NULL TETRAD, SPINOR AND HELICITY FORMALISMS FOR ALL 4-DIMENSIONAL RIEMANNIAN MANIFOLDS. I. NULL TETRADS AND SPINORS

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Null tetrad and spinor formalisms for all 4-dimensional real or complex Riemannian manifolds are given. Global and local aspects of these formalisms are analysed, and their equivalence for suitably oriented spaces is shown.

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1. *Introduction*

A great importance of the null tetrad, spinor or bivector ($\equiv$ helicity) formalisms in the theory of relativity is well known. Persuading anyone of this might look like a provocation. In fact one can hardly imagine the modern theory of relativity without those formalisms; moreover, their role in the complex relativity or in the theory of gravitational instantons seems to be even greater than in relativity with Lorentzian metric. Therefore, it is reasonable to give a compact and uniform description of those formalisms for *all* 4-dimensional real or complex Riemannian manifolds. This is just a purpose of our work.

It is evident that one can expect three formalisms considered to be locally equivalent. However, there is a filling consisting in general theory that they are also *globally* equivalent for suitably oriented spaces. In the present work we make this point proven. For this purpose we abandon the idea of the spinor structure and we deal with "spinor formalism without spinor structure". The mathematical language we use is the theory of fibre bundles. This is the language which seems to be most convenient and general for the problems considered [25, 26].

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Another question we deal with in this work is the possibility of decomposition of, so called, Einsteinian structures into their sub-structures. We find these decompositions for complex Riemannian manifolds or real Riemannian manifolds with metrics of signatures $(\pm 1, \pm 1, \pm 1, \pm 1)$ or $(1, 1, -1, -1)$. Last but not the least purpose of our work is to give some new formulation of the action principle leading to Einstein equations. The result obtained seems to be important in Yang-Mills fields theory. This problem is now intensively investigated by the present authors.

Our work is divided into two parts. The present one is devoted to the null tetrad (Section 2) and spinor (Section 3) formalisms. Our intention is to give only the mathematical foundations of these formalisms. The details important for the practical applications can be found in Refs [8, 13, 17, 18, 24]. We want also the formalisms presented here to be identical, at least locally, with the ones from our previous works. The only differences lie in the factor used for the definition of 1-forms $g^{A\dot{B}}$, i.e., $g^{A\dot{B}}$ as defined in the present paper is equal to $\frac{1}{\sqrt{2}} g^{A\dot{B}}$ from the previous works, and in the definition of $\nabla_{A\dot{B}}$ (see 3.94)).

We do not list all the literature concerning the problem. We have decided to mention only these works which have been inspiring for the present article. Thus in the analysis of the null tetrad formalism and in the considerations on the spinor formalism we refer to Refs [1–24].

2. *Null tetrads*

Let M be a four-dimensional real of class C^∞ or complex analytic differentiable manifold. $T(M)$ denotes the complexified tangent bundle over M (if M is real) or the holomorphic tangent bundle (if M is complex). Analogously $T^*(M)$ is the complexified or holomorphic cotangent bundle over M, respectively. We assume that M is endowed with metric g, i.e., $g \in \mathscr{E}(T^*(M) \underset{s}{\otimes} T^*(M))$ is nowhere degenerated tensor field on M which is real, $\bar{g} = g$, for real M.

(In the present paper $\mathscr{E}(\ldots)$ denotes the set of all cross sections of the vector bundle defined in the parenthesis.)

Thus (M, g) is a four-dimensional real or complex Riemannian manifold.

We have the following cases

(a) "Complex Relativity" (CR): M is complex.

(b) "Hyperbolic Relativity" (HR): M is real; the metric g is of signature $(+++-)$ (HR_+) or $(---+)$ (HR_-).

(c) "Ultra-hyperbolic Relativity" (UR): M is real; the metric g is of signature $(++--)$.

(d) "Euclidean Relativity" (ER): M is real; the metric g is of signature $(++++)$ (ER_+) or $(----)$ (ER_-).

The cases (b), (c) and (d) define "Real Relativity" (RR). Let p be a point of M. Then, four linearly independent vectors $(e_1, e_2, e_3, e_4)_p$, $e_a \in T_p(M)$, $a = 1, \ldots, 4$, such that

$$g = g_{ab} e^a \otimes e^b \tag{2.1}$$

at p, where

$$\|g_{ab}\| = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}. \tag{2.2}$$

$(e^1, e^2, e^3, e^4)_p$, $e^a \in T_p^*(M)$, constitutes a basis dual to $(e_1, e_2, e_3, e_4)_p$, and moreover

$$\text{HR}: \begin{cases} \text{HR}_+: \bar{e}_1 = \bar{e}_2, \bar{e}_3 = e_3, \bar{e}_4 = e_4, \\ \text{HR}_-: \bar{e}_1 = -e_2, \bar{e}_3 = -e_3, \bar{e}_4 = -e_4, \end{cases} \tag{2.3}$$

$$\text{UR}: \bar{e}_a = e_a, \quad a = 1, \ldots, 4, \tag{2.4}$$

$$\text{ER}: \begin{cases} \text{ER}_+: \bar{e}_1 = e_2, \bar{e}_3 = e_4, \\ \text{ER}_-: \bar{e}_1 = -e_2, \bar{e}_3 = -e_4, \end{cases} \tag{2.5}$$

are called a null tetrad at p.

The set $\bigcup_{p \in M} \{(p, (e_a)_p)\}$, where $(e_a)_p := (e_1, e_2, e_3, e_4)_p$, with naturally defined structure of the principal fibre bundle is called a bundle of null tetrads and will be denoted by $NL(M)$.

The structure group of $NL(M)$ we denote by $\mathcal{G}$ and call a tetrad group. One can easily find the following isomorphisms $\mathcal{G}$

$$\text{CR}: \mathcal{G} \cong 0(4; \boldsymbol{C}), \qquad \text{HR}: \mathcal{G} \cong 0(3, 1; \boldsymbol{R}),$$

$$\text{UR}: \mathcal{G} \cong 0(2, 2; \boldsymbol{R}), \qquad \text{ER}: \mathcal{G} \cong 0(4; \boldsymbol{R}). \tag{2.6}$$

Of course $NL(M)$ is a $\mathcal{G}$-structure on M, i.e., it is a reduction of the bundle of linear frames (holomorphic if M is complex) $L(M)$ to the subgroup $\mathcal{G}$.

Let $\mathcal{G}$ acts on the left on $\boldsymbol{C}^4$ as follows: if $l := \|l^{a'}{}_a\| \in \mathcal{G}$ and $\xi := (\xi^2, \xi^2, \xi^3, \xi^4) \in \boldsymbol{C}^4$, then

$$l\xi = \xi' = (\xi^{1'}, \xi^{2'}, \xi^{3'}, \xi^{4'}) \in \boldsymbol{C}^4, \qquad \xi^{a'} := l^{a'}{}_a \xi^a. \tag{2.7}$$

(In the present paper we assume that small Latin indices a, b, c and d run through 1, 2, 3, 4.) Having this, one can define in a standard way [25] a vector bundle $E(M)$ associated with $NL(M)$. We have the obvious isomorphism $E(M) \cong T(M)$ and very often we consider objects on $E(M)$ as ones on $T(M)$ and vice versa. The metric g on M defines a fibre metric, also denoted by g, on $E(M)$. For an arbitrary point $p \in M$ and arbitrary two vectors $X = X^a e_a$, $Y = Y^a e_a \in \Pi_{E(M)}^{-1}(p)$, where $\Pi_{E(M)}: E(M) \to M$ is the projection of $E(M)$ onto M, one has

$$g(X, Y) = g_{ab} X^a Y^b, \tag{2.8}$$

with g_{ab} defined by (2.2).

Now we intend to define a connection on $E(M)$ compatible with the fibre metric g. We must proceed with caution because our differentiable manifold M is not assumed to be paracompact and, moreover, if M is complex analytic, then there arises the problem of analytic extensions of local cross sections of $E(M)$, even for paracompact M.

Let $\mathscr{E}_{\text{loc}}(\ldots)$ denote the set of all local cross sections of a vector bundle defined in parenthesis. Then, we define a connection D on $E(M)$ to be a mapping

$$D\colon \mathscr{E}_{\text{loc}}(E(M)) \to \mathscr{E}_{\text{loc}}(E(M) \otimes T^*(M)) \tag{2.9}$$

such that for each open set $U \subset M$ and for each $X \in \mathscr{E}(E(U))$, $f \in \mathscr{E}(U \times C)$

(*i*)
$$DX \in \mathscr{E}(E(U) \otimes T^*(U)), \tag{2.10}$$

(*ii*)
$$D(fX) = dfX + fDX \tag{2.11}$$

and, moreover, for any open sets U, $V \subset M$, $U \cap V \neq \phi$, and any local sections $X \in \mathscr{E}(E(U))$, $Y \in \mathscr{E}(E(U))$, $Z \in \mathscr{E}(E(U \cap V))$ such that

$$X|_{U\cap V} = Y|_{U\cap V} = Z \tag{2.12}$$

(where by $|_{U\cap V}$ we mean the restriction of the cross section to the set $U \cap V$) is

(*iii*)
$$DX|_{U\cap V} = DY|_{U\cap V} = DZ. \tag{2.13}$$

A connection D on $E(M)$ is said to be compatible with the fibre metric g on $E(M)$ (i.e., it is a metric connection) iff for every open set $U \subset M$ and any X, $Y \in \mathscr{E}(E(U))$

$$dg(X, Y) = g(DX, Y) + g(X, DY). \tag{2.14}$$

If (e_a), $e_a \in \mathscr{E}(E(U))$, is a null tetrad on an open set $U \subset M$, then from (2.10) and (2.11) one has

$$De_a = \Gamma^b{}_a e_b \tag{2.15}$$

and

$$DX = (dX^a + \Gamma^a{}_b X^b)e_a \tag{2.16}$$

for any $X = X^a e_a \in \mathscr{E}(E(U))$. The 1-forms $\Gamma^a{}_b \in \mathscr{E}(T^*(U))$ are called components of a connection form over U associated with the connection D with respect to the null tetrad (e_a), or briefly, components of a connection form over U. Using (2.16) one finds that a connection D on $E(M)$ is compatible with the fibre metric g on $E(M)$ iff for each open set $U \subset M$ and each null tetrad (e_a) on U

$$g_{cb}\Gamma^c{}_a + g_{ac}\Gamma^c{}_b = 0, \tag{2.17}$$

or, equivalently, with the "lowered" indices

$$\Gamma_{ab} + \Gamma_{ba} = 0. \tag{2.18}$$

Notice that the conditions (2.17) say exactly that $\Gamma^a{}_b \in \mathscr{E}(T^*(U))$ are components of the 1-form on $U \subset M$ with the values in the complexified Lie algebra $\mathfrak{g} \otimes_R C$ of the tetrad group $\mathscr{G}$. We say that a connection D on $E(M)$ is real if M is real and if for any local real cross section $X = \bar{X} \in \mathscr{E}(E(U))$

$$\overline{DX} = DX. \tag{2.19}$$

With the use of (2.3)–(2.5) and (2.16) one easily finds that for real M a connection D on $E(M)$ is real iff for each point $p \in M$

$$\text{HR}: \overline{\Gamma^a{}_b} = \Gamma^c{}_d, \tag{2.20}$$

where indices c, d are connected with a, b according to the scheme $1 \to 2$, $2 \to 1$, $3 \to 3$, $4 \to 4$;

$$\text{UR}: \overline{\Gamma^a{}_b} = \Gamma^a{}_b; \tag{2.21}$$

$$\text{ER}: \overline{\Gamma^a{}_b} = \Gamma^c{}_d, \tag{2.22}$$

where c, d and a, b are connected as follows $1 \to 2$, $2 \to 1$, $3 \to 4$, $4 \to 3$. A connection D on $E(M)$ defines the exterior covariant differentiation on $E(M)$ which we also denote by the symbol D, and which is a mapping

$$D: \mathscr{E}_{\text{loc}}(E(M) \otimes \Lambda^r T^*(M)) \to \mathscr{E}_{\text{loc}}(E(M) \otimes \Lambda^{r+1} T^*(M)) \tag{2.23}$$

for $0 \leqslant r \leqslant 4$, such that for an arbitrary open set $U \subset M$

(a) $$D\sigma \in \mathscr{E}(E(U) \otimes \Lambda^{r+1} T^*(U)) \quad \text{for} \quad \sigma \in \mathscr{E}(E(U) \otimes \Lambda^r T^*(U)), \tag{2.24}$$

(b) $$D(\alpha\sigma_1 + \beta\sigma_2) = \alpha D\sigma_1 + \beta D\sigma_2 \quad \text{for} \quad \alpha, \beta \in C; \sigma_1, \sigma_2 \in \mathscr{E}(E(U) \otimes \Lambda^r T^*(U)) \tag{2.25}$$

(c) $$D(X \otimes \omega) = DX \wedge \omega + Xd\omega \quad \text{for} \quad X \in \mathscr{E}(E(U)), \omega \in \mathscr{E}(\Lambda^r T^*(U)). \tag{2.26}$$

Let $\theta \in \mathscr{E}(E(M) \otimes T^*(M))$ be the canonical form of $E(M)$, i.e., for any point $p \in M$ and any vector $X \in T_p(M)$

$$\theta(X) = i(X) \in \Pi^{-1}_{E(M)}(p), \tag{2.27}$$

where

$$i: T(M) \to E(M) \tag{2.28}$$

is the natural isomorphism. (Notice that as a rule we use the same symbols for the objects from $T(M)$ and $E(M)$ if this identification does not lead to any misunderstanding).

If (e_a), $e_a \in \mathscr{E}(E(U))$, is a null tetrad on an open set $U \subset M$ and (e^a), $e^a \in \mathscr{E}(T^*(U))$, is a dual null tetrad of (e_a), then by (2.27) one has

$$\theta = e^a e_a, \text{ on } U. \tag{2.29}$$

Now, the torsion form of a connection D on $E(M)$ is defined to be a cross section $\mathscr{T} \in \mathscr{E}(E(M) \otimes \Lambda^2 T^*(M))$

$$\mathscr{T} := D\theta. \tag{2.30}$$

The curvature form of D is the cross section $\mathscr{R} \in \mathscr{E}(E(M) \otimes E^*(M) \otimes \Lambda^2 T^*(M))$ defined locally on an open set $U \subset M$ with respect to a null tetrad (e_a), $e_a \in \mathscr{E}(E(U))$, as follows

$$\mathscr{R}^a{}_b := d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b. \tag{2.31}$$

Eq. (2.31) is the local representation of the following (symbolic) expression

$$\mathcal{R} := D\Gamma. \tag{2.32}$$

Eqs (2.30) and (2.32) are the 1st and the 2nd Cartan structure equations, respectively. From (2.30) and (2.32) one finds easily the Bianchi's identities
1st identity:

$$D\mathcal{T} = \mathcal{R} \wedge \theta (\overset{\text{locally}}{\Leftrightarrow} D\mathcal{T}^a = \mathcal{R}^a{}_b \wedge e^b), \tag{2.33}$$

2nd identity:

$$D\mathcal{R} = 0. \tag{2.34}$$

With the use of (2.31) we prove that for a real connection D, the 2-forms $\mathcal{R}^a{}_b$ fulfil the analogous relations with respect to complex conjugation as the 1-forms $\Gamma^a{}_b$ (see (2.20)–(2.22)).

In what follows we deal with a connection D on $E(M)$ which is real for real M, compatible with the fibre metric g and its torsion form vanishes. For a given $E(M)$ there exists one and only one connection of these properties. We call it Riemannian or Levi-Civita connection on $E(M)$. The 1st Cartan structure equation reads then

$$D\theta = 0, \tag{2.35}$$

and the 1st Bianchi's identity takes the form of

$$\mathcal{R} \wedge \theta = 0 (\overset{\text{locally}}{\Leftrightarrow} \mathcal{R}^a{}_b \wedge e^b = 0). \tag{2.36}$$

The functions $R^a{}_{bcd} \in \mathcal{E}(U \times C)$ defined by

$$\mathcal{R}^a{}_b = \tfrac{1}{2} R^a{}_{bcd} e^c \wedge e^d \tag{2.37}$$

constitute components of the curvature tensor of Riemannian manifold (M, g) on an open set $U \subset M$ with respect to the basis (e_a), $e_a \in \mathcal{E}(T(U))$ (we identify $i^{-1}(e_a) \equiv e_a$).

By (2.17) and (2.31) one has

$$R_{abcd} = -R_{bacd}. \tag{2.38}$$

From the definition (2.37) we have of course

$$R_{abcd} = -R_{abdc}. \tag{2.39}$$

Then the 1st Bianchi's identity is locally equivalent to the following relation

$$R_{abcd} + R_{adbc} + R_{acdb} = 0. \tag{2.40}$$

The formulae (2.38), (2.39) and (2.40) yield

$$R_{abcd} = R_{cdab}. \tag{2.41}$$

In a standard way we define:
(a) The scalar curvature of (M, g), $R \in \mathscr{E}(M \times C)$, which is given locally by

$$R := R^{ab}{}_{ab}, \tag{2.42}$$

(b) The traceless Ricci tensor of (M, g), $C \in \mathscr{E}(T^*(M) \underset{s}{\otimes} T^*(M))$,

with local components

$$C_{ab} := R^{c}{}_{abc} - \tfrac{1}{4} R g_{ab}, \tag{2.43}$$

(c) The conformal curvature tensor of (M, g) (the Weyl tensor of (M, g)), $W \in \mathscr{E}(T(M) \otimes T^*(M) \otimes \Lambda^2 T^*(M))$, with local components

$$C^{a}{}_{bcd} := R^{a}{}_{bcd} + C^{a}{}_{[c} g_{d]b} + \delta^{a}{}_{[c} C_{d]b} + \frac{R}{6} \delta^{a}{}_{[c} \mu_{d]b}. \tag{2.44}$$

The components of the Weyl tensor satisfy the conditions analogous to (2.38)–(2.41), and moreover

$$C^{c}{}_{abc} = 0. \tag{2.45}$$

One finds easily that the complex conjugation of the components $R^{a}{}_{bcd}$, $C^{a}{}_{bcd}$, C_{ab}, R etc., can be accomplished according to the scheme explained in (2.20)–(2.22).

Define the following objects

$$C^{(5)} := 2C_{4242} = 2R_{4242},\ C^{(4)} := C_{1242} + C_{3442} = R_{1242} + R_{3442},$$

$$C^{(3)} := 2C_{4231} = 2R_{4231} + \frac{R}{6},\ C^{(2)} := C_{1231} + C_{3431} = R_{1231} + R_{3431},$$

$$C^{(1)} := 2C_{3131} = 2R_{3131} \tag{2.46}$$

and

$$\dot{C}^{(5)} := 2C_{4141} = 2R_{4141},\ \dot{C}^{(4)} := C_{2141} + C_{3441} = R_{2141} + R_{3441},$$

$$\dot{C}^{(3)} := 2C_{4132} = 2R_{4132} + \frac{R}{6},\ \dot{C}^{(2)} := C_{2132} + C_{3432} = R_{2132} + R_{3432},$$

$$\dot{C}^{(1)} := 2C_{3232} = 2R_{3232}. \tag{2.47}$$

We have the following relations

$$\text{HR:}\ \overline{C^{(\alpha)}} = \dot{C}^{(\alpha)}, \tag{2.48}$$

$$\text{UR:}\ \overline{C^{(\alpha)}} = C^{(\alpha)},\ \overline{\dot{C}^{(\alpha)}} = \dot{C}^{(\alpha)}, \tag{2.49}$$

$$\text{ER:}\ \overline{C^{(5)}} = C^{(1)},\ \overline{C^{(4)}} = -C^{(2)},\ \overline{C^{(3)}} = C^{(3)}$$

$$\overline{\dot{C}^{(5)}} = \dot{C}^{(1)},\ \overline{\dot{C}^{(4)}} = -\dot{C}^{(2)},\ \overline{\dot{C}^{(3)}} = \dot{C}^{(3)}, \tag{2.50}$$

where $\alpha = 1, \ldots, 5$.

Now we write down the 2nd Cartan structure equations in the form of

$$d\Gamma_{42}+\Gamma_{42}\wedge(\Gamma_{12}+\Gamma_{34}) = \tfrac{1}{2}C^{(5)}e^4\wedge e^2+\tfrac{1}{2}C^{(4)}(e^1\wedge e^2+e^3\wedge e^4)$$
$$+\left(\tfrac{1}{2}C^{(3)}-\frac{R}{12}\right)e^3\wedge e^1-\tfrac{1}{2}C_{44}e^4\wedge e^1-\tfrac{1}{2}C_{42}(-e^1\wedge e^2+e^3\wedge e^4)$$
$$-\tfrac{1}{2}C_{22}e^3\wedge e^2 = \mathcal{R}_{42}, \tag{2.51}$$

$$d(\Gamma_{12}+\Gamma_{34})+2\Gamma_{42}\wedge\Gamma_{31} = C^{(4)}e^4\wedge e^2+\left(C^{(3)}+\frac{R}{12}\right)(e^1\wedge e^2+e^3\wedge e^4)$$
$$+C^{(2)}e^3\wedge e^1-C_{41}e^4\wedge e^1-C_{12}(-e^1\wedge e^2+e^3\wedge e^4)$$
$$+C_{32}e^3\wedge e^2 = \mathcal{R}_{12}+\mathcal{R}_{34}, \tag{2.52}$$

$$d\Gamma_{31}+(\Gamma_{12}+\Gamma_{34})\wedge\Gamma_{31} = \left(\tfrac{1}{2}C^{(3)}-\frac{R}{12}\right)e^4\wedge e^2+\tfrac{1}{2}C^{(2)}(e^1\wedge e^2+e^3\wedge e^4)$$
$$+\tfrac{1}{2}C^{(1)}e^3\wedge e^1-\tfrac{1}{2}C_{11}e^4\wedge e^1+\tfrac{1}{2}C_{31}(-e^1\wedge e^2+e^3\wedge e^4)$$
$$-\tfrac{1}{2}C_{33}e^3\wedge e^2 = \mathcal{R}_{31}, \tag{2.53}$$

$$d\Gamma_{41}+\Gamma_{41}\wedge(-\Gamma_{12}+\Gamma_{34}) = \tfrac{1}{2}\dot{C}^{(5)}e^4\wedge e^1+\tfrac{1}{2}\dot{C}^{(4)}(-e^1\wedge e^2+e^3\wedge e^4)$$
$$+\left(\tfrac{1}{2}\dot{C}^{(3)}-\frac{R}{12}\right)e^3\wedge e^2-\tfrac{1}{2}C_{44}e^4\wedge e^2-\tfrac{1}{2}C_{41}(e^1\wedge e^2+e^3\wedge e^4)$$
$$-\tfrac{1}{2}C_{11}e^3\wedge e^1 = \mathcal{R}_{41}, \tag{2.54}$$

$$d(-\Gamma_{12}+\Gamma_{34})+2\Gamma_{41}\wedge\Gamma_{32} = \dot{C}^{(4)}e^4\wedge e^1+\left(\dot{C}^{(3)}+\frac{R}{12}\right)(-e^1\wedge e^2+e^3\wedge e^4)$$
$$+\dot{C}^{(2)}e^3\wedge e^2-C_{42}e^4\wedge e^2-C_{12}(e^1\wedge e^2+e^3\wedge e^4)$$
$$+C_{31}e^3\wedge e^1 = -\mathcal{R}_{12}+\mathcal{R}_{34}, \tag{2.55}$$

$$d\Gamma_{32}+(-\Gamma_{12}+\Gamma_{34})\wedge\Gamma_{32} = \left(\tfrac{1}{2}\dot{C}^{(3)}-\frac{R}{12}\right)e^4\wedge e^1+\tfrac{1}{2}\dot{C}^{(2)}(-e^1\wedge e^2+e^3\wedge e^4)$$
$$+\tfrac{1}{2}\dot{C}^{(1)}e^3\wedge e^2-\tfrac{1}{2}C_{22}e^4\wedge e^2+\tfrac{1}{2}C_{32}(e^1\wedge e^2+e^3\wedge e^4)-\tfrac{1}{2}C_{33}e^3\wedge e^1 = \mathcal{R}_{32}. \tag{2.56}$$

(Notice that Eqs (2.54)–(2.56) can be obtained from (2.51)–(2.53) by changing $1\to 2$, $2\to 1$, $3\to 3$, $4\to 4$ and $C^{(\alpha)}\to\dot{C}^{(\alpha)}$.) We intend to give a deeper interpretation of the formulae (2.51)–(2.56). For this purpose we have to define some new objects.

First, a metric g on M defines in a natural manner a fiber metric on the vector bundle $\bigoplus_{r=0}^{4} \Lambda^r T^*(M)$, $(\Lambda^0 T^*(M) := M \times C)$ as follows.

For any $\omega, \sigma \in \Pi^{-1}_{\Lambda^r T^*(M)}(p)$, $p \in M$, of the form $\omega = \omega^1 \wedge \ldots \wedge \omega^r$, $\sigma = \sigma^1 \wedge \ldots \wedge \sigma^r$ with $\omega^i, \sigma^j \in \Pi^{-1}_{T^*(M)}(p)$, $i, j = 1, \ldots, r$, we define an inner product of ω and σ, $\overset{(r)}{\gamma}(\omega, \sigma)$, to be a number

$$\overset{(r)}{\gamma}(\omega, \sigma) := r! \det \|g(\omega^i, \sigma^j)\| \tag{2.57}$$

for $0 < r \leqslant 4$, and

$$\overset{(0)}{\gamma}(\omega, \sigma) := \omega\sigma \tag{2.58}$$

for $r = 0$. A C-linear extension of the formulae (2.57), (2.58), defines the inner product on the fibre $\Pi^{-1}_{\Lambda^r T^*(M)}(p)$. As p is an arbitrary point of M we obtain the fibre metric $\overset{(r)}{\gamma} \in \mathscr{E}(\Lambda^r T^*(M) \underset{s}{\otimes} \Lambda^r T^*(M))$ on $\Lambda^r T^*(M)$.

Let now $\omega := \omega_0 + \ldots + \omega_4$, $\sigma := \sigma_0 + \ldots + \sigma_4$, where $\omega_r, \sigma_r \in \Pi^{-1}_{\Lambda^r T^*(M)}(p)$, $p \in M$. Then one defines

$$\gamma(\omega, \sigma) := \sum_{r=0}^{4} \overset{(r)}{\gamma}(\omega_r, \sigma_r). \tag{2.59}$$

A C-linear extension of (2.59) for each point $p \in M$ determines the fibre metric $\gamma \in \mathscr{E}(\Lambda T^*(M) \underset{s}{\otimes} \Lambda T^*(M))$. Assume that Riemannian manifold (M, g) is oriented. In the present formalism it means that the structure group of $NL(M)$, $\mathscr{G}$, has been reduced to "special tetrad group" $S\mathscr{G}$

$$S\mathscr{G} := \{l = \|l^a{}_b\| \in \mathscr{G} : \det\|l^a{}_b\| = 1\}. \tag{2.60}$$

It is equivalent to the statement that one has defined a 4-form $V \in \mathscr{E}(\Lambda^4 T^*(M))$ represented locally in the form of

$$V = e^1 \wedge e^2 \wedge e^3 \wedge e^4. \tag{2.61}$$

Now we are in a position to introduce the Hodge $*$-operator to be a bundle isomorphism

$$* : \Lambda T^*(M) \to \Lambda T^*(M), \tag{2.62}$$

which restricted to $\Lambda^r T^*(M)$, $0 \leqslant r \leqslant 4$, is defined to be a bundle isomorphism

$$* : \Lambda^r T^*(M) \to \Lambda^{4-r} T^*(M), \tag{2.63}$$

such that for any point $p \in M$ and $\sigma, \omega \in \Pi^{-1}_{\Lambda^r T^*(M)}(p)$

$$\sigma \wedge *\omega = -\exp\left\{\frac{i\pi}{2} r(4-r)\right\} \cdot \frac{1}{r!} \gamma(\sigma, \omega) V. \tag{2.64}$$

(The factor "-exp {...}" makes the present definition of the Hodge $*$-operator identical with the one given in our previous works [8, 15, 16, 21]).

By this factor our definition (2.64) differs from those of Atiyah et al. [27], Friedrich [28] or Flanders [29]).

One finds easily that

$$** = \text{identity on } \Lambda T^*(M). \tag{2.65}$$

Let $\Lambda^2_\pm T^*(M)$ be the eigenspaces of the $*$-operator belonging to ± 1 eigenvalues, respectively.

Now, as from (2.64) one has

$$\sigma \wedge * \omega = (-1)^{r(4-r)} \omega \wedge * \sigma = * \sigma \wedge \omega, \tag{2.66}$$

the decomposition

$$\Lambda^2 T^*(M) = \Lambda^2_+ T^*(M) + \Lambda^2_- T^*(M) \tag{2.67}$$

is orthogonal one. $\Lambda^2_+ T^*(M)$ is called a bundle of self-dual 2-forms and $\Lambda^2_- T^*(M)$ a bundle of anti-self-dual 2-forms.

The curvature tensor of (M, g) can be understood as a bundle morphism

$$\mathscr{R}: \Lambda^2 T^*(M) \to \Lambda^2 T^*(M), \tag{2.68}$$

locally defined as follows

$$\mathscr{R}(e^a \wedge e^b) := \tfrac{1}{2} R^{ab}{}_{cd} e^c \wedge e^d, \tag{2.69}$$

where (e^a), $e^a \in T^*(U)$ is a dual null tetrad on an open set $U \subset M$.

Define local bases of $\Lambda^2_+ T^*(U)$ and $\Lambda^2_- T^*(U)$:

$$S^{11} := 2e^4 \wedge e^2,\ S^{12} := e^1 \wedge e^2 + e^3 \wedge e^4,\ S^{22} := 2e^3 \wedge e^1,$$

$$S^{11}, S^{12}, S^{22} \in \mathscr{E}(\Lambda^2_+ T^*(U)); \tag{2.70}$$

$$S^{\dot{1}\dot{1}} := 2e^4 \wedge e^1,\ S^{\dot{1}\dot{2}} := -e^1 \wedge e^2 + e^3 \wedge e^4,\ S^{\dot{2}\dot{2}} := 2e^3 \wedge e^2,$$

$$S^{\dot{1}\dot{1}}, S^{\dot{1}\dot{2}}, S^{\dot{2}\dot{2}} \in \mathscr{E}(\Lambda^2_- T^*(U)). \tag{2.71}$$

Using them we find the following orthonormal bases of $\Lambda_\pm T^*(U)$:
for CR, HR, ER:

$$\mathscr{S}^1 := \tfrac{1}{2}(S^{11} + S^{22}),\ \mathscr{S}^2 := \frac{i}{2} S^{12},\ \mathscr{S}^3 := \frac{i}{2}(S^{11} - S^{22}), \tag{2.72a}$$

$$\mathscr{S}^{\dot{1}} := \tfrac{1}{2}(S^{\dot{1}\dot{1}} + S^{\dot{2}\dot{2}}),\ \mathscr{S}^{\dot{2}} := -\frac{i}{2} S^{\dot{1}\dot{2}},\ \mathscr{S}^{\dot{3}} := -\frac{i}{2}(S^{\dot{1}\dot{1}} - S^{\dot{2}\dot{2}}), \tag{2.72b}$$

$$\gamma(\mathscr{S}^u, \mathscr{S}^v) = \delta^u{}_v,\ \gamma(\mathscr{S}^{\dot{u}}, \mathscr{S}^{\dot{v}}) = \delta^{\dot{u}}{}_{\dot{v}},\ \gamma(\mathscr{S}^u, \mathscr{S}^{\dot{v}}) = 0. \tag{2.72c}$$

for UR:

$$\mathscr{S}^1 := \tfrac{1}{2}(S^{11}+S^{22}),\ \mathscr{S}^2 := \tfrac{1}{2}S^{12},\ \mathscr{S}^3 := \tfrac{1}{2}(S^{11}-S^{22}), \tag{2.73a}$$

$$\mathscr{S}^{\dot 1} := \tfrac{1}{2}(S^{\dot 1\dot 1}+S^{\dot 2\dot 2}),\ \mathscr{S}^{\dot 2} := \tfrac{1}{2}S^{\dot 1\dot 2},\ \mathscr{S}^{\dot 3} := \tfrac{1}{2}(S^{\dot 1\dot 1}-S^{\dot 2\dot 2}), \tag{2.73b}$$

$$\gamma(\mathscr{S}^u, \mathscr{S}^v) = \operatorname{diag}\|1, -1, -1\| = \gamma(\mathscr{S}^{\dot u}, \mathscr{S}^{\dot v}),\ \gamma(\mathscr{S}^u, \mathscr{S}^{\dot v}) = 0. \tag{2.73c}$$

(From now on we assume that Latin letters u, t, v, w run through 1, 2, 3!).

Then we have

$$\mathscr{R}(\mathscr{S}^u) = R^u{}_v\mathscr{S}^v + C^u{}_{\dot v}\mathscr{S}^{\dot v},$$

$$\mathscr{R}(\mathscr{S}^{\dot u}) = R^{\dot u}{}_{\dot v}\mathscr{S}^{\dot v} + C^{\dot u}{}_v\mathscr{S}^v. \tag{2.74}$$

From Eqs (2.51)–(2.56), (2.70), (2.71), (2.72a, b) and (2.73a, b) one finds the following formulae

CR, HR, ER:

$$R^u{}_v = C^u{}_v - \frac{R}{12}\delta^u{}_v, \quad R^{\dot u}{}_{\dot v} = C^{\dot u}{}_{\dot v} - \frac{R}{12}\delta^{\dot u}{}_{\dot v}, \tag{2.75}$$

where

$$\|C^u{}_v\| = \tfrac{1}{2}\begin{Vmatrix} \tfrac{1}{2}(C^{(1)}+C^{(5)})+C^{(3)}, & -i(C^{(2)}+C^{(4)}), & \frac{i}{2}(C^{(1)}-C^{(5)}) \\ -i(C^{(2)}+C^{(4)}), & -2C^{(3)}, & C^{(2)}-C^{(4)} \\ \frac{i}{2}(C^{(1)}-C^{(5)}), & C^{(2)}-C^{(4)}, & -\tfrac{1}{2}(C^{(1)}+C^{(5)})+C^{(3)} \end{Vmatrix}, \tag{2.76}$$

$$\|C^{\dot u}{}_{\dot v}\| = \tfrac{1}{2}\begin{Vmatrix} \tfrac{1}{2}(\dot C^{(1)}+\dot C^{(5)})+\dot C^{(3)}, & i(\dot C^{(2)}+\dot C^{(4)}), & -\frac{i}{2}(\dot C^{(1)}-\dot C^{(5)}) \\ i(\dot C^{(2)}+\dot C^{(4)}), & -2\dot C^{(3)}, & \dot C^{(2)}-\dot C^{(4)} \\ -\frac{i}{2}(\dot C^{(1)}-\dot C^{(5)}), & \dot C^{(2)}-\dot C^{(4)}, & -\tfrac{1}{2}(\dot C^{(1)}+\dot C^{(5)})+\dot C^{(3)} \end{Vmatrix}, \tag{2.77}$$

and

$$\|C^u{}_{\dot v}\| = \tfrac{1}{2}\begin{Vmatrix} -\tfrac{1}{2}(C_{11}+C_{22}+C_{33}+C_{44}), & i(C_{31}-C_{42}), & \frac{i}{2}(C_{33}+C_{22}-C_{44}-C_{11}) \\ i(C_{41}-C_{32}), & -2C_{12}, & -(C_{41}+C_{32}) \\ \frac{i}{2}(C_{44}+C_{22}-C_{33}-C_{11}), & -(C_{42}+C_{31}), & \tfrac{1}{2}(C_{11}+C_{22}-C_{33}-C_{44}) \end{Vmatrix}$$

$$= \|C^{\dot u}{}_v\|^{\mathrm{T}}. \tag{2.78}$$

UR:
One has (2.75) with

$$\|C^u{}_v\| = \frac{1}{2}\begin{Vmatrix} \frac{1}{2}(C^{(1)}+C^{(5)})+C^{(3)}, & C^{(2)}+C^{(4)}, & \frac{1}{2}(C^{(5)}-C^{(1)}) \\ -(C^{(2)}+C^{(4)}), & -2C^{(3)}, & C^{(2)}-C^{(4)} \\ \frac{1}{2}(C^{(1)}-C^{(5)}), & C^{(2)}-C^{(4)}, & -\frac{1}{2}(C^{(1)}+C^{(5)})+C^{(3)} \end{Vmatrix}, \tag{2.79}$$

$\|C^{\dot u}{}_{\dot v}\|$ is defined by the matrix (2.79) after changing $C^{(\alpha)} \to \dot C^{(\alpha)}$, $\alpha = 1, \ldots, 5$. Moreover,

$$\|C^u{}_{\dot v}\| = \frac{1}{2}\begin{Vmatrix} -\frac{1}{2}(C_{11}+C_{22}+C_{33}+C_{44}), & C_{31}-C_{42}, & \frac{1}{2}(C_{33}+C_{22}-C_{44}-C_{11}) \\ C_{41}-C_{32}, & 2C_{12}, & C_{41}+C_{32} \\ \frac{1}{2}(C_{44}+C_{22}-C_{33}-C_{11}), & C_{42}+C_{31}, & \frac{1}{2}(C_{33}+C_{44}-C_{11}-C_{22}) \end{Vmatrix} \tag{2.80}$$

and $\|C^{\dot v}{}_u\|$ can be obtained from the matrix (2.80) by changing $1 \to 2$, $2 \to 1$, $3 \to 3$, $4 \to 4$.

Gathering all above considerations we conclude:

The endomorphism $\mathscr{R}$: $\Lambda^2T^*(M) \to \Lambda^2T^*(M)$ locally defined by (2.69) can be represented in the following form relative to the decomposition (2.67):

$$\mathscr{R} = \begin{Vmatrix} \mathscr{W}_+, & 0 \\ 0, & \mathscr{W}_- \end{Vmatrix} + \left(-\frac{R}{12}\right)\begin{Vmatrix} 1, & 0 \\ 0, & 1 \end{Vmatrix} + \begin{Vmatrix} 0, & \mathscr{C}' \\ \mathscr{C}, & 0 \end{Vmatrix}, \tag{2.81}$$

where $\mathscr{W}_+ \in \mathrm{End}\, \Lambda^2_+T^*(M)$, $\mathscr{W}_- \in \mathrm{End}\, \Lambda^2_-T^*(M)$, $\mathscr{C} \in \mathrm{Hom}\,(\Lambda^2_+T^*(M),\ \Lambda^2_-T^*(M))$, $\mathscr{C}' \in \mathrm{Hom}\,(\Lambda^2_-T^*(M), \Lambda^2_+T^*(M))$, and 1 is the identity transformation of $\Lambda^2_+T^*(M)$ or $\Lambda^2_-T^*(M)$. Then

$$Tr\mathscr{W}_+ = 0 = Tr\mathscr{W}_-. \tag{2.82}$$

With respect to the orthonormal bases defined by (2.72a, b), (2.73a, b), the bundle morphisms $\mathscr{W}_+$, $\mathscr{W}_-$, $\mathscr{C}$ and $\mathscr{C}'$ are represented by the matrices $\|C^u{}_v\|$, $\|C^{\dot u}{}_{\dot v}\|$, $\|C^u{}_{\dot v}\|$ and $\|C^{\dot u}{}_v\|$ respectively. Using these representations one finds easily the following relations

$$\gamma(\omega_+, \mathscr{W}_+\sigma_+) = \gamma(\mathscr{W}_+\omega_+, \sigma_+), \tag{2.83a}$$

$$\gamma(\omega_-, \mathscr{W}_-\sigma_-) = \gamma(\mathscr{W}_-\omega_-, \sigma_-), \tag{2.83b}$$

$$\gamma(\omega_-, \mathscr{C}\sigma_+) = \gamma(\mathscr{C}'\omega_-, \sigma_+), \tag{2.83c}$$

$$\gamma(\omega, \mathscr{R}\sigma) = \gamma(\mathscr{R}\omega, \sigma), \tag{2.83d}$$

for an arbitrary point $p \in M$ and any $\omega_+, \sigma_+ \in \overset{-1}{\Pi}_{\Lambda^2{}_+T^*(M)}(p)$, $\omega_-, \sigma_- \in \overset{-1}{\Pi}_{\Lambda^2{}_-T^*(M)}(p)$, ω, $\sigma \in \overset{-1}{\Pi}_{\Lambda^2T^*(M)}(p)$.

The formulae (2.81)–(2.83) are generalizations of the facts well known in the case of ER (see [27, 28, 30, 35]).

Concluding the present Section we would like to notice that the 2nd Bianchi's identity (2.34) can be easily represented in terms of the null tetrad formalism (see the formulae (A.3a)–(A.4d) in [16] and the formulae arising from (A.3a)–(A.3h) after changing $1 \to 2$, $2 \to 1$, $3 \to 3$, $4 \to 4$ and $C^{(\alpha)} \to \dot C^{(\alpha)}$ for $\alpha = 1, \ldots, 5$).

3. Spinors

In this Section we present "a spinor formalism without spinor structure" for 4-dimensional Riemannian manifolds. We assume that our 4-dimensional Riemannian manifold (M, g) is oriented. Moreover, for HR or UR we need also the reduction of $S\mathscr{G}$-group to $S\mathscr{G}^{\dagger}$, where $S\mathscr{G}^{\dagger}$ is the connected component of the unity element. (For CR or ER, $S\mathscr{G}^{\dagger} = S\mathscr{G}$.)

In the case of HR this reduction means the time orientation of (M, g) and one has

$$\text{HR}: S\mathscr{G}^{\dagger} = \{\|l^a{}_b\| \in S\mathscr{G};\ \tfrac{1}{2}(l^4{}_4 + l^3{}_3 - l^4{}_3 - l^3{}_4) \geq 1\}. \tag{3.1}$$

For the case of UR the group $S\mathscr{G}^{\dagger}$ is analysed in the part II of our work.

$S\mathscr{G}^{\dagger}$-structure on M will be denoted by $SNL(M)$. In what follows we do not assume that M admits a spinor structure (see [24, 31–35]). Let $(p, (e_a)_p) \in SNL(M)$, $p \in M$. Define

$$(p, (e_{1\dot{1}}, e_{1\dot{2}}, e_{2\dot{1}}, e_{2\dot{2}})_p) := (p, (e_4, e_2, e_1, -e_3)_p). \tag{3.2}$$

One can easily verify that if $(p, (e_{a'})_p) \in SNL(M)$, $e_{a'} = l^a{}_{a'} e_a$, $\|l^a{}_{a'}\| \in S\mathscr{G}^{\dagger}$, and if

$$(p, (e_{1'\dot{1}'}, e_{1'\dot{2}'}, e_{2'\dot{1}'}, e_{2'\dot{2}'})_p) := (p, (e_{4'}, e_{2'}, e_{1'}, -e_{3'})_p), \tag{3.3}$$

then

$$e_{A'\dot{B}'} = l^A{}_{A'} l^{\dot{B}}{}_{\dot{B}'} e_{A\dot{B}} \tag{3.4}$$

(capital Latin letters A, B, C ... etc., run through 1, 2), where

$$\begin{aligned}
&\text{CR}: && \|l^A{}_{A'}\| \in \mathrm{SL}(2; \boldsymbol{C}), && \|l^{\dot{A}}{}_{\dot{A}'}\| \in \mathrm{SL}(2; \boldsymbol{C}),\\
&\text{HR}: && \|l^A{}_{A'}\| \in \mathrm{SL}(2; \boldsymbol{C}), && \|l^{\dot{A}}{}_{\dot{A}'}\| = \overline{\|l^A{}_{A'}\|},\\
&\text{UR}: && \|l^A{}_{A'}\| \in \mathrm{SL}(2; \boldsymbol{R}), && \|l^{\dot{A}}{}_{\dot{A}'}\| \in \mathrm{SL}(2; \boldsymbol{R}),\\
&\text{ER}: && \|l^A{}_{A'}\| \in \mathrm{SU}(2), && \|l^{\dot{A}}{}_{\dot{A}'}\| \in \mathrm{SU}(2).
\end{aligned} \tag{3.5}$$

Thus, (3.3), (3.4) and (3.5) yield the following group isomorphisms

$$\begin{aligned}
&\text{CR}: \ \mathrm{SO}(4; \boldsymbol{C}) \cong S\mathscr{G}^{\dagger} \cong \mathrm{SL}(2; \boldsymbol{C}) \otimes \mathrm{SL}(2; \boldsymbol{C}) \cong \mathrm{SL}(2; \boldsymbol{C}) \times \mathrm{SL}(2; \boldsymbol{C})/Z_2,\\
&\text{HR}: \ \mathrm{SO}^{\dagger}(3, 1; \boldsymbol{R}) \cong S\mathscr{G}^{\dagger} \cong \mathrm{SL}(2; \boldsymbol{C}) \otimes \overline{\mathrm{SL}(2; \boldsymbol{C})} \cong \mathrm{SL}(2; \boldsymbol{C})/Z_2,
\end{aligned} \tag{3.6}$$

where

$$\mathrm{SL}(2; \boldsymbol{C}) \otimes \overline{\mathrm{SL}(2; \boldsymbol{C})} := \{\|l^A{}_B l^{\dot{C}}{}_{\dot{D}}\|;\ \|l^A{}_B\| \in \mathrm{SL}(2; \boldsymbol{C}),\ l^{\dot{A}}{}_{\dot{B}} = \overline{l^A{}_B}\},$$

$$\text{UR}: \ \mathrm{SO}^{\dagger}(2, 2; \boldsymbol{R}) \cong S\mathscr{G}^{\dagger} \cong \mathrm{SL}(2; \boldsymbol{R}) \otimes \mathrm{SL}(2; \boldsymbol{R}) \cong \mathrm{SL}(2; \boldsymbol{R}) \times \mathrm{SL}(2; \boldsymbol{R})/Z_2,$$

$$\text{ER}: \ \mathrm{SO}(4; \boldsymbol{R}) \cong S\mathscr{G}^{\dagger} \cong \mathrm{SU}(2) \otimes \mathrm{SU}(2) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)/Z_2,$$

where $Z_2 := \{1, -1\}$ is the cyclic group. (Remember that for CR, ER: $S\mathscr{G}^{\dagger} = S\mathscr{G}$!).

Therefore, the sum $\bigcup_{p\in M}\{(p, (e_{A\dot{B}})_p\}$ defines a principal fibre bundle with the structure group $\mathscr{G}_{sp}$, where

$$\text{CR: } \mathscr{G}_{sp} := \mathrm{SL}(2; \boldsymbol{C}) \otimes \mathrm{SL}(2; \boldsymbol{C}), \quad \text{HR: } \mathscr{G}_{sp} := \mathrm{SL}(2; \boldsymbol{C}) \otimes \overline{\mathrm{SL}(2; \boldsymbol{C})},$$

$$\text{UR: } \mathscr{G}_{sp} := \mathrm{SL}(2; \boldsymbol{R}) \otimes \mathrm{SL}(2; \boldsymbol{R}), \quad \text{ER: } \mathscr{G}_{sp} := \mathrm{SU}(2) \otimes \mathrm{SU}(2). \tag{3.7}$$

This principal fibre bundle will be denoted by $SP^{1\dot{1}}(M)$ and called the bundle of spinor frames over M. One has obviously

$$SP^{1\dot{1}}(M) \cong SNL(M). \tag{3.8}$$

Let $\mathscr{G}_{sp}$ acts on C^4 on the left as follows: if $l = ||l^{A'}{}_A l^{\dot{B}'}{}_{\dot{B}}|| \in \mathscr{G}_{sp}$,

(i) $\xi = (\xi^{A\dot{B}}) \in C^4 \Rightarrow l\xi = \xi' = (\xi^{A'\dot{B}'}), \quad \xi^{A'\dot{B}'} := l^{A'}{}_A l^{\dot{B}'}{}_{\dot{B}} \xi^{A\dot{B}},$

(ii) $\xi = (\xi^{AB}) \in C^4 \Rightarrow l\xi = \xi' = (\xi^{A'B'}), \quad \xi^{A'B'} := l^{A'}{}_A l^{B'}{}_B \xi^{AB},$

(iii) $\xi = (\xi^{\dot{A}\dot{B}}) \in C^4 \Rightarrow l\xi = \xi' = (\xi^{\dot{A}'\dot{B}'}), \quad \xi^{\dot{A}'\dot{B}'} := l^{\dot{A}'}{}_{\dot{A}} l^{\dot{B}'}{}_{\dot{B}} \xi^{\dot{A}\dot{B}},$

(iv) $\xi = (\xi^{A}{}_{B}) \in C^4 \Rightarrow l\xi = \xi' = (\xi^{A'}{}_{B'}), \quad \xi^{A'}{}_{B'} := l^{A'}{}_A l^{-1B}{}_{B'} \xi^{A}{}_{B},$

(v) $\xi = (\xi^{\dot{A}}{}_{\dot{B}}) \in C^4 \Rightarrow l\xi = \xi' = (\xi^{\dot{A}'}{}_{\dot{B}'}), \quad \xi^{\dot{A}'}{}_{\dot{B}'} := l^{\dot{A}'}{}_{\dot{A}} l^{-1\dot{B}}{}_{\dot{B}'} \xi^{\dot{A}}{}_{\dot{B}},$

(vi) $\xi = (\xi^{A}{}_{\dot{B}}) \in C^4 \Rightarrow l\xi = \xi' = (\xi^{A'}{}_{\dot{B}'}), \quad \xi^{A'}{}_{\dot{B}'} := l^{A'}{}_A l^{-1\dot{B}}{}_{\dot{B}'} \xi^{A}{}_{\dot{B}}.$

Then, one can define in a standard manner vector bundles associated with the principal fibre bundle $SP^{1\dot{1}}(M)$:

(i) $S^{1\dot{1}}(M)$ — the spinor bundle of type $((1, \dot{1}), (0, \dot{0}))$ over M,
(ii) $S^2(M)$ — the spinor bundle of type $((2, \dot{0}), (0, \dot{0}))$ over M,
(iii) $S^{\dot{2}}(M)$ — the spinor bundle of type $((0, \dot{2}), (0, \dot{0}))$ over M,
(iv) $S^1{}_1(M)$ — the spinor bundle of type $((1, \dot{0}), (1, \dot{0}))$ over M,
(v) $S^{\dot{1}}{}_{\dot{1}}(M)$ — the spinor bundle of type $((0, \dot{1}), (0, \dot{1}))$ over M,
(vi) $S^1{}_{\dot{1}}(M)$ — the spinor bundle of type $((1, 0), (0, \dot{1}))$ over M.

The dual vector bundles to $S^{1\dot{1}}(M)$, $S^2(M)$, $S^{\dot{2}}(M)$ or $S^1{}_{\dot{1}}(M)$ are

(i)* $S_{1\dot{1}}(M)$ — the spinor bundle of type $((0, 0), (1, \dot{1}))$ over M,
(ii)* $S_2(M)$ — the spinor bundle of type $((0, \dot{0}), (2, \dot{0}))$ over M,
(iii)* $S_{\dot{2}}(M)$ — the spinor bundle of type $((0, \dot{0}), (0, \dot{2}))$ over M,
(vi)* $S_1{}^{\dot{1}}(M)$ — the spinor bundle of type $((0, \dot{1}), (1, \dot{0}))$ over M,

respectively.

By taking the tensor products of above defined vector bundles we obtain the spinor bundles of the form $S^{p\dot{q}}_{r\dot{s}}(M)$. $S^{p\dot{q}}_{r\dot{s}}(M)$ is called the spinor bundle of type $((p, \dot{q}), (r, \dot{s}))$ over M.

From our construction of $S^{1\dot{1}}(M)$ one finds easily an isomorphism

$$j\colon S^{1\dot{1}}(M) \to E(M). \tag{3.9}$$

Namely, j is defined according to the formula

$$j\colon (e_{1\dot{1}}, e_{1\dot{2}}, e_{2\dot{1}}, e_{2\dot{2}})_p \to (e_4, e_2, e_1, -e_3)_p \tag{3.10}$$

for any point $p \in M$, with $(e_{A\dot{B}})_p$ $(e_{A\dot{B}} \in \Pi^{-1}_{S^{1\dot{1}}(M)}(p))$ being a spinor frame at p and $(e_a)_p$ $(e_a \in \Pi^{-1}_{E(M)}(p))$ being a null tetrad at p which understood as an element of $\Pi^{-1}_{NL(M)}(p)$ belongs to $\Pi^{-1}_{SNL(M)}(p)$ (see (3.2)).

(From now on we use null tetrads in the sense of $SNL(M)$!). Then, for an arbitrary vector $X^{A\dot{B}}e_{A\dot{B}} \in \Pi^{-1}_{S^{1\dot{1}}(M)}(p)$

$$j(X^{A\dot{B}}e_{A\dot{B}}) = X^a e_a \in \Pi^{-1}_{E(M)}(p),$$

$$X^{1\dot{1}} = X^4, \quad X^{1\dot{2}} = X^2, \quad X^{2\dot{1}} = X^1, \quad X^{2\dot{2}} = -X^3. \tag{3.11}$$

If M is real then $X^a e_a \in \Pi^{-1}_{E(M)}(p)$ is real iff

$$\mathrm{HR}_\pm;\quad \overline{X^{A\dot{B}}} = \pm X^{B\dot{A}}, \tag{3.12a}$$

$$\mathrm{UR}:\quad \overline{X^{A\dot{B}}} = X^{A\dot{B}}, \tag{3.12b}$$

$$\mathrm{ER}_\pm:\quad \overline{X^{1\dot{1}}} = \mp X^{2\dot{2}}, \quad \overline{X^{1\dot{2}}} = \pm X^{2\dot{1}}. \tag{3.12c}$$

Let D be a connection on $E(M)$. The isomorphism $j^{-1}: E(M) \to S^{1\dot{1}}(M)$ defines uniquely the connection $\tilde{D}$ on $S^{1\dot{1}}(M)$ such that for any open set $U \subset M$ and any $X \in \mathscr{E}(E(U))$ the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\;D\;} & DX \\ {\scriptstyle j^{-1}}\downarrow & & \downarrow{\scriptstyle j^{-1}\otimes 1} \\ j^{-1}(X) & \xrightarrow{\;\tilde{D}\;} & \tilde{D}j^{-1}(X) \end{array} \tag{3.13}$$

commutes; where 1: $T^*(M) \to T^*(M)$ is the identity mapping. With the use of (2.15), (3.10) and (3.13) one can easily find, that the components $\Gamma^{A\dot{B}}{}_{C\dot{D}} \in \mathscr{E}(T^*(U))$ of the connection 1-form, over an open set $U \subset M$, associated with the connection $\tilde{D}$ with respect to a local spinor frame $(e_{A\dot{B}})$, $e_{A\dot{B}} \in \mathscr{E}(S^{1\dot{1}}(U))$, defined by

$$\tilde{D}e_{C\dot{D}} = \Gamma^{A\dot{B}}{}_{C\dot{D}}e_{A\dot{B}}, \tag{3.14}$$

are related to the components $\Gamma^a{}_b \in \mathscr{E}(T^*(U))$ as follows

$$\Gamma^{1\dot{1}}{}_{1\dot{1}} = \Gamma^4{}_4,\ \Gamma^{1\dot{2}}{}_{1\dot{1}} = \Gamma^2{}_4,\ \Gamma^{2\dot{1}}{}_{1\dot{1}} = \Gamma^1{}_4,\ \Gamma^{2\dot{2}}{}_{1\dot{1}} = -\Gamma^3{}_4,$$

$$\Gamma^{1\dot{1}}{}_{1\dot{2}} = \Gamma^4{}_2,\ \Gamma^{1\dot{2}}{}_{1\dot{2}} = \Gamma^2{}_2,\ \Gamma^{2\dot{1}}{}_{1\dot{2}} = \Gamma^1{}_2,\ \Gamma^{2\dot{2}}{}_{1\dot{2}} = -\Gamma^3{}_2,$$

$$\Gamma^{1\dot{1}}{}_{2\dot{1}} = \Gamma^4{}_1,\ \Gamma^{1\dot{2}}{}_{2\dot{1}} = \Gamma^2{}_1,\ \Gamma^{2\dot{1}}{}_{2\dot{1}} = \Gamma^1{}_1,\ \Gamma^{2\dot{2}}{}_{2\dot{1}} = -\Gamma^3{}_1,$$

$$\Gamma^{1\dot{1}}{}_{2\dot{2}} = -\Gamma^4{}_3,\ \Gamma^{1\dot{2}}{}_{2\dot{2}} = -\Gamma^2{}_3,\ \Gamma^{2\dot{1}}{}_{2\dot{2}} = -\Gamma^1{}_3,\ \Gamma^{2\dot{2}}{}_{2\dot{2}} = \Gamma^3{}_3, \tag{3.15}$$

or, in the compact form

$$\Gamma^{A\dot{B}}{}_{C\dot{D}} = -g^{A\dot{B}}{}_{a} g_{C\dot{D}}{}^{b} \Gamma^{a}{}_{b}, \tag{3.16}$$

where $g^{A\dot{B}}{}_{a}$ and $g_{C\dot{D}}{}^{b}$ are defined later on by the formulae (3.82)–(3.84). The fibre metric g on $E(M)$ and the isomorphism (3.9) define the fibre metric $\tilde{g}$ on $S^{1\dot{1}}(M)$ as follows

$$\tilde{g}(j^{-1}X, j^{-1}Y) := g(X, Y) \tag{3.17}$$

for any point $p \in M$ and any $X, Y \in \Pi^{-1}_{E(M)}(p)$. Hence, for a spinor frame $(e_{A\dot{B}})_p$ at p

$$\tilde{g}(e_{A\dot{B}}, e_{C\dot{D}}) = -\varepsilon_{AB}\varepsilon_{\dot{B}\dot{D}}, \tag{3.18}$$

where

$$\|\varepsilon_{AB}\| := \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} =: \|\varepsilon_{\dot{A}\dot{B}}\|. \tag{3.19}$$

From the definition of the connection $\tilde{D}$ on $S^{1\dot{1}}(M)$ one easily concludes that this connection is compatible with the fibre metric $\tilde{g}$ on $S^{1\dot{1}}(M)$ iff the corresponding connection D on $E(M)$ is compatible with g.

Then, by the definition, we find that $\tilde{D}$ is compatible with $\tilde{g}$ iff for each point $p \in M$

$$\varepsilon_{EC}\varepsilon_{\dot{F}\dot{D}}\Gamma^{E\dot{F}}{}_{A\dot{B}} + \varepsilon_{AE}\varepsilon_{\dot{B}\dot{F}}\Gamma^{E\dot{F}}{}_{C\dot{D}} = 0, \tag{3.20}$$

or with "lowered indices"

$$\Gamma_{C\dot{D}A\dot{B}} + \Gamma_{A\dot{B}C\dot{D}} = 0. \tag{3.21}$$

(The spinorial indices are to be manipulated according to the scheme

$$\begin{aligned} &\varepsilon_{AC}\psi^{C\cdots}{}_{\cdots} = \psi_{A}{}^{\cdots}{}_{\cdots}, \qquad \varepsilon_{\dot{A}\dot{C}}\psi^{\dot{C}\cdots}{}_{\cdots} = \psi_{\dot{A}}{}^{\cdots}{}_{\cdots} \\ &\varepsilon^{CA}\psi_{C}{}^{\cdots}{}_{\cdots} = \psi^{A\cdots}{}_{\cdots}, \qquad \varepsilon^{\dot{C}\dot{A}}\psi_{\dot{C}}{}^{\cdots}{}_{\cdots} = \psi^{\dot{A}\cdots}{}_{\cdots}, \end{aligned} \tag{3.22}$$

where

$$\|\varepsilon^{AB}\| := \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} =: \|\varepsilon^{\dot{A}\dot{B}}\|.) \tag{3.23}$$

We now prove an important theorem.

Theorem 2.1. A connection $\tilde{D}$ on $S^{1\dot{1}}(M)$ is compatible with the fibre metric $\tilde{g}$ on $S^{1\dot{1}}(M)$ iff for each point $p \in M$ there exist 1-forms $\Gamma_{AB} = \Gamma_{BA}$, $\Gamma_{\dot{A}\dot{B}} = \Gamma_{\dot{B}\dot{A}}$ at p such that

$$\Gamma_{A\dot{B}C\dot{D}} = \Gamma_{AC}\varepsilon_{\dot{B}\dot{D}} + \Gamma_{\dot{B}\dot{D}}\varepsilon_{AC}. \tag{3.24}$$

The 1-forms Γ_{AB} and $\Gamma_{\dot{A}\dot{B}}$ are uniquely defined by $\Gamma_{A\dot{B}C\dot{D}}$.

Proof. Assume that a connection $\tilde{D}$ on $S^{1\dot{1}}(M)$ is compatible with $\tilde{g}$. Therefore (3.21) holds, and also

$$-\Gamma_{C\dot{B}A\dot{D}} - \Gamma_{A\dot{D}C\dot{B}} = 0. \tag{3.25}$$

From (3.21) and (3.25) one finds

$$(\Gamma_{A\dot{B}C\dot{D}}-\Gamma_{A\dot{D}C\dot{B}})+(\Gamma_{C\dot{D}A\dot{B}}-\Gamma_{C\dot{B}A\dot{D}}) = (\Gamma_A{}^{\dot{E}}{}_{C\dot{E}}-\Gamma_C{}^{\dot{E}}{}_{A\dot{E}})\varepsilon_{\dot{B}\dot{D}} = 0. \tag{3.26}$$

Hence, defining

$$\Gamma_{AC} := \tfrac{1}{2}\Gamma_A{}^{\dot{E}}{}_{C\dot{E}}, \tag{3.27}$$

we have

$$\Gamma_{A\dot{B}C\dot{D}}-\Gamma_{A\dot{D}C\dot{B}} = 2\Gamma_{AC}\varepsilon_{\dot{B}\dot{D}}, \tag{3.28}$$

$$\Gamma_{AC} = \Gamma_{CA}. \tag{3.29}$$

Writing the left-hand side of (3.26) in another form one obtains

$$(\Gamma_{A\dot{B}C\dot{D}}-\Gamma_{C\dot{B}A\dot{D}})+(\Gamma_{C\dot{D}A\dot{B}}-\Gamma_{A\dot{D}C\dot{B}}) = (\Gamma^E{}_{\dot{B}E\dot{D}}-\Gamma^E{}_{\dot{D}E\dot{B}})\varepsilon_{AC} = 0. \tag{3.30}$$

Thus,

$$\Gamma_{A\dot{B}C\dot{D}}-\Gamma_{C\dot{B}A\dot{D}} = 2\Gamma_{\dot{B}\dot{D}}\varepsilon_{AC}, \tag{3.31}$$

$$\Gamma_{\dot{B}\dot{D}} = \Gamma_{\dot{D}\dot{B}}, \tag{3.32}$$

where

$$\Gamma_{\dot{B}\dot{D}} := \tfrac{1}{2}\Gamma^E{}_{\dot{B}E\dot{D}}. \tag{3.33}$$

Adding (3.28) and (3.31), using also (3.25) we conclude that (3.24) holds. Conversely, if (3.24) holds, then the relation (3.21) is satisfied and, consequently, $\tilde{D}$ is compatible with $\tilde{g}$. Thus, the proof is completed. ■

Now, (3.24) is equivalent to the following relation

$$\Gamma^{A\dot{B}}{}_{C\dot{D}} = \Gamma^A{}_C\delta^{\dot{B}}{}_{\dot{D}}+\Gamma^{\dot{B}}{}_{\dot{D}}\delta^A{}_C,$$
$$\Gamma^A{}_A = 0 = \Gamma^{\dot{A}}{}_{\dot{A}}. \tag{3.34}$$

The formula (3.34) means exactly that $\Gamma^{A\dot{B}}{}_{C\dot{D}} \in T^*_p(M)$ are the components of a 1-form at $p \in M$ with values in the complex Lie algebra of the complex Lie group SL(2; C) ⊗ SL(2; C), and $\Gamma^A{}_B$, $\Gamma^{\dot{A}}{}_{\dot{B}} \in T^*_p(M)$ are the components of 1-forms at $p \in M$ with values in the complex Lie algebra sl(2; C).

In what follows we deal with a connection $\tilde{D}$ on $S^{1\dot{1}}(M)$ compatible with the fibre metric $\tilde{g}$. Then, from (3.15) and (3.34) one finds

$$\|\Gamma^A{}_B\| = -\tfrac{1}{2}\begin{Vmatrix}2\Gamma_{42}, & \Gamma_{12}+\Gamma_{34}\\ \Gamma_{12}+\Gamma_{34}, & 2\Gamma_{31}\end{Vmatrix}, \quad \|\Gamma^{\dot{A}}{}_{\dot{B}}\| = -\tfrac{1}{2}\begin{Vmatrix}2\Gamma_{41}, & -\Gamma_{12}+\Gamma_{34}\\ -\Gamma_{12}+\Gamma_{34}, & 2\Gamma_{32}\end{Vmatrix} \tag{3.35}$$

From (2.20)–(2.22) and (3.35) it follows that a connection D on $E(M)$ is real iff for an arbitrary point $p \in M$

$$\text{HR:}\quad \overline{\Gamma_{AB}} = \Gamma_{\dot{A}\dot{B}},$$
$$\text{UR:}\quad \overline{\Gamma_{AB}} = \Gamma_{AB},\quad \overline{\Gamma_{\dot{A}\dot{B}}} = \Gamma_{\dot{A}\dot{B}},$$
$$\text{ER:}\quad \overline{\Gamma_{AB}} = \Gamma^{AB},\quad \overline{\Gamma_{\dot{A}\dot{B}}} = \Gamma^{\dot{A}\dot{B}}. \tag{3.36}$$

The formulae (3.36) are equivalent to the following statements

$$\begin{aligned}
&\text{HR:} \quad \|\Gamma^A{}_B\| \in \mathrm{sl}(2;C) \otimes T_p^{*R}(M),\ \|\Gamma^{\dot{A}}{}_{\dot{B}}\| = \overline{\|\Gamma^A{}_B\|},\\
&\text{UR:} \quad \|\Gamma^A{}_B\| \in \mathrm{sl}(2;R) \otimes T_p^{*R}(M),\ \|\Gamma^{\dot{A}}{}_{\dot{B}}\| \in \mathrm{sl}(2;R) \otimes T_p^{*R}(M),\\
&\text{ER:} \quad \|\Gamma^A{}_B\| \in \mathrm{su}(2) \otimes T_p^{*R}(M),\ \|\Gamma^{\dot{A}}{}_{\dot{B}}\| \in \mathrm{su}(2) \otimes T_p^{*R}(M),
\end{aligned} \tag{3.37}$$

where $T_p^{*R}(M)$ is the real cotangent space at the point $p \in M$. One has also

$$\text{CR:} \quad \|\Gamma^A{}_B\| \in \mathrm{sl}(2;C) \otimes T_p^*(M),\ \|\Gamma^{\dot{A}}{}_{\dot{B}}\| \in \mathrm{sl}(2;C) \otimes T_p^*(M). \tag{3.38}$$

A connection $\tilde{D}$ on $S^{1\dot{1}}(M)$ defines the exterior covariant differentiation $\tilde{D}$ on $S^{1\dot{1}}(M)$ in the analogous manner as a connection D on $E(M)$ defines the exterior covariant differentiation on $E(M)$ (see (2.23)–(2.26)).

We easily find that for any open set $U \subset M$ and any cross section $\sigma \in \mathscr{E}(E(U) \otimes \Lambda^r T^*(U))$, $0 \leqslant r \leqslant 4$, the following diagram

$$\begin{array}{ccc}
\sigma & \xrightarrow{\ D\ } & D\sigma \\
{\scriptstyle j^{-1}\otimes 1}\downarrow & & \downarrow{\scriptstyle j^{-1}\otimes 1} \\
\tilde{\sigma} & \xrightarrow{\ \tilde{D}\ } & \tilde{D}\tilde{\sigma}
\end{array} \tag{3.39}$$

commutes.

The decomposition of a connection $\tilde{D}$ on $S^{1\dot{1}}(M)$ given by (3.34) enables us to define the connection on an arbitrary spinor bundle $S^{p\dot{q}}_{r\dot{s}}(M)$. This connection will be denoted by $\tilde{D}$ and it is locally defined as follows

$$\begin{aligned}
\tilde{D}e_{C\ldots\dot{D}\ldots}^{A\ldots\dot{B}\ldots} &:= \Gamma^E{}_C e_{E\ldots\dot{D}\ldots}^{A\ldots\dot{B}\ldots} + \ldots + \Gamma^{\dot{E}}{}_{\dot{D}} e_{C\ldots\dot{E}\ldots}^{A\ldots\dot{B}\ldots} + \ldots \\
&\quad - \Gamma^A{}_E e_{C\ldots\dot{D}\ldots}^{E\ldots\dot{B}\ldots} - \ldots - \Gamma^{\dot{B}}{}_{\dot{E}} e_{C\ldots\dot{D}\ldots}^{A\ldots\dot{E}\ldots} - \ldots,
\end{aligned} \tag{3.40}$$

where $(e_{C\ldots\dot{D}\ldots}^{A\ldots\dot{B}\ldots})$, $e_{C\ldots\dot{D}\ldots}^{A\ldots\dot{B}\ldots} \in \mathscr{E}(S^{p\dot{q}}_{r\dot{s}}(U))$, is a spinor frame on an open set $U \subset M$.

Then, one can immediately define the exterior covariant differentiation $\tilde{D}$ on $S^{p\dot{q}}_{r\dot{s}}(M)$.

Now we find easily from (3.14) and (3.34) that if $(e_{A\dot{B}})$ and $(e_{A'\dot{B}'})$, $e_{A\dot{B}}, e_{A'\dot{B}'} \in \mathscr{E}(S^{1\dot{1}}(U))$, are two spinor frames on some open set $U \subset M$ related by the transformation (3.4) with (3.5), then

$$\Gamma^{A'}{}_{B'} = l^{-1A'}{}_A l^B{}_{B'} \Gamma^A{}_B + l^{-1A'}{}_A dl^A{}_{B'} \tag{3.41}$$

and

$$\Gamma^{\dot{A}'}{}_{\dot{B}'} = l^{-1\dot{A}'}{}_{\dot{A}} l^{\dot{B}}{}_{\dot{B}'} \Gamma^{\dot{A}}{}_{\dot{B}} + l^{-1\dot{A}'}{}_{\dot{A}} dl^{\dot{A}}{}_{\dot{B}'}. \tag{3.42}$$

Let θ be the canonical form of $E(M)$. Then, the canonical form $\tilde{\theta}$ of $S^{1\dot{1}}(M)$ is defined as follows

$$\mathscr{E}(S^{1\dot{1}}(M) \otimes T^*(M)) \ni \tilde{\theta} := (j^{-1} \otimes 1)\theta. \tag{3.43}$$

With the use of (3.39) one concludes that the torsion form $\tilde{\mathscr{T}}$ of a connection $\tilde{D}$ on $S^{1\dot{1}}(M)$

$$\mathscr{E}(S^{1\dot{1}}(M) \otimes \Lambda^2 T^*(M))\tilde{\mathscr{T}} := \tilde{D}\tilde{\theta} \tag{3.44}$$

is related to the torsion form $\mathscr{T}$ of the connection D on $E(M)$ (see (2.30)) as follows

$$\tilde{\mathscr{T}} = (j^{-1} \otimes 1)\mathscr{T}. \tag{3.45}$$

In the local spinor frame $(e_{A\dot{B}})$, $e_{A\dot{B}} \in \mathscr{E}(S^{1\dot{1}}(U))$, on an open set $U \subset M$, we have

$$\tilde{\theta} = g^{A\dot{B}} e_{A\dot{B}}, \tag{3.46}$$

where $g^{A\dot{B}} \in \mathscr{E}(T^*(U))$

$$\|g^{A\dot{B}}\| := \begin{Vmatrix} e^4 & e^2 \\ e^1 & -e^3 \end{Vmatrix}. \tag{3.47}$$

One can express the metric g on U in terms of $g^{A\dot{B}}$ as follows

$$g = -g^{A\dot{B}} \underset{s}{\otimes} g_{A\dot{B}}. \tag{3.48}$$

Then, the 1st Cartan structure equation (3.44) restricted to U gives

$$dg^{A\dot{B}} + \Gamma^A{}_C \wedge g^{C\dot{B}} + \Gamma^{\dot{B}}{}_{\dot{C}} \wedge g^{A\dot{C}} = \mathscr{T}^{A\dot{B}}, \tag{3.49}$$

with $\tilde{\mathscr{T}} = \mathscr{T}^{A\dot{B}} e_{A\dot{B}}$.

The curvature form of $\tilde{D}$ is the cross section $\tilde{\mathscr{R}} \in \mathscr{E}(S^{1\dot{1}}(M) \otimes S_{1\dot{1}}(M) \otimes \Lambda^2 T^*(M))$ which restricted to an open set $U \subset M$ and with respect to a spinor frame $(e_{A\dot{B}})$ on U, $e_{A\dot{B}} \in \mathscr{E}(S^{1\dot{1}}(U))$, possesses the components

$$\mathscr{R}^{A\dot{B}}{}_{C\dot{D}} := d\Gamma^{A\dot{B}}{}_{C\dot{D}} + \Gamma^{A\dot{B}}{}_{E\dot{F}} \wedge \Gamma^{E\dot{F}}{}_{C\dot{D}}. \tag{3.50}$$

This is the local representation of the following (symbolic) expression

$$\tilde{\mathscr{R}} := \tilde{D}\tilde{\Gamma}. \tag{3.51}$$

Then, the Bianchi's identities read

1st identity: $$\tilde{D}\tilde{\mathscr{T}} = \tilde{\mathscr{R}} \wedge \tilde{\theta} \left(\overset{\text{locally}}{\Leftrightarrow} \tilde{D}\tilde{\mathscr{T}}^{A\dot{B}} = \mathscr{R}^{A\dot{B}}{}_{C\dot{D}} \wedge g^{C\dot{D}} \right), \tag{3.52}$$

2nd identity: $$\tilde{D}\tilde{\mathscr{R}} = 0. \tag{3.53}$$

Using (3.34) one finds easily that

$$\mathscr{R}^{A\dot{B}}{}_{C\dot{D}} = \mathscr{R}^A{}_C \delta^{\dot{B}}{}_{\dot{D}} + \mathscr{R}^{\dot{B}}{}_{\dot{D}} \delta^A{}_C, \tag{3.54}$$

where

$$\mathscr{R}^A{}_B := d\Gamma^A{}_B + \Gamma^A{}_C \wedge \Gamma^C{}_B, \tag{3.55}$$

and

$$\mathscr{R}^{\dot{A}}{}_{\dot{B}} := d\Gamma^{\dot{A}}{}_{\dot{B}} + \Gamma^{\dot{A}}{}_{\dot{C}} \wedge \Gamma^{\dot{C}}{}_{\dot{B}}. \tag{3.56}$$

Hence,

$$\tilde{\mathscr{R}} = \tilde{\mathscr{R}}_+ \otimes \dot{1} + 1 \otimes \tilde{\mathscr{R}}_-, \tag{3.57}$$

where $\tilde{\mathscr{R}}_+ \in \mathscr{E}(S^1{}_1 \otimes \Lambda^2 T^*(M))$, $\tilde{\mathscr{R}}_- \in \mathscr{E}(S^{\dot{1}}{}_{\dot{1}}(M) \otimes \Lambda^2 T^*(M))$ are (locally) defined by (3.55), (3.56), respectively, and $1 \in \mathscr{E}(S^1{}_1(M))$, $\dot{1} \in (S^{\dot{1}}{}_{\dot{1}}(M))$ are the Kronecker's δ's.

By (3.29) and (3.32) one has

$$\mathscr{R}_{AB} = \mathscr{R}_{BA}, \mathscr{R}_{\dot{A}\dot{B}} = \mathscr{R}_{\dot{B}\dot{A}}. \tag{3.58}$$

Eqs (3.55), (3.56) can be written globally as follows

$$\tilde{\mathscr{R}}_+ := \tilde{D}\tilde{\Gamma}_+, \tag{3.59}$$

$$\tilde{\mathscr{R}}_- := \tilde{D}\tilde{\Gamma}_-, \tag{3.60}$$

respectively, where $\tilde{\Gamma}_+$, $\tilde{\Gamma}_-$ are the components of the connection 1-form associated with the connection $\tilde{D}$; $\tilde{\Gamma}_+$, $\tilde{\Gamma}_-$ are locally represented by $\Gamma^A{}_B$, $\Gamma^{\dot{A}}{}_{\dot{B}}$, respectively. Now the Bianchi's identities take the forms of

1st identity (locally):

$$\tilde{D}\tilde{\mathscr{T}}^{A\dot{B}} = \mathscr{R}^A{}_C \wedge g^{C\dot{B}} + \mathscr{R}^{\dot{B}}{}_{\dot{C}} \wedge g^{A\dot{C}}, \tag{3.61}$$

2nd identity:

$$\tilde{D}\tilde{\mathscr{R}}_+ = 0 \text{ and } \tilde{D}\tilde{\mathscr{R}}_- = 0. \tag{3.62}$$

From now on we assume that the connection D on $E(M)$ is Riemannian. This occurs iff

$$\tilde{D}\tilde{\theta} = 0 \tag{3.63}$$

(see (2.35), (3.39) and (3.43)).

Then the 1st Bianchi's identity reads

$$\tilde{\mathscr{R}} \wedge \tilde{\theta} = 0 (\overset{\text{locally}}{\Leftrightarrow} \mathscr{R}^A{}_C \wedge g^{C\dot{B}} + \mathscr{R}^{\dot{B}}{}_{\dot{C}} \wedge g^{A\dot{C}} = 0). \tag{3.64}$$

Let $\tilde{S} \in \mathscr{E}(S^2(M) \otimes \Lambda^2 T^*(M))$, $\dot{\tilde{S}} \in \mathscr{E}(S^{\dot{2}}(M) \otimes \Lambda^2 T^*(M))$ be spinor fields on M locally defined as follows

$$S^{AB} := \varepsilon_{\dot{C}\dot{D}} g^{A\dot{C}} \wedge g^{B\dot{D}} = S^{BA}, \tag{3.65}$$

$$S^{\dot{A}\dot{B}} := \varepsilon_{CD} g^{C\dot{A}} \wedge g^{D\dot{B}} = S^{\dot{B}\dot{A}}. \tag{3.66}$$

One finds easily that S^{11}, $S^{12} = S^{21}$, S^{22} are exactly the self-dual 2-forms defined by (2.70), and $S^{\dot{1}\dot{1}}$, $S^{\dot{1}\dot{2}} = S^{\dot{2}\dot{1}}$, $S^{\dot{2}\dot{2}}$ are the anti-self-dual 2-forms defined by (2.71). As a consequence of (3.63) we have

$$\tilde{D}\tilde{S} = 0, \quad \tilde{D}\dot{\tilde{S}} = 0. \tag{3.67}$$

Let $U \subset M$ be an open set and $S^{AB} \in \mathscr{E}(\Lambda^2{}_+T^*(U))$, $S^{\dot{A}\dot{B}} \in \mathscr{E}(\Lambda^2{}_-T^*(U))$ be as in (3.65), (3.66). Then consider the following decompositions of $\mathscr{R}_{AB}$ and $\mathscr{R}_{\dot{A}\dot{B}}$ on U

$$\mathscr{R}_{AB} = R_{ABCD}S^{CD} + R_{AB\dot{C}\dot{D}}S^{\dot{C}\dot{D}}, \tag{3.68}$$

$$\mathscr{R}_{\dot{A}\dot{B}} = \dot{R}_{CD\dot{A}\dot{B}}S^{CD} + R_{\dot{A}\dot{B}\dot{C}\dot{D}}S^{\dot{C}\dot{D}}, \tag{3.69}$$

where $R_{ABCD} = R_{(AB)(CD)}$, $R_{\dot{A}\dot{B}\dot{C}\dot{D}} = R_{(\dot{A}\dot{B})(\dot{C}\dot{D})}$, $R_{AB\dot{C}\dot{D}} = R_{(AB)(\dot{C}\dot{D})}$, $\dot{R}_{AB\dot{C}\dot{D}} = \dot{R}_{(AB)(\dot{C}\dot{D})}$ are the components of spinor fields on U. Now we decompose the spinor fields on U defined by R_{ABCD} or $R_{\dot{A}\dot{B}\dot{C}\dot{D}}$ into irreducible objects according to the formulae

$$R_{ABCD} = \psi_{ABCD} + \phi_{D(A}\varepsilon_{B)C} + \Lambda(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}), \tag{3.70}$$

$$R_{\dot{A}\dot{B}\dot{C}\dot{D}} = \psi_{\dot{A}\dot{B}\dot{C}\dot{D}} + \phi_{\dot{D}(\dot{A}}\varepsilon_{\dot{B})\dot{C}} + \dot{\Lambda}(\varepsilon_{\dot{A}\dot{C}}\varepsilon_{\dot{B}\dot{D}} + \varepsilon_{\dot{A}\dot{D}}\varepsilon_{\dot{B}\dot{C}}), \tag{3.71}$$

where $\psi_{ABCD} = \psi_{(ABCD)}$, $\psi_{\dot{A}\dot{B}\dot{C}\dot{D}} = \psi_{(\dot{A}\dot{B}\dot{C}\dot{D})}$, $\phi_{AB} = \phi_{(AB)}$, $\phi_{\dot{A}\dot{B}} = \phi_{(\dot{A}\dot{B})}$, Λ, $\dot{\Lambda} \in \mathscr{E}(U \times C)$ define some spinor fields on U.

We prove an important proposition

Proposition 2.1. The 1st Bianchi's identity on U, (3.64), is equivalent to the following formulae

$$R_{AB\dot{C}\dot{D}} = \dot{R}_{AB\dot{C}\dot{D}}, \quad \phi_{AB} = 0, \quad \phi_{\dot{A}\dot{B}} = 0, \quad \Lambda = \dot{\Lambda} \tag{3.72}$$

on U.

Proof. First, one finds that the 1st Bianchi's identity on U, (3.64), is equivalent to the formulae

$$(\mathscr{R}^A{}_C \wedge g^{C\dot{B}} + R^{\dot{B}}{}_{\dot{C}} \wedge g^{A\dot{C}}) \wedge g^{D\dot{E}} = 0 \tag{3.73}$$

on U. Then, using (3.68)–(3.71), (3.65), (3.66), (3.47) and (2.61) we conclude that

$$\{(3.73) \Leftrightarrow [(R_{AB\dot{C}\dot{D}} - \dot{R}_{AB\dot{C}\dot{D}}) - 2\phi_{AB}\varepsilon_{\dot{C}\dot{D}} - 2\varepsilon_{AB}\phi_{\dot{C}\dot{D}}$$
$$-3(\Lambda - \dot{\Lambda})\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}}]V = 0 \Leftrightarrow (3.72)\}. \blacksquare \tag{3.74}$$

Finally, denoting

$$C_{ABCD} := -2\psi_{ABCD}, \quad C_{\dot{A}\dot{B}\dot{C}\dot{D}} := -2\psi_{\dot{A}\dot{B}\dot{C}\dot{D}}, \quad C_{AB\dot{C}\dot{D}} := 2R_{AB\dot{C}\dot{D}}. \tag{3.75}$$

and using (3.55), (3.56), (3.35), (3.68)–(3.71) one finds, by comparing with (2.51)–(2.56), the following relations

$$C_{1111} = \tfrac{1}{2}C^{(5)}, \quad C_{1112} = \tfrac{1}{2}C^{(4)}, \quad C_{1122} = \tfrac{1}{2}C^{(3)},$$
$$C_{1222} = \tfrac{1}{2}C^{(2)}, \quad C_{2222} = \tfrac{1}{2}C^{(1)}, \tag{3.76}$$

$$C_{\dot{1}\dot{1}\dot{1}\dot{1}} = \tfrac{1}{2}\dot{C}^{(5)}, \quad C_{\dot{1}\dot{1}\dot{1}\dot{2}} = \tfrac{1}{2}\dot{C}^{(4)}, \quad C_{\dot{1}\dot{1}\dot{2}\dot{2}} = \tfrac{1}{2}\dot{C}^{(3)},$$
$$C_{\dot{1}\dot{2}\dot{2}\dot{2}} = \tfrac{1}{2}\dot{C}^{(2)}, \quad C_{\dot{2}\dot{2}\dot{2}\dot{2}} = \tfrac{1}{2}\dot{C}^{(1)}, \tag{3.77}$$

$$C_{AB\dot{C}\dot{D}} = \tfrac{1}{2}\, g_{A\dot{C}}{}^{a} g_{B\dot{D}}{}^{b} C_{ab}, \tag{3.78}$$

$$\Lambda = \frac{R}{48}; \tag{3.79}$$

for $\mathscr{R}_{AB}$ and $\mathscr{R}_{\dot{A}\dot{B}}$ we have

$$\mathscr{R}_{AB} = -\tfrac{1}{2}\, C_{ABCD}S^{CD} + \frac{R}{24} S_{AB} + \tfrac{1}{2}\, C_{AB\dot{C}\dot{D}}S^{\dot{C}\dot{D}}, \tag{3.80}$$

$$\mathscr{R}_{\dot{A}\dot{B}} = -\tfrac{1}{2}\, C_{\dot{A}\dot{B}\dot{C}\dot{D}}S^{\dot{C}\dot{D}} + \frac{R}{24} S_{\dot{A}\dot{B}} + \tfrac{1}{2}\, C_{CD\dot{A}\dot{B}}S^{CD}. \tag{3.81}$$

An object (spin-tensor) $g_{A\dot{C}}{}^{a}$ appearing in (3.78) is defined as follows: according to (3.47) we put

$$g^{A\dot{B}} = g^{A\dot{B}}{}_{a} e^{a} \tag{3.82}$$

with (the Infeld-van der Waerden matrices)

$$g^{A\dot{B}}{}_{1} := \begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix}, \quad g^{A\dot{B}}{}_{2} := \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix},$$
$$g^{A\dot{B}}{}_{3} := \begin{Vmatrix} 0 & 0 \\ 0 & -1 \end{Vmatrix}, \quad g^{A\dot{B}}{}_{4} := \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}. \tag{3.83}$$

Then

$$g_{A\dot{B}}{}^{a} := \varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}} g^{ab} g^{C\dot{D}}{}_{b}. \tag{3.84}$$

Now, the isomorphism $j\colon S^{1\dot{1}}(M) \to E(M)$ defined by (3.10) can be naturally extended on $S_{1\dot{1}}(M)$. Namely, in the obvious notation

$$j(e^{A\dot{B}}) := g^{A\dot{B}}{}_{a} e^{a} \Rightarrow j^{-1}(e^{a}) = -g_{A\dot{B}}{}^{a} e^{A\dot{B}}. \tag{3.85}$$

Notice that (3.10) can be written in the form of

$$j(e_{A\dot{B}}) = -g_{A\dot{B}}{}^{a} e_{a} \Rightarrow j^{-1}(e_{a}) = g^{A\dot{B}}{}_{a} e_{A\dot{B}}. \tag{3.86}$$

Using (3.85) and (3.86) we are in a position to define a bundle isomorphism

$$j^{-1}\colon T^{r}_{s}(M) \cong E^{r}_{s}(M) \to S^{r\dot{r}}_{s\dot{s}}(M). \tag{3.87}$$

Indeed, if p is any point of M and

$$T := T^{a_1\dots a_r}{}_{b_1\dots b_s} e_{a_1} \otimes \dots \otimes e_{a_r} \otimes e^{b_1} \otimes \dots \otimes e^{b_s} \in \Pi^{-1}_{E_s{}^r(M)}(p)$$

then

$$j^{-1}(T) := T^{a_1\dots a_r}{}_{b_1\dots b_s} j^{-1}(e_{a_1}) \otimes \dots \otimes j^{-1}(e_{a_r}) \otimes j^{-1}(e^{b_1}) \otimes \dots$$
$$\otimes\, j^{-1}(e^{b_s}) \in \Pi^{-1}_{S_{s\dot{s}}{}^{r\dot{r}}(M)}(p). \tag{3.88}$$

Thus

$$j^{-1}(T) = T^{A_1\dot{B}_1\ldots A_r\dot{B}_r}{}_{C_1\dot{D}_1\ldots C_s\dot{D}_s} e_{A_1\dot{B}_1} \otimes \ldots \otimes e_{A_r\dot{B}_r} \otimes e^{C_1\dot{D}_1} \otimes \ldots$$
$$\otimes\, e^{C_s\dot{D}_s} \in \Pi^{-1}_{S_{s\dot{s}}{}^{r\dot{r}}(M)}(p), \tag{3.89}$$

where

$$T^{A_1\dot{B}_1\ldots A_r\dot{B}_r}{}_{C_1\dot{D}_1\ldots C_s\dot{D}_s} = (-1)^s g^{A_1\dot{B}_1}{}_{a_1} \ldots g^{A_r\dot{B}_r}{}_{a_r} g_{C_1\dot{D}_1}{}^{b_1} \ldots g_{C_s\dot{D}_s}{}^{b_s} \cdot T^{a_1\ldots a_r}{}_{b_1\ldots b_r}. \tag{3.90}$$

The spinor arising on the right-hand side of (3.88) or (3.89) is called a spinor image of the tensor T. Thus, one can easily find that the covariant curvature tensor of (M, g) with the local components R_{abcd} possesses the spinor image belonging to $\mathscr{E}(S_{4\dot{4}}(M))$ with local components

$$R_{A\dot{B}C\dot{D}E\dot{F}G\dot{H}} = C_{ACEG}\varepsilon_{\dot{B}\dot{D}}\varepsilon_{\dot{F}\dot{H}} + C_{\dot{B}\dot{D}\dot{F}\dot{H}}\varepsilon_{AC}\varepsilon_{EG} - C_{AC\dot{F}\dot{H}}\varepsilon_{EG}\varepsilon_{\dot{B}\dot{D}} - C_{EG\dot{B}\dot{D}}\varepsilon_{AC}\varepsilon_{\dot{F}\dot{H}}$$
$$-\frac{R}{12}(\varepsilon_{AC}\varepsilon_{EG}\varepsilon_{\dot{B}\dot{H}}\varepsilon_{\dot{D}\dot{F}} + \varepsilon_{AE}\varepsilon_{CG}\varepsilon_{\dot{B}\dot{D}}\varepsilon_{\dot{F}\dot{H}}). \tag{3.91}$$

The spinor field on M with local components $C_{ACEG}\varepsilon_{\dot{B}\dot{D}}\varepsilon_{\dot{F}\dot{H}} + C_{\dot{B}\dot{D}\dot{F}\dot{H}}\varepsilon_{AC}\varepsilon_{EG}$ appears to be the spinor image of the covariant Weyl tensor of (M, g). The spinor fields on M with local components $C_{ACEG}\varepsilon_{\dot{B}\dot{D}}\varepsilon_{\dot{F}\dot{H}}$ or $C_{\dot{B}\dot{D}\dot{F}\dot{H}}\varepsilon_{AC}\varepsilon_{EG}$ are the spinor images of the self-dual part or anti-self-dual part, respectively, of the covariant Weyl tensor of (M, g). Thus the spinor fields on M with local components C_{ABCD} or $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ are called the undotted ($\equiv$ left) or dotted ($\equiv$ right), respectively, Weyl spinor of (M, g). The spinor field on M defined by $C_{AB\dot{C}\dot{D}}$ is called the traceless Ricci spinor of (M, g). (From the group theoretical point of view, the formula (3.91) defines the decomposition of $R_{A\dot{B}C\dot{D}E\dot{F}G\dot{H}}$ into irreducible objects: $D(2, 0)$, $D(0, 2)$, $D(1, 1)$ and $D(0, 0)$.)

Then, with (3.76)–(3.78), we find easily the following relations

$$\begin{aligned}
&\text{CR}: C_{ABCD}, C_{\dot{A}\dot{B}\dot{C}\dot{D}}, C_{AB\dot{C}\dot{D}}, R - \text{complex},\\
&\text{HR}: \overline{C_{ABCD}} = C_{\dot{A}\dot{B}\dot{C}\dot{D}}, \overline{C_{AB\dot{C}\dot{D}}} = C_{CD\dot{A}\dot{B}}, \bar{R} = R,\\
&\text{UR}: C_{ABCD}, C_{\dot{A}\dot{B}\dot{C}\dot{D}}, C_{AB\dot{C}\dot{D}}, R - \text{real},\\
&\text{ER}: \overline{C_{ABCD}} = C^{ABCD}, \overline{C_{\dot{A}\dot{B}\dot{C}\dot{D}}} = C^{\dot{A}\dot{B}\dot{C}\dot{D}}, \overline{C_{AB\dot{C}\dot{D}}} = C^{AB\dot{C}\dot{D}}, \bar{R} = R.
\end{aligned} \tag{3.92}$$

Now, the 2nd Bianchi's identity (3.62) can be written locally in the form of

$$\begin{aligned}
\nabla^E{}_{\dot{A}} C_{BCDE} + \nabla_{(B}{}^{\dot{E}} C_{CD)\dot{A}\dot{E}} &= 0,\\
\nabla_A{}^{\dot{E}} C_{\dot{B}\dot{C}\dot{D}\dot{E}} + \nabla^E{}_{(\dot{B}} C_{|AE|\dot{C}\dot{D})} &= 0,\\
\nabla^{C\dot{D}} C_{AC\dot{B}\dot{D}} + \tfrac{1}{8}\nabla_{A\dot{B}} R &= 0,
\end{aligned} \tag{3.93}$$

where $\nabla_{A\dot{B}}$ is defined by the formula

$$\tilde{D} = g^{A\dot{B}} \nabla_{A\dot{B}}. \tag{3.94}$$

Finally, for any spinor valued r-form on an open set $U \subset M$ one has the Ricci's identities

$$\tilde{D}\tilde{D}\phi^{A\ldots\dot{B}\ldots}_{C\ldots\dot{D}\ldots} = \Gamma^{A}{}_{E} \wedge \phi^{E\ldots\dot{B}\ldots}_{C\ldots\dot{D}\ldots} + \ldots + \Gamma^{\dot{B}}{}_{\dot{E}} \wedge \phi^{A\ldots\dot{E}\ldots}_{C\ldots\dot{D}\ldots}$$

$$+ \ldots - \Gamma^{E}{}_{C} \wedge \phi^{A\ldots\dot{B}\ldots}_{E\ldots\dot{D}\ldots} - \ldots - \Gamma^{\dot{E}}{}_{\dot{D}} \wedge \phi^{A\ldots\dot{B}\ldots}_{C\ldots\dot{E}\ldots} - \ldots . \tag{3.95}$$

Up to now we have dealt with the spinor formalism without spinor structure. However, if (M, g) admits an appropriate spinor structure, then one can define the spinor bundle $S^1(M)$ of type $((1, \dot{0}), (0, \dot{0}))$ over M, the spinor bundle $S^{\dot{1}}(M)$ of type $((0, \dot{1}), (\dot{0}, \dot{0}))$ over M, and the dual vector bundles $S_1(M)$ and $S_{\dot{1}}(M)$, respectively. Then the spinor bundles $S^{p\dot{q}}_{r\dot{s}}(M)$ appear to be the tensor products of above mentioned vector bundles. Moreover, $\Gamma^{A}{}_{B}$ or $\Gamma^{\dot{A}}{}_{\dot{B}}$ define the connections on $S^1(M)$ or $S^{\dot{1}}(M)$, respectively, and $\tilde{\mathscr{R}}_{+}$ or $\mathscr{R}_{-}$, respectively, are the curvature forms of these connections.

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