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ON THE EQUATIONS OF STATE AND ON FLOW OF PERFECT FLUIDS IN GENERAL RELATIVITY
(COMMENTS TO TWO PAPERS BY V. I. OBOZOV)

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The claims made by V. I. Obozov, that Einstein's equations together with the Bianchi identities and their consequences exclude certain types of equations of state and prohibit certain kinds of flow of a perfect fluid, are mostly contradicted by existing explicit solutions of Einstein's equations. The errors in Obozov's arguments are pointed out.

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1. The incorrect claims of V. I. Obozov and their counterexamples

This note has the purpose of correcting certain errors made by V. I. Obozov in his two articles [1–2], referred to as Papers 1 and 2.

In Paper 1, the author claims that the Einstein field equations, the Bianchi identities and some conclusions from them exclude the following situations for a perfect fluid:

I. $$\varrho_{,i} = hu_i, \quad p_{,k}(\delta^k{}_i - u^k u_i) \neq 0, \quad \varrho_{,i}p_{,j} \neq 0,$$

II. $$p_{,i} = gu_i, \quad \varrho_{,k}(\delta^k{}_i - u^k u_i) \neq 0, \quad \varrho_{,i}p_{,j} \neq 0,$$

III. $$\varrho = \text{const.} \quad p_{,i} = gu_i,$$

IV. $$\varrho_{,k}(\delta^k{}_i - u^k u_i) \neq 0, \quad p = \text{const},$$

where ϱ is the mass-density of the fluid, p is its pressure, u_i is the velocity field, $h = \varrho_{,k}u^k$, $g = p_{,k}u^k$. In fact, statements I, II and IV are contradicted by known explicit solutions of Einstein's equations.

Counterexamples to I are e.g.:

(a) The Stephani Universe [3–5] which is the most general conformally flat perfect fluid solution of the Einstein's equations with nonzero expansion (Obozov's references 1 and 3 contain special cases thereof).

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(b) The special case of the Kustaanheimo-Qvist (K-Q) class of solutions [6–7] considered by Kustaanheimo [8], Taub [9] and Glass [10] which is defined by the arbitrary function $f(x)$ being proportional to $x^{-5/2}$ (the K-Q class are rotation-free, shearfree, expanding, spherically symmetric perfect fluid spacetimes).

Both (a) and (b) have mass-density depending only on time in the comoving coordinate system and the gradient of pressure not aligned with the velocity field.

Counterexamples to II are some of the solutions found by Szafron [11]. They have, in the comoving reference frame, pressure dependent only on time and inhomogeneous mass-density. The Szafron class is defined by the following set of properties: 1. Rotation and acceleration are absent, 2. The 3-spaces orthogonal to the fluid flow are conformally flat and their Ricci tensor has a double eigenvalue. 3. The shear tensor has a double eigenvalue [12].

Counterexamples to IV are the special cases of the Szafron solutions found earlier by Szekeres [13] (where $p = \Lambda = 0$ in addition to the properties listed above) and Lemaitre [14] (where $p = 0 \neq \Lambda$ and the spacetime is spherically symmetric). (For unknown reasons the latter class of solutions is referred to in literature as the "Tolman-Bondi model" although Tolman [15] reobtained the results of Ref. [14] and quoted Lemaitre, and Bondi [16] quoted Tolman, none of them claiming priority).

Thus, of the above list only statement III survives. The errors in the arguments leading to the incorrect statements I, II and IV will be pinpointed in the next Section.

Two more claims of Paper 1 require corrections:

V. "... as follows from the thermodynamics of a perfect fluid its flow is isentropic (...) the equations of state of the form $\varrho \neq \varrho(p)$ describe physically unrealistic models of the perfect fluid". (Introduction to Paper 1).

For a general perfect fluid, the entropy per particle is *constant along the flow-lines* (see e.g. Ref. [17], Eq. (1.8)). In an *isentropic* perfect fluid, the entropy per particle is a *universal constant*, and only this requirement is equivalent to a barotropic equation of state $\varrho = \varrho(p)$ (see again Ref. [17]). Thus there is no reason to call those solutions which do not admit such an equation of state "physically unrealistic"; on the contrary, those with $\varrho = \varrho(p)$ are rather special.

VI. "In nonstationary gravitational fields of an irrotational perfect fluid the acceleration of the fluid particles is zero" (Section 4 of Paper 1).

This statement is contradicted by all the nonstationary solutions of Barnes [18] which include the Stephani Universe [3–5] and the K-Q class [6] as subcases. The erroneous argument is that $t_i u_k - t_k u_i = \eta_{,k}\phi_{,i} - \eta_{,i}\phi_{,k}$ implies $u_{i,k} = u_{k,i}$ and $t_{i,k} = t_{k,i}$; this is not true.

In Paper 2, the following statements require corrections:

VII. "The velocity field of a perfect fluid with geodesic flow is defined by:

$$u_{i,k} = \tfrac{1}{3}\,\theta(g_{ij} - u_i u_k)\text{''}, \tag{1}$$

where θ is the scalar of expansion (theorem 2.1 in Paper 2).

VIII. Eq. (1) above "is the necessary and sufficient condition for the gravitational field of a perfect fluid to be conformal to a flat spacetime" (theorem 3.1).

IX. Eq. (1) is also the necessary and sufficient condition for conformal flatness of a viscous fluid spacetime in which mass-density and the viscosity coefficients are functions of pressure only (theorem 4.1).

Statement VII is contradicted by the solutions of Szafron [11] in which the perfect fluid moves geodesically, but with nonzero shear, and also by the solutions of Lanczos [19] and Gödel [20] (see also Ref. [17]) in which the fluid (actually dust) moves geodesically, but with nonzero rotation.

In statement VIII, Eq. (1) is a sufficient, but not necessary condition for conformal flatness. If (1) holds, then $\omega = \sigma = 0 = \dot{u}_i$, and the only perfect fluid solutions with these properties are the Friedman–Lemaitre–Robertson–Walker (FLRW) solutions (this follows e.g. from the paper by Barnes [18], quoted in Paper 1). They are only a subset of the most general conformally flat perfect fluid spacetimes which were found by Stephani [3, 21].

No counterexample to statement IX is known to me, but its justification contains errors which invalidate the conclusion (see next Section).

Since VII is false, also the following conclusion from VII and VIII, formulated in the introduction to Paper 2, is false:

X. "The flow-lines are geodesics only in the conformally flat gravitational fields of a perfect and viscous fluid". Counterexamples to this are again the Szafron solutions [11] which are of Petrov type D although the flow-lines of the perfect fluid are geodesics.

The errors in deriving statements VII, VIII and IX are discussed in the next Section.

2. *The errors in the arguments*

The errors which led to statements I, II and IV are of the same type. The author found out that a certain tensor T_{ijk} (different in each case) has the properties $T^i{}_{jk}u_i = T^i{}_{jk}\varrho_{,i} = 0$ in cases II and IV, and $T^i{}_{jk}u_i = T^i{}_{jk}p_{,i} = 0$ in case I. From this he concluded that $T_{ijk} = 0$. However, $(u_i, \varrho_{,i})$ or $(u_i, p_{,i})$ is a set of just two linearly independent vectors and if the projections of a certain tensor onto both vectors vanish, it does not imply that the tensor is zero. Thus the errors are hidden in the paragraphs following Eqs (3.8), (3.10) and in the second paragraph of section 3d in Paper 1.

Statement VII says in effect that if acceleration vanishes for a perfect fluid, then rotation and shear must also vanish. The conclusion that $\dot{u}_i = 0$ implies $\omega = 0$ is drawn from Eqs (2.2)–(2.3) in Paper 2 which are correct. However, those equations actually imply that *if* $\dot{u}_i = 0$, then *either* $\omega = 0$ *or* $p_{,n}u^n = 0$. The second case includes the solutions by Lanczos and Gödel [19–20]. The further claim that $\dot{u}_i = 0 = \omega$ implies $\sigma = 0$ is based on the author's previous unpublished work [22], so it is not possible to pinpoint the error in the argument. However, the counterexamples mentioned in Section 1 (Szafron's solutions [11]) are a sufficiently convincing disproof of the claim.

In deriving statement VIII, an additional assumption is fed into the argument, namely that the equation of state is of the form $\varrho = \varrho(p)$. With this additional condition imposed, the Stephani Universe [3–5] reduces to a FLRW Universe [5]. Thus statement VIII becomes correct if the expression "perfect fluid" is replaced by "barotropic perfect fluid".

One of the arguments leading to statement IX was the false conclusion from Eqs (2.2)–(2.3) in Paper 2. This in itself invalidates the claim. However, the reasoning involves one more error. The fact that the metric can be written in a manifestly conformally flat form does not itself imply $\omega = 0$ (after Eq. (4.7) in Paper 2). This implication is true for a perfect fluid via the field equations (see Ref. [21]), but for a viscous fluid a proof would have to be supplied (if the implication holds at all).

REFERENCES

[1] V. I. Obozov, *Acta Phys. Pol.* **B13**, 17 (1982).
[2] V. I. Obozov, *Acta Phys. Pol.* **B13**, 501 (1982).
[3] H. Stephani, *Commun. Math. Phys.* **4**, 137 (1967).
[4] A. Krasiński, *Gen. Relativ. Gravitation* **13**, 1021 (1981).
[5] A. Krasiński, *Gen. Relativ. Gravitation* **15**, 673 (1983).
[6] P. Kustaanheimo, B. Qvist, *Societas Scientiarum Fennica Commentationes Physico- Mathematicae* **XIII**, no 16 (1948).
[7] H. Stephani, *J. Phys.* **A16**, 3529 (1983).
[8] P. Kustaanheimo, *Soc. Sci. Fennica Comment. Phys. Math.* **XIII**, no 12 (1947).
[9] A. H. Taub, *Ann. Inst. Henri Poincare* **A9**, 153 (1968).
[10] E. N. Glass, *J. Math. Phys.* **20**, 1508 (1979).
[11] D. A. Szafron, *J. Math. Phys.* **18**, 1673 (1977).
[12] C. B. Collins, D. A. Szafron, *J. Math. Phys.* **20**, 2354 (1979).
[13] P. Szekeres, *Commun. Math. Phys.* **41**, 55 (1975).
[14] G. Lemaitre, *Ann. Soc. Sci. Bruxelles* **53A**, 51 (1933).
[15] R. C. Tolman, *Proc. Natl. Acad. Sci. USA* **20**, 169 (1934).
[16] H. Bondi, *Mon. Not. R. Astr. Soc.* **107**, 410 (1947).
[17] A. Krasiński, *Acta Phys. Pol.* **B5**, 411 (1974).
[18] A. Barnes, *Gen. Relativ. Gravitation* **4**, 105 (1973).
[19] K. Lanczos, *Z. Phys.* **21**, 73 (1924).
[20] K. Gödel, *Rev. Mod. Phys.* **21**, 447 (1949).
[21] D. Kramer, H. Stephani, E. Herlt, M. A. H. MacCallum, *Exact solutions of Einstein's field equations*, Cambridge University Press 1980, p. 371 (theorem 32.15).
[22] V. I. Obozov, Dep. in VINTI (?), reg. dep. N4781-80 (1980).