ANOMALIES IN QUANTUM MECHANICS*

By P. Kosiński

Institute of Physics, University of Łódź**

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The problem of anomalies in quantum mechanics is discussed. It is shown that they can be treated in a way completely analogous to the Fujikawa approach in field theory.

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It is sometimes very useful to study some interesting phenomena in field theory using simple models. Recently Elitzur et al. [1] have succeeded in inventing quantum mechanical models exhibiting global anomalies. Their paper shows how far can one proceed with simplifications still retaining some nontrivial properties of quantum theory. They produced two kinds of models: those having classical gauge invariance which is inherently broken on quantum level and others, in which the gauge symmetry can be restored at the expense of breaking some global symmetry.

Here I want to add some further remarks to show that the analogy with the field-theoretical situation can be pursued even further. Apart from any serious motivation it is done also for fun.

Let us take the simplest quantum-mechanical model one can imagine — the "theory" of one fermionic degree of freedom. This is nothing but the 2×2 matrix algebra in disguise. We will work in the coordinate representation. To this end let us introduce the Grassman variable ζ and its complex conjugate $\bar{\zeta}$. The wave function reads

$$\phi(\zeta) = a_0 + a_1\zeta, \qquad \overline{\phi(\zeta)} = \bar{a}_0 + \bar{a}_1\bar{\zeta},$$

with complex a_0 and a_1, which, in general may depend on other variables (see [2, 3]). Adopting the integration rules $\int d\zeta = \int d\bar{\zeta} = 0$, $\int \zeta d\zeta = \int \bar{\zeta} d\bar{\zeta} = 1$ one can write the following scalar product [4]

$$(\phi, \psi) \equiv \int \overline{\phi(\zeta)}\psi(\zeta)e^{-\zeta\bar{\zeta}}d\zeta d\bar{\zeta} = \bar{a}_0 b_0 + \bar{a}_1 b_1.$$

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** Address: Instytut Fizyki, Uniwersytet Łódzki, Nowotki 149/153, 90-236 Łódź, Poland.

The basic operators are ζ and $\dfrac{\partial}{\partial\zeta}$

$$\zeta\phi = \zeta(a_0+a_1\zeta) = a_0\zeta, \qquad \frac{\partial}{\partial\zeta}\phi = \frac{\partial}{\partial\zeta}(a_0+a_1\zeta) = a_1.$$

They fulfill the relations

$$\left\{\zeta, \frac{\partial}{\partial\zeta}\right\} = 1, \qquad \zeta^+ = \frac{\partial}{\partial\zeta}.$$

All other operators can be constructed out of ζ and $\dfrac{\partial}{\partial\zeta}$. The physically interesting ones are the fermion number operator F and Witten index operator $(-1)^F$

$$F = \frac{\partial}{\partial\zeta}\zeta, \qquad (-1)^F = \zeta\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\zeta}\zeta.$$

We shall also need in the sequel two other operators: the "charge conjugation"

$$C = \zeta + \frac{\partial}{\partial\zeta}, \qquad C^+ = C^{-1} = C$$

which fulfills

$$C\zeta C^+ = \frac{\partial}{\partial\zeta}, \qquad C\frac{\partial}{\partial\zeta}C^+ = \zeta,$$

and the operator realizing gauge transformations

$$U(\alpha) = e^{i\alpha\zeta\frac{\partial}{\partial\zeta}} = 1+(e^{i\alpha}-1)\zeta\frac{\partial}{\partial\zeta},$$

$$U(\alpha)\zeta U^+(\alpha) = e^{i\alpha}\zeta,$$

$$U(\alpha)\frac{\partial}{\partial\zeta}U^+(\alpha) = e^{-i\alpha}\frac{\partial}{\partial\zeta}, \qquad U(\alpha)\phi = a_0+\iota$$

Let us now couple our fermions to the external field. The lagrangian reads

$$L = \psi^+(t)\,(i\partial_t+A(t))\psi(t).$$

The theory has two formal symmetries [1]. First, there is a gauge invariance

$$\psi(t) \to e^{i\alpha(t)}\psi(t), \qquad \psi^+(t) \to \psi^+(t)e^{-i\alpha(t)}, \qquad A(t) \to A(t)+\partial_t\alpha(t).$$

There is also a discrete symmetry, charge conjugation,

$$\psi(t) \leftrightarrow \psi^+(t), \quad A(t) \to -A(t).$$

Elitzur et al. [1] have shown that the above symmetries cannot coexist on quantum level. This point will be discussed at length below.

The hamiltonian of the theory reads

$$H = -A(t)\,(\psi^+(t)\psi(t)+\beta).$$

Here β is an arbitrary real number connected with the problem of operator ordering. In the Schrödinger picture and coordinate representation one can take

$$H = -A(t)\left(\frac{\partial}{\partial\zeta}\zeta+\beta\right).$$

Solving the Schrödinger equation one gets

$$\phi(\zeta, t) = a_0 \exp\left(i(1+\beta)\int_0^t d\tau A(\tau)\right)+a_1\zeta \exp\left(i\beta\int_0^t d\tau A(\tau)\right). \tag{1}$$

As Elitzur et al. have noticed, if we demand the charge conjugation to be the symmetry operation, i.e.

$$CHC^+ = H(-A),$$

then β is uniquely fixed to $\beta = -\frac{1}{2}$. On the other hand the gauge symmetry is then broken, as it will be checked below.

In the Euclidean approach the lagrangian reads ($t \to it$, $A \to -iA$)

$$L = \eta(\partial_t - iA(t))\zeta. \tag{2}$$

Again the gauge transformations $\zeta \to e^{i\alpha(t)}\zeta$, $\eta \to e^{-i\alpha(t)}\eta$, $A(t) \to A(t)+\partial_t\alpha(t)$ and charge conjugation $\eta \leftrightarrow \zeta$, $A(t) \to -A(t)$ are the symmetries of the classical theory.

Let us denote by $K_A(\zeta, t|\zeta', 0)$ the propagation function $\phi(\zeta, t) = \int K_A(\zeta, t|\zeta', 0)\phi(\zeta')d\zeta'$. Then because of the fermion number conservation K_A takes the form

$$K_A(\zeta, t|\zeta', 0) = K_0(t)\zeta'+K_1(t)\zeta.$$

Another representation is

$$\phi(\zeta, t) = \int \tilde{K}_A(\zeta, t|\bar{\zeta}', 0)\phi(\zeta')e^{-\zeta'\bar{\zeta}'}d\zeta' d\bar{\zeta}',$$

with

$$\tilde{K}_A(\zeta, t|\bar{\zeta}', 0) = K_0(t)+K_1(t)\zeta\bar{\zeta}'.$$

The main point we want to emphasize is that calculating K_A we may proceed in two ways following the treatment given by Fujikawa [5].

(A) Non-gauge-theoretical formulation

Our starting point is the Euclidean lagrangian, Eq. (2). Following Refs. [5] and [6] we assume smooth continuation $iA \to A$ in Eq. (2) which takes the form

$$L = \eta(\partial_t - A(t))\zeta. \tag{3}$$

Note that such a continuation usually either changes the gauge group (for example from $U_L(N) \times U_R(N)$ to $GL(N, C)$ [6]) or breaks it. In our case the group changes from U(1) to $\boldsymbol{R}$:

$$\zeta \to e^{\alpha(t)}\zeta, \quad \eta \to \eta e^{-\alpha(t)}, \quad A(t) \to A(t) + \partial_t\alpha(t).$$

In Ref. [3] we have shown that the propagation kernel for the Euclidean lagrangian (3) can be obtained from the suitably defined path integral

$$\int_{\text{b.c.}} D\zeta D\eta \exp(\eta(-\partial_t + A(t))\zeta) = \zeta \exp(-\tfrac{1}{2}\int_0^t d\tau A(\tau)) + \zeta' \exp(\tfrac{1}{2}\int_0^t d\tau A(\tau)),$$

where b.c. denote the free boundary conditions for η (η is the momentum!) and $\zeta(0) = -\zeta'$, $\zeta(t) = \zeta$ [1]. Continuing back to the starting Euclidean lagrangian (2) and then to "Minkowski space" we obtain

$$K_A(\zeta, t|\zeta', 0) = \zeta \exp\left(-\frac{i}{2}\int_0^t d\tau A(\tau)\right) + \zeta' \exp\left(\frac{i}{2}\int_0^t d\tau A(\tau)\right),$$

or

$$\tilde{K}_A(\zeta, t|\bar{\zeta}', 0) = \zeta\bar{\zeta}' \exp\left(-\frac{i}{2}\int_0^t d\tau A(\tau)\right) + \exp\left(\frac{i}{2}\int_0^t d\tau A(\tau)\right).$$

Comparing with Eq. (1) we conclude that it corresponds to the choice $\beta = -\frac{1}{2}$, i.e. the theory is charge conjugation invariant

$$C^{+'}\tilde{K}_A C = \tilde{K}_{A^c}, \quad A^c = -A.$$

On the other hand local gauge symmetry is lost; it is easy to check that

$$U^+(\alpha(t))\tilde{K}_A(t)U(\alpha(0)) \neq \tilde{K}_{A^\alpha}(t),$$

with $A^\alpha(t) = A(t) + \partial_t\alpha(t)$, $\alpha(0) \neq \alpha(t)$. Note that passing from the partition function of Ref. [1] to the propagator we exhibit the gauge symmetry breaking even for "perturbative" gauge transformations. The "chiral charge" is, however, conserved

$$(\psi(t), Q\phi(t)) \equiv \left(\psi(t), \zeta\frac{\partial}{\partial\zeta}\phi(t)\right) = \text{const.}$$

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[1] The choice $\zeta(0) = \zeta'$, made in Ref. [3], differs from the one made above because of different rule of Grassman integration.

because it generates the gauge transformations of the first kind which are still the symmetries of the theory.

Taking the trace of the propagator K_A corresponding to the Euclidean lagrangian (2) we may calculate the partition function

$$Z = \mathrm{Tr}\, \tilde{K}_A = \sum_{i=1}^{2} (\phi_i, \tilde{K}_A \phi_i) = 2\cos\left(\tfrac{1}{2}\int_0^t d\tau A(\tau)\right), \quad \phi_1 = 1, \quad \phi_2 = \zeta$$

in agreement with Ref. [1].

(B) Gauge-theoretical approach

Again we start with the original euclidean lagrangian (2) and define the integral

$$\int_{\mathrm{b.c.}} D\zeta D\eta \exp\left(\eta(-\partial_t + iA(t))\zeta\right)$$

proceeding exactly along the lines of Ref. [3]. Denote

$$D = -\partial_t + iA(t)$$

and let $-D^2 = D^+ D$ be the positive definite self-adjoint operator on $(0, t)$ defined by the boundary conditions

$$\varphi(0) = 0, \quad \varphi(t) = 0.$$

Denote also by φ_n, and λ_n, $n = 1, 2, \ldots$, the eigenfunctions and eigenvalues of $-D^2$, respectively. Then $\{\lambda_n^{-\frac{1}{2}} D\varphi_n\}$ is also an orthonormal set. We add one function φ_0 such that

$$D\varphi_0 = 0, \quad (\varphi_0, \varphi_0) \equiv \int_0^t d\tau |\varphi_0(\tau)|^2 = 1,$$

i.e.

$$\varphi_0(t') = t^{-\frac{1}{2}} \int \exp\left(i \int_0^{t'} d\tau A(\tau)\right)$$

to form an orthonormal complete set $\{\varphi_0, \{\lambda_n^{-\frac{1}{2}} D\varphi_n\}_1^\infty\}$. Now we may expand

$$\eta(\tau) = \eta_0 \varphi_0(\tau) + \sum_{n=1}^{\infty} \lambda_n^{-\frac{1}{2}} D\varphi_n(\tau)\eta_n,$$

with Grassman variables η_n. Similarly we put

$$\zeta(\tau) = \zeta_c(\tau) + \sum_{k=1}^{\infty} \varphi_k(\tau)\zeta_k,$$

with $\zeta_c(\tau)$ obeying

$$D^2\zeta_c(\tau) = 0, \quad \zeta_c(t) = \zeta, \quad \zeta_c(0) = -\zeta'.$$

Our integral becomes

$$\int_{\mathrm{b.c.}} D\zeta D\eta \exp \eta D\zeta = N^{-1}(t)\,(\varphi_0, D\zeta_c)\,\mathrm{Det}^{\frac{1}{2}}(-D^2).$$

Now

$$(\varphi_0, D\zeta_c) = \int_0^t d\tau \bar{\varphi}_0(\tau) D\zeta_c(\tau) = t^{-\frac{1}{2}}(\zeta' + \zeta e^{-i\int_0^t d\tau A(\tau)})$$

and using Coleman theorem [7, 3] we check that $\mathrm{Det}^{\frac{1}{2}}(-D^2)t^{-\frac{1}{2}}$ does not depend on $A(t)$ and therefore we can put

$$N(t) = t^{-\frac{1}{2}}\,\mathrm{Det}^{\frac{1}{2}}(-D^2).$$

The only point one has to take care of is that $-D^2$ is not of the form $-\partial_\tau^2 + f(\tau)$, in order to apply Coleman theorem the first-order derivative term must be eliminated. Then we arrive at the operator $-\partial_\tau^2$ and Coleman theorem is trivial in this case.

Thus our final expression for the propagator reads (after going back to Minkowski space)

$$K_A(\zeta, t|\zeta', 0) = \zeta' + \zeta \exp\left(-i \int_0^t d\tau A(\tau)\right). \tag{4}$$

Comparing Eqs (4) and (1) we conclude that our propagator corresponds to the choice $\beta = -1$. Obviously the charge conjugation is no longer a symmetry

$$C^+ \tilde{K}_A C \neq \tilde{K}_{A^c}.$$

On the other hand theory is gauge invariant

$$U^+(\alpha(t))\tilde{K}_A(t)U(\alpha(0)) \neq \tilde{K}_{A^\alpha}(t)$$

for any $\alpha(t)$. It is to be expected because we maintained the gauge invariance at each stage of derivation. We can also compute the partition function

$$Z = 1 + \exp i \int_0^t d\tau A(\tau).$$

Let us note that contrary to the case of global gauge symmetries [1] the choice $\beta = -1$ is here unique.

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