. On the Finslerian structure

In this Section, in order to recall our situations and to make this paper consistent, we shall summarize the essential points of our theory developed in the previous paper [1].

In our case, the vector $y(= y^i;\ i = 1, 2, 3, 4)$ is attached, as the internal variable, to each point $x(= x^\kappa;\ \kappa = 1, 2, 3, 4)$, so that there appear two fields: One is the external (x)-field spanned by points $\{x\}$, which is nothing but the gravitational field in Einstein's sense, and the other is the internal (y)-field spanned by vectors $\{y\}$, which is likened to the so-called internal space. The former is governed by the Riemann metric $\gamma_{\lambda\kappa}(x)$, while the latter is assumed to be governed by the Riemann metric $h_{ij}(y)$, in general.

From a vector bundle-like viewpoint [2, 5], the (y)-field may be regarded as a fibre at the point x of the base (x)-field and the total space with 8-dimensional Riemannian structure may be considered a unified field between the (x)- and (y)-fields. This unified field is governed by the unified Riemann metric $G_{AB}(X)$, where $X(= X^A = (x^\kappa, y^i); A, B = 1, 2, \ldots, 8)$ is the unified coordinate (see Section 1 of [1]).

At this stage, in order to set the base and dual base in the unified field, we must, first of all, geometrize the intrinsic behaviour of y such as the nonlinear gauge transformation [6] $\bar{y}^i = K^i_j(x, y)y^j$. The result is represented in the form of intrinsic connection (or parallelism) of y as follows (cf. [7]):

$$\begin{aligned} \delta y^i &= dy^i + K_j{}^i{}_\mu y^j dx^\mu + L_j{}^i{}_k y^j dy^k \\ &\equiv P^i_k dy^k + Q^i_\mu dx^\mu, \end{aligned} \tag{2.1}$$

where $K_j{}^i{}_\mu = -\dfrac{\partial K^i_j}{\partial x^\mu}$, $L_j{}^i{}_k = -\dfrac{\partial K^i_j}{\partial y^k}$, $P^i_k = \delta^i_k + L_j{}^i{}_k y^j$ and $Q^i_\mu = K_j{}^i{}_\mu y^j$.
Next, by use of (2.1), the connection relation in the unified field is given by, e.g.,

$$\begin{aligned} DV^\kappa &= dV^\kappa + \Gamma_\lambda{}^\kappa{}_\mu V^\lambda dx^\mu + \Gamma_\lambda{}^\kappa{}_i V^\lambda dy^i \\ &= dV^\kappa + F_\lambda{}^\kappa{}_\mu V^\lambda dx^\mu + \Theta_\lambda{}^\kappa{}_i V^\lambda dy^i, \end{aligned} \tag{2.2}$$

where $F_\lambda{}^\kappa{}_\mu = \Gamma_\lambda{}^\kappa{}_\mu - N^i_\mu \Gamma_\lambda{}^\kappa{}_i$ and $\Theta^\kappa_{\lambda i} = (P^{-1})^j_i \Gamma_\lambda{}^\kappa{}_j (N^i_\mu \equiv (P^{-1})^i_j Q^j_\mu)$. (The quantity N^i_μ plays the role of nonlinear connection (cf. [2, 3, 5]). P^i_k is assumed to be non-singular.) From (2.2), the covariant derivatives are defined by, e.g.,

$$\begin{aligned} V^\kappa{}_{/\mu} &= \frac{\delta V^\kappa}{\delta x^\mu} + F_\lambda{}^\kappa{}_\mu V^\lambda, \\ V^\kappa{}_{/i} &= \frac{\delta V^\kappa}{\delta y^i} + \Theta_\lambda{}^\kappa{}_i V^\lambda, \end{aligned} \tag{2.3}$$

where $\dfrac{\delta}{\delta x^\mu} = \dfrac{\partial}{\partial x^\mu} - N^i_\mu \dfrac{\partial}{\partial y^i}$ and $\dfrac{\delta}{\delta y^i} = (P^{-1})^j_i \dfrac{\partial}{\partial y^j}$. Therefore, from (2.2) and (2.3), the base $\left(\dfrac{\delta}{\delta x^\mu}, \dfrac{\delta}{\delta y^i}\right)$ and the dual base $(dx^\kappa, \delta y)$ can be set (see Section 2 of [1]).

The above-introduced base and dual base are prescribed by such decomposition processes as $\dfrac{\delta}{\delta x^\mu} = A^B_\mu \dfrac{\partial}{\partial X^B}$, $\delta y^i = B^i_A dX^A$, etc., so that the following decomposition factors $\boldsymbol{A}$, $\boldsymbol{B}$ can be determined:

$$\begin{aligned} A^B_\lambda &= (\delta^\kappa_\lambda, \quad -N^i_\lambda), & A^\kappa_B &= (\delta^\kappa_\lambda, \quad 0), \\ B^A_i &= (0, \quad (P^{-1})^j_i), & B^i_A &= (Q^i_\lambda, \quad P^i_j). \end{aligned} \tag{2.4}$$

By use of (2.4), G_{AB} is decomposed as, e.g.,

$$g_{\lambda\kappa}(x, y) = A^A_\lambda A^B_\kappa G_{AB}$$
$$= G_{\lambda\kappa} - N^i_\kappa G_{\lambda i} - N^i_\lambda G_{i\kappa} + N^i_\lambda N^j_\kappa G_{ij}. \tag{2.5}$$

In this case, if it is assumed that $G_{\lambda\kappa} = \gamma_{\lambda\kappa}(x)$, $G_{\lambda i} = G_{i\lambda} = 0$ and $G_{ij} = h_{ij}(y)$, as in the vacuum state without fluctuations in the sense of generalized Kaluza-Klein theory of gravity (cf. [8], see Section 3 of [1]), then (2.5) is reduced to $g_{\lambda\kappa}(x, y) = \gamma_{\lambda\kappa}(x) + N^i_\lambda N^j_\kappa h_{ij}(y)$.

Under these conditions, if we want to return to the 4-dimensional Finslerian structure (F_4), we must reduce the dimension number from 8 to 4 by taking account of some mapping process of the (y)-field on the (x)-field. So, we shall focus again our attention on the metric obtained above, i.e.,

$$g_{\lambda\kappa}(x, y) = \gamma_{\lambda\kappa}(x) + h_{\lambda\kappa}(x, y); \qquad h_{\lambda\kappa}(x, y) \equiv N^i_\lambda N^j_\kappa h_{ij}(y) \tag{2.6}$$

and adopt this as our F_4-metric. In this case, the nonlinear connection $N(x, y)$ is found to play the role of mapping operator. Then, by this N-mapping, (2.1) and (2.2) are formally brought to their corresponding F_4-formulas as follows:

$$\delta y^\kappa = N^\kappa_i \delta y^i = dy^\kappa + K_\lambda{}^\kappa{}_\mu y^\lambda dx^\mu + L_\lambda{}^\kappa{}_\mu y^\lambda dy^\mu$$
$$\equiv P^\kappa_\mu dy^\mu + Q^\kappa_\mu dx^\mu, \tag{2.7}$$

$$DV^\kappa = dV^\kappa + \Gamma_\lambda{}^\kappa{}_\mu V^\lambda dx^\mu + C_\lambda{}^\kappa{}_\mu V^\lambda dy^\mu$$
$$= dV^\kappa + F_\lambda{}^\kappa{}_\mu V^\lambda dx^\mu + \Theta_\lambda{}^\kappa{}_\mu V^\lambda \delta y^\mu \tag{2.8}$$

(see Section 4 of [1]).

Finally, it is necessary to obtain the relation of two connections δ and D. In our case, the intrinsic connection δ is treated as known from physical conditions such as (2.1), so that the relation of D with δ must be obtained. δ is assumed, from the beginning, to be metrical for $h_{\lambda\kappa}$ (i.e., $\delta h_{\lambda\kappa} = 0$) under $\delta N = 0$, but not metrical for $g_{\lambda\kappa}$ (i.e., $\delta g_{\lambda\kappa} \neq 0$). On the other hand, D is metrical for $g_{\lambda\kappa}$ (i.e., $Dg_{\lambda\kappa} = 0$), but not metrical for $h_{\lambda\kappa}$ (i.e., $Dh_{\lambda\kappa} \neq 0$). Therefore, for our purpose, the relations ($Dg_{\lambda\kappa} = 0$ & $\delta g_{\lambda\kappa} \neq 0$) must be reconsidered as follows: D is a metrical connection for $g_{\lambda\kappa}$ derived from the non-metrical δ. Then, by virtue of Kawaguchi's theorem [4], the desired relation can be obtained:

$$Dy^\kappa = \delta y^\kappa + \tfrac{1}{2} g^{\kappa\nu}(\delta g_{\nu\lambda}) y^\lambda. \tag{2.9}$$

With the aid of (2.9), our Finslerian structure based on (2.6) can be completely clarified (see Section 4 of [1]).

Thus, it turns out that our F_4-structure has the metric (2.6) and the connection (2.8). In the following Sections, we shall extract some physical aspects underlying this F_4-structure.

3. *Physical aspects — I*

First, it should be remarked that this kind of metric (2.6) is now called the generalized Lagrange metric (cf. [2]), which does not admit any fundamental function $\mathscr{L}(x, y)$ from which (2.6) is derived in the form $g_{\lambda\kappa} = \frac{1}{2}\frac{\partial^2 \mathscr{L}}{\partial y^\lambda \partial y^\kappa}$, so that (2.6) does not satisfy any homogeneity condition with respect to y. Therefore, (2.6) is quite different from the original Finsler metric (cf. [3]), which is defined by the homogeneous fundamental function $L(x, y)$ in the form $g_{\lambda\kappa} = \frac{1}{2}\frac{\partial^2 L}{\partial y^\lambda \partial y^\kappa}$ and becomes positively homogeneous of degree 0 in y.

Now, it is seen from (2.6) that the metric $g_{\lambda\kappa}(x, y)$ deviates from the Riemann metric $\gamma_{\lambda\kappa}(x)$ by the amount of $h_{\lambda\kappa}(x, y)$. Therefore, the Finslerian light-cone (i.e., $ds_F^2 = g_{\lambda\kappa}(x, y)dx^\kappa dx^\lambda = (\gamma_{\lambda\kappa}(x)+h_{\lambda\kappa}(x, y))dx^\kappa dx^\lambda = 0$) does not coincide with the Riemannian light-cone (i.e., $ds_R^2 = \gamma_{\lambda\kappa}(x)dx^\kappa dx^\lambda = 0$). That is to say, $ds_F^2 = 0$ is not compatible with $ds_R^2 = 0$, because $h_{\lambda\kappa}dx^\kappa dx^\lambda \neq 0$, in general. In a word, $g_{\lambda\kappa}(x, y)$ breaks (or does not preserve) the conformal structure underlying the Riemannian or Einsteinian general relativity (cf. [9]). This is one Finslerian result (or effect) caused by $h_{\lambda\kappa}(x, y)$.

Next, as mentioned in Section 2, the metrical conditions $Dg_{\lambda\kappa} = 0$ hold good, so that the horizontal connection coefficient $F_\lambda{}^\kappa{}_\mu$ can be formally written as

$$F_\lambda{}^\kappa{}_\mu = \begin{Bmatrix} \kappa \\ \lambda\ \mu \end{Bmatrix} + \Delta_\lambda{}^\kappa{}_\mu, \tag{3.1}$$

where $\begin{Bmatrix} \kappa \\ \lambda\ \mu \end{Bmatrix}$ means the Christoffel three-index symbol formed with $\gamma_{\lambda\kappa}(x)$ and $\Delta_\lambda{}^\kappa{}_\mu$ is defined as the rest, the latter being essentially constructed from $h_{\lambda\kappa}(x, y)$. From this form of $F_\lambda{}^\kappa{}_\mu$, it is found that the Finslerian geodesic composed of $F_\lambda{}^\kappa{}_\mu$ does not coincide with the Riemannian one composed of $\begin{Bmatrix} \kappa \\ \lambda\ \mu \end{Bmatrix}$, due to the term $\Delta_\lambda{}^\kappa{}_\mu$. That is to say, $h_{\lambda\kappa}(x, y)$, summarized as $\Delta_\lambda{}^\kappa{}_\mu$, gives rise to the difference of Finslerian and Riemannian geodesics. In a word, the connection $F_\lambda{}^\kappa{}_\mu$ breaks (or does not preserve) the projective structure underlying the Riemannian or Einsteinian general relativity (cf. [9]). This is another Finslerian result caused by $h_{\lambda\kappa}(x, y)$.

Concerning the above-mentioned Riemannian or Einsteinian conformal and projective structures, it has been shown recently [10] that they can be preserved simultaneously, even if we choose such a Finsler metric $g_{\lambda\kappa}(x, y)$ as is conformal to the Riemann metric $\gamma_{\lambda\kappa}(x)$, i.e., $g_{\lambda\kappa} = \gamma_{\lambda\kappa}(x) \exp(2\sigma(x, y))$, where the condition $\sigma_{/\mu} = \frac{\delta\sigma}{\delta x^\mu} = 0$ must be imposed to reduce $F_\lambda{}^\kappa{}_\mu$ to $\begin{Bmatrix} \kappa \\ \lambda\ \mu \end{Bmatrix}$. This kind of metric (without homogeneity), however, has already been proposed by the author and some physical problems have been considered (cf. [11]). And the spatial structure of the (generalized) Finsler space based on this metric has been fully investigated by several authors (cf. [12]). By the way, it should be noticed that if $h_{\lambda\kappa}(x, y)$ is assumed to be given by $h_{\lambda\kappa} = \gamma_{\lambda\kappa}(x) \exp(2\phi(x, y))$, then our $g_{\lambda\kappa}(x, y)$ (2.6) is also changed to the conformally Riemannian form, i.e., $g_{\lambda\kappa} = \gamma_{\lambda\kappa}(x)(1+\exp(2\phi(x, y))$.

4. *Physical aspects — II*

Now, from $F_\lambda{}^\kappa{}_\mu$ (3.1), the third curvature tensor $R_\nu{}^\kappa{}_{\lambda\mu}$ can be written in the form, by its definition [3],

$$R_\nu{}^\kappa{}_{\lambda\mu} = K_\nu{}^\kappa{}_{\lambda\mu}(\{\ \}) + L_\nu{}^\kappa{}_{\lambda\mu}(\Delta), \tag{4.1}$$

where $K_\nu{}^\kappa{}_{\lambda\mu}$ is the purely Riemannian curvature derived from $\left\{ {\kappa \atop \lambda\ \mu} \right\}$ and $L_\nu{}^\kappa{}_{\lambda\mu}$ is defined as the rest. Therefore, by use of (4.1), one kind of Einsteinian field equation for the Finslerian field can be constructed as follows:

$$R_{\nu\lambda} - \tfrac{1}{2} R g_{\nu\lambda} = K_{\nu\lambda} - \tfrac{1}{2} K \gamma_{\nu\lambda} + M_{\nu\lambda} = \tau_{\nu\lambda}, \tag{4.2}$$

where $R_{\nu\lambda} = R_\nu{}^\kappa{}_{\lambda\kappa}$, $R = R_{\nu\lambda} g^{\nu\lambda}$, etc. and $M_{\nu\lambda}$ denotes all the remaining parts and $\tau_{\nu\lambda}$ means the energy-momentum tensor for this case. Therefore, the term $M_{\nu\lambda}$ constructed essentially by $L_\nu{}^\kappa{}_{\lambda\mu}(\Delta)$ of (4.1) summarizes all the Finslerian (F_4) contributions caused by $h_{\lambda\kappa}(x, y)$. That is to say, $M_{\nu\lambda}$ summarizes the deviation from the Riemannian (R_4) structure. (The symbols F_4 and R_4 mean the 4-dimensional Finslerian and Riemannian structures, respectively). In the case of Finslerian vacuum with $\tau_{\nu\lambda} = 0$, $M_{\nu\lambda}$ plays the role of source term for the Riemannian field represented by $K_{\nu\lambda}$ (see also below).

In (4.2), $K_{\nu\lambda} \neq 0$ (non-vacuum in R_4-field), even if $\tau_{\nu\lambda} = 0$ in F_4-vacuum, owing to $M_{\nu\lambda} \neq 0$. Conversely, $\tau_{\nu\lambda} \neq 0$ (non-vacuum in F_4-field), even if $K_{\nu\lambda} = 0$ in R_4-vacuum, owing to $M_{\nu\lambda} \neq 0$. This means the difference in character of the F_4- and R_4-vacuum states. The latter case (i.e., $K_{\nu\lambda} = 0$ and $M_{\nu\lambda} \neq 0$) embodies the concept of complete compactification of the internal space with the flat-background, because as mentioned above, the quantity $M_{\nu\lambda}$ absorbs all the y-dependent internal contributions (cf. [8]).

5. *Physical aspects — III*

The field equation (4.2) represents the x-dependence of the Finslerian field, which resembles Einstein's gravitational field equation. On the other hand, the corresponding field equation which represents the y-dependence can also be considered. In fact, several authors [13] have proposed the following field equation:

$$S_{\nu\lambda} - \tfrac{1}{2} S g_{\nu\lambda} = \mu_{\nu\lambda}, \tag{5.1}$$

where $S_{\nu\lambda}$ is the Ricci-tensor derived from the first curvature $S_\nu{}^\kappa{}_{\lambda\mu}$ (i.e., $S_{\nu\lambda} = S_\nu{}^\kappa{}_{\lambda\kappa}$, see [3]), and $S(= S_{\nu\lambda} g^{\nu\lambda})$ is the scalar curvature and $\mu_{\nu\lambda}$ represents the energy-momentum tensor for this case.

Concerning (5.1), it has been known [13] that if $\mu_{\nu\lambda} = 0$ (F_4-vacuum state), then $S_{\nu\lambda} = 0$ holds good, which implies $S_\nu{}^\kappa{}_{\lambda\mu} = 0$ by Matsumoto's theorem [14]. And if the conditions that $S_\nu{}^\kappa{}_{\lambda\mu} = 0$ and $L(x, y) = L(x, -y)$ hold good, where L is the Finslerian fundamental function from which $g_{\lambda\kappa}(x, y)$ is derived, then the Finsler space reduces to Riemannian due to Brickell's theorem [15]. Therefore, it is found that the empty region in

(5.1) presents an almost Riemannian structure. This shows that the field equation $S_{\nu\lambda} = 0$ is somewhat unsuitable from our Finslerian standpoint.

In our case based on (2.6), too, the field equation $S_{\nu\lambda} = 0$ is found to be unsuitable from a physical viewpoint, as will be seen in the following. For that purpose, we shall actually calculate $S_{\nu\lambda}$ in the first order approximation with respect to $h_{\lambda\kappa}(x, y)$ (i.e., neglecting higher order terms $\geqslant O(h^2)$): First, as the inverse of $g_{\lambda\kappa}(x, y)$, we shall put

$$g^{\kappa\lambda}(x, y) = \gamma^{\kappa\lambda}(x) - h^{\kappa\lambda}(x, y); \qquad h^{\kappa\lambda} = \gamma^{\kappa\alpha}\gamma^{\lambda\beta}h_{\alpha\beta}. \tag{5.2}$$

Next, by use of the metrical conditions $Dg_{\lambda\kappa} = 0$, the vertical connection coefficient $C_{\lambda}{}^{\kappa}{}_{\mu}$ becomes (cf. [3])

$$C_{\lambda}{}^{\kappa}{}_{\mu} = \tfrac{1}{2}\gamma^{\kappa\nu}\left(\frac{\partial h_{\nu\lambda}}{\partial y^{\mu}} + \frac{\partial h_{\mu\nu}}{\partial y^{\lambda}} - \frac{\partial h_{\lambda\mu}}{\partial y^{\nu}}\right). \tag{5.3}$$

Finally, the first curvature $S_{\nu}{}^{\kappa}{}_{\lambda\mu}$ is approximated as follows:

$$S_{\nu}{}^{\kappa}{}_{\lambda\mu} = \mathfrak{U}_{\lambda\mu}\left\{\gamma^{\kappa\alpha}\left(\frac{\partial^2 h_{\alpha\nu}}{\partial y^{\mu}\partial y^{\lambda}} + \frac{\partial^2 h_{\lambda\alpha}}{\partial y^{\mu}\partial y^{\nu}} - \frac{\partial^2 h_{\nu\lambda}}{\partial y^{\mu}\partial y^{\alpha}}\right)\right\}, \tag{5.4}$$

from which $S_{\nu\lambda}$ is obtained in the form

$$S_{\nu\lambda} = \gamma^{\kappa\alpha}\left(\frac{\partial^2 h_{\lambda\alpha}}{\partial y^{\kappa}\partial y^{\nu}} - \frac{\partial^2 h_{\nu\lambda}}{\partial y^{\kappa}\partial y^{\alpha}} - \frac{\partial^2 h_{\kappa\alpha}}{\partial y^{\lambda}\partial y^{\nu}} + \frac{\partial^2 h_{\nu\kappa}}{\partial y^{\lambda}\partial y^{\alpha}}\right). \tag{5.5}$$

(In (5.4), the symbol $\mathfrak{U}_{\lambda\mu}$ means interchange of λ, μ and subtraction). At this stage, if it is assumed that there exists a function $\varepsilon(x, y)$ (Finsler energy [2, 16]) such that

$$h_{\lambda\kappa}(x, y) = \tfrac{1}{2}\frac{\partial^2 \varepsilon(x, y)}{\partial y^{\lambda}\partial y^{\kappa}}, \tag{5.6}$$

then $C_{\lambda}{}^{\kappa}{}_{\mu}$ (5.3) becomes $C_{\lambda}{}^{\kappa}{}_{\mu} = \tfrac{1}{2}\gamma^{\kappa\nu}\dfrac{\partial h_{\nu\lambda}}{\partial y^{\mu}} = \tfrac{1}{4}\gamma^{\kappa\nu}\dfrac{\partial^3 \varepsilon}{\partial y^{\mu}\partial y^{\nu}\partial y^{\lambda}}$, by which $S_{\nu}{}^{\kappa}{}_{\lambda\mu} = 0$ in (5.4) and $S_{\nu\lambda} = 0$ in (5.5). That is to say, in this special case, the field equation $S_{\nu\lambda} = 0$ holds good identically for any value of $h_{\lambda\kappa}(x, y)$ (5.6). Therefore, it may be said that the vacuum field equation $S_{\nu\lambda} = 0$ can always be satisfied for any Finslerian perturbation $h_{\lambda\kappa}(x, y)$ given by (5.6). This is unsuitable from a physical point of view. The field equation (5.1) itself, therefore, should be reconsidered in future (cf. [16]).

6. Physical aspects — IV

In the same manner as in Section 5, we shall finally take up again the field equation (4.2) (with $\tau_{\nu\lambda} = 0$) derived from the third curvature $R_{\nu}{}^{\kappa}{}_{\lambda\mu}$. First, in our case, the horizontal connection coefficient $F_{\lambda}{}^{\kappa}{}_{\mu}$ is given by (3.1), where $\Delta_{\lambda}{}^{\kappa}{}_{\mu}$ is put in the first order approximation with respect to $h_{\lambda\kappa}(x, y)$,

$$\Delta_{\lambda}{}^{\kappa}{}_{\mu} = \tfrac{1}{2}\gamma^{\kappa\nu}\left(\frac{\partial h_{\nu\lambda}}{\partial x^{\mu}} + \frac{\partial h_{\mu\nu}}{\partial x^{\lambda}} - \frac{\partial h_{\lambda\mu}}{\partial x^{\nu}}\right), \tag{6.1}$$

because $F_\lambda{}^\kappa{}_\mu$ is constructed from $g^{\kappa\nu}\frac{\delta g_{\nu\lambda}}{\delta x^\mu} \doteqdot \gamma^{\kappa\nu}\left(\frac{\partial g_{\nu\lambda}}{\partial x^\mu} - N^\alpha_\mu \frac{\partial g_{\nu\lambda}}{\partial y^\alpha}\right)$ and $N^\kappa_\mu \doteqdot \frac{\partial}{\partial y^\mu}(F_\alpha{}^\kappa{}_\beta y^\alpha y^\beta)$ is already the first order quantity (cf. [3]). Then, $R_\nu{}^\kappa{}_{\lambda\mu}$ is again given by the form of (4.1) and the field equation (4.2) (with $\tau_{\nu\lambda} = 0$) remains formally as it is, because the term $L_\nu{}^\kappa{}_{\lambda\mu}(\Delta)$ does not vanish in this approximation (see (6.1)). That is to say, even in the first order approximation, the Finslerian perturbation $h_{\lambda\kappa}(x, y)$ exerts effective influence on the field equation (4.2) (with $\tau_{\nu\lambda} = 0$). This conclusion is not changed, even if (5.6) is assumed. This is quite different from the field equation (5.1) (with $\mu_{\nu\lambda} = 0$).

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