NONMINIMAL DESCRIPTION OF SPIN $\frac{3}{2}$*

By W. Tybor

Institute of Physics, University of Łódź**

(Received March 14, 1988)

The nonminimal description (with the help of the antisymmetric tensor-bispinor) of the spin $\frac{3}{2}$, equivalent to the Rarita-Schwinger theory, is given. The variational principle is formulated.

PACS numbers: 11.10.Ef, 11.10.Qr, 11.15.-q

1. Introduction

In the previous papers [1-3] we have discussed the nonminimal description of the boson field (with spin 2). Now we go to the fermion case (with the spin $\frac{3}{2}$ as an example).

To describe, in economical way [1], a massive fermion with spin $\frac{3}{2}$ (and definite parity) we use Lorentz spin tensors carrying the maximal spin $s_{\max} = \frac{3}{2}$. In this case, the highest representations being contained in such spin tensors are $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$ and $(\frac{3}{2}, 0) \oplus (0, \frac{3}{2})$. The theory based on the representation $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$ is the well known theory of Rarita and Schwinger (the spin vector ψ^{μ} is a field variable). We will refer to this description as to the minimal one. The description using the spin tensor $\phi^{\mu\nu} = -\phi^{\nu\mu}$ including the highest representation $(\frac{3}{2}, 0) \oplus (0, \frac{3}{2})$ we call the nonminimal one.

In the present paper we discuss the nonminimal description of the spin $\frac{3}{2}$, equivalent to the Rarita-Schwinger theory. We assume that the field equation in the nonminimal formulation is, as in the Rarita-Schwinger case, of the first order. In Section 2 it is shown that the nonminimal description is possible, but a combination $(\frac{3}{2}, 0) \oplus (0, \frac{3}{2}) \oplus (1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$, as the highest representation, must be used. The admixture of the $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$ representation is necessary, since there exists no first order equation based on the $(\frac{3}{2}, 0) + (0, \frac{3}{2})$ representation only [4]. In Section 3 the variational principle is formulated.

* Supported by CPBP 01.03.

** Address: Instytut Fizyki, Uniwersytet Łódzki, Nowotki 149/153, 90-236 Łódź, Poland.

2. *The field equations*

Let us start with the Rarita–Schwinger equation[1] for the spin vector ψ^μ transforming under the Lorentz group as $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1) \oplus (0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ representation

$$\varepsilon_{\alpha\beta\lambda\kappa}\gamma_5\gamma^\kappa\partial^\beta\psi^\lambda - m(\psi_\alpha - \gamma_\alpha\gamma^\beta\psi_\beta) = 0. \tag{2.1}$$

From Eq. (2.1) the supplementary conditions result

$$\gamma_\mu\psi^\mu = 0, \tag{2.2}$$

$$\partial_\mu\psi^\mu = 0. \tag{2.3}$$

Inserting these conditions to Eq. (2.1) we get

$$(i\gamma\cdot\partial - m)\psi^\mu = 0 \quad (\text{or } (\Box + m^2)\psi^\mu = 0).$$

So, the field ψ^μ has the mass m. The supplementary conditions restrict the number of spin variables. Indeed, in the momentum space in the rest system ($p = (m, 0, 0, 0)$) the spin vector $\psi^\mu(\vec{p})$ has only $2\cdot 4$ components: $\vec{\psi} + \frac{1}{3}\vec{\gamma}(\vec{\gamma}\vec{\psi})$. So, the field ψ^μ, obeying Eq. (2.1), carries the spin $\frac{3}{2}$.

To introduce the nonminimal description of the spin $\frac{3}{2}$ with the help of the spin tensor we rewrite the Rarita–Schwinger equation in the form of a set of two equations. It can be done (in alternative way) in two manners:

a)

$$\tfrac{1}{2}\varepsilon_{\alpha\beta\lambda\kappa}\gamma_5\gamma^\kappa\phi^{\beta\lambda} - (g_{\alpha\lambda} - \gamma_\alpha\gamma_\lambda)\psi^\lambda = 0, \tag{2.4a}$$

$$m\phi^{\beta\lambda} = \partial^\beta\psi^\lambda - \partial^\lambda\psi^\beta; \tag{2.4b}$$

b)

$$\partial^\beta\psi_{\beta\alpha} + m(g_{\alpha\lambda} - \gamma_\alpha\gamma_\lambda)\psi^\lambda = 0, \tag{2.5a}$$

$$\psi_{\beta\alpha} = \varepsilon_{\lambda\beta\alpha\kappa}\gamma_5\gamma^\kappa\psi^\lambda; \tag{2.5b}$$

where $\phi^{\beta\lambda}$ and $\psi^{\beta\lambda}$ are antysimmetric spin tensors: $\phi^{\beta\lambda} = -\phi^{\lambda\beta}$, $\psi^{\beta\lambda} = -\psi^{\lambda\beta}$. The set (2.4) is unique up to the point transformation

$$\phi^{\alpha\beta} \to \phi^{\alpha\beta} + A(\gamma^\alpha\gamma_\sigma\phi^{\sigma\beta} - \gamma^\beta\gamma_\sigma\phi^{\sigma\alpha}) + B\sigma^{\alpha\beta}\sigma\cdot\phi,$$

where $\sigma\cdot\phi \equiv \sigma_{\mu\nu}\phi^{\mu\nu}$ and $(1+2A)(1+6A+12B) \neq 0$, and to the scaling

$$\phi^{\alpha\beta} \to \lambda\phi^{\alpha\beta}.$$

The same is valid for the set (2.5).

Let us discuss the set (2.4). Excluding ψ^α, we get the equation for the field $\phi^{\alpha\beta}$:

$$\tfrac{1}{6}(\partial^\alpha\gamma^\beta - \partial^\beta\gamma^\alpha)\sigma\cdot\phi + i(\partial^\alpha\gamma_\sigma\phi^{\sigma\beta} - \partial^\beta\gamma_\sigma\phi^{\sigma\alpha}) = m\phi^{\alpha\beta}. \tag{2.6}$$

---

[1] See, for example, the paper [5], where the full analysis of the first order equation for a spin vector ψ^μ is given.

From Eq. (2.6) we obtain the supplementary conditions

$$\sigma \cdot \phi = 0, \tag{2.7}$$

$$\varepsilon_{\mu\nu\alpha\beta}\partial^{\nu}\phi^{\alpha\beta} = 0. \tag{2.8}$$

Taking into account these conditions one gets from Eq. (2.6)

$$(i\gamma \cdot \partial - m)\phi^{\alpha\beta} = 0.$$

In the momentum space, in the rest system, the nonvanishing components of $\phi^{\alpha\beta}$ are ϕ^{0i}, $\gamma_i\phi^{0i} = 0$. So, Eq. (2.6) describes the spin $\frac{3}{2}$. Using the decomposition (A.1) (see Appendix) we conclude that $W^{0i} = E^{0i}$, $\gamma_i E^{0i} = \gamma_i W^{0i} = 0$. So, the representation $(\frac{3}{2}, 0) \oplus (0, \frac{3}{2})$ can be used as the highest one only in the combination with $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$. We note that this situation is similar to the case of the spin 2, where description with the help of the 4-th rank tensor is possible if we accept an admixture of (1, 1) representation to the highest one (2, 0)+(0, 2) [1].

Let us discuss the set (2.5). From Eq. (2.5b) we get

$$\psi^{\lambda} = \tfrac{1}{6}\gamma^{\lambda}(\sigma \cdot \psi) + \frac{i}{2}\gamma_{\beta}\psi^{\beta\lambda} \tag{2.9}$$

and

$$\psi^{\alpha\beta} = \tfrac{1}{2}(\gamma^{\alpha}\gamma_{\sigma}\psi^{\sigma\beta} - \gamma^{\beta}\gamma_{\sigma}\psi^{\sigma\alpha}) - \tfrac{1}{6}\sigma^{\alpha\beta}(\sigma \cdot \psi). \tag{2.10}$$

So, the highest representation is $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$. Inserting Eq. (2.9) to Eq. (2.5a) we get the equation for the field $\psi^{\alpha\beta}$

$$2i\partial^{\beta}\psi_{\beta\alpha} - m\gamma^{\beta}\psi_{\beta\alpha} = 0. \tag{2.11}$$

Eq. (2.11) gives the following supplementary conditions

$$\sigma \cdot \psi = 0, \tag{2.12}$$

$$\partial^{\alpha}\gamma^{\beta}\psi_{\alpha\beta} = 0. \tag{2.13}$$

Using Eqs. (2.10–13) we get from Eq. (2.11)

$$(i\gamma \cdot \partial - m)\gamma_{\sigma}\psi^{\sigma\alpha} = 0 \text{ (and } (\Box + m^2)\psi^{\alpha\beta} = 0).$$

We see that the field variable is actually the spin vector $\chi^{\alpha} \equiv i\gamma_{\sigma}\psi^{\sigma\alpha}$. Using Eqs. (2.10) and (2.11) we obtain the equation for χ^{α}:

$$i(\gamma \cdot \partial)\chi^{\alpha} - i\gamma^{\alpha}\partial_{\beta}\chi^{\beta} - \frac{i}{3}(\gamma \cdot \partial)\gamma^{\alpha}(\gamma \cdot \chi) + \frac{i}{3}\partial^{\alpha}(\gamma \cdot \chi) = m\chi^{\alpha}.$$

It turns into the Rarita-Schwinger equation after the point transformation $\chi^{\alpha} = \psi^{\alpha} - \gamma^{\alpha}(\gamma \cdot \psi)$.

3. *The variational principle*

We start with the action

$$I = \int dx[\tfrac{1}{2}\bar{\psi}_{\beta\lambda}\partial^{\beta}\psi^{\lambda}+\tfrac{1}{2}\partial^{\beta}\bar{\psi}^{\lambda}\psi_{\beta\lambda}-m(\bar{\psi}_{\alpha}\psi^{\alpha}-\bar{\psi}_{\alpha}\gamma^{\alpha}\gamma_{\beta}\psi^{\beta})$$
$$-\tfrac{1}{4}m(\bar{\psi}_{\beta\lambda}-\bar{\psi}^{\alpha}\varepsilon_{\alpha\beta\lambda\kappa}\gamma_5\gamma^{\kappa})\phi^{\beta\lambda}-\tfrac{1}{4}m\bar{\phi}_{\beta\lambda}(\psi^{\beta\lambda}-\varepsilon^{\alpha\beta\lambda\kappa}\gamma_5\gamma_{\kappa}\psi_{\alpha})]. \tag{3.1}$$

From $\delta I = 0$ we obtain the set of the equations

$$\partial^{\beta}\psi^{\lambda}-\partial^{\lambda}\psi^{\beta} = m\phi^{\beta\lambda}, \tag{3.2}$$

$$\psi^{\beta\lambda} = \varepsilon^{\alpha\beta\lambda\kappa}\gamma_5\gamma_{\kappa}\psi_{\alpha}, \tag{3.3}$$

$$-\tfrac{1}{2}\partial^{\beta}\psi_{\beta\lambda}+\tfrac{1}{4}m\varepsilon_{\lambda\beta\alpha\kappa}\gamma_5\gamma^{\kappa}\phi^{\beta\alpha}-m(\psi_{\lambda}-\gamma_{\lambda}\gamma^{\sigma}\psi_{\sigma}) = 0, \tag{3.4}$$

and the one of the Dirac conjugated equations. From these equations we obtain immediately the sets (2.4) and (2.5).

We observe that the fields $\psi_{\beta\lambda}$, $\bar{\psi}_{\beta\lambda}$, $\phi_{\beta\lambda}$ and $\bar{\phi}_{\beta\lambda}$ in the action (3.1) are Lagrange multipliers and they can be eliminated from the action. With the help of Eqs. (3.2) and (3.3) we get

$$I = \int dx[\tfrac{1}{2}\bar{\psi}^{\alpha}\varepsilon_{\alpha\beta\lambda\kappa}\gamma_5\gamma^{\kappa}\partial^{\beta}\psi^{\lambda}+\tfrac{1}{2}\partial^{\beta}\bar{\psi}^{\lambda}\varepsilon_{\alpha\beta\lambda\kappa}\gamma_5\gamma^{\kappa}\psi^{\alpha}-m(\bar{\psi}_{\alpha}\psi^{\alpha}-\bar{\psi}_{\alpha}\gamma^{\alpha}\gamma_{\beta}\psi^{\beta})]$$

what is the symmetric form of the Rarita-Schwinger action.

Performing integration by parts in the action (3.1) we convert ψ^{α}, $\bar{\psi}^{\alpha}$ into Lagrange multipliers that can be removed using Eq. (3.3). So, we obtain the action in terms of $\psi^{\alpha\beta}$ and $\phi^{\alpha\beta}$ fields:

$$I = \int dx\{-\tfrac{1}{4}\partial^{\beta}\bar{\psi}_{\beta\lambda}[\tfrac{1}{3}\gamma^{\lambda}(\sigma\cdot\psi)+i\gamma_{\sigma}\psi^{\sigma\lambda}]$$
$$-\tfrac{1}{4}[\tfrac{1}{3}(\bar{\psi}\cdot\sigma)\gamma^{\lambda}+i\bar{\psi}^{\lambda\sigma}\gamma_{\sigma}]\partial^{\beta}\psi_{\beta\lambda}$$
$$+\tfrac{1}{4}m[\bar{\psi}^{\lambda\mu}\gamma_{\mu}\gamma^{\nu}\psi_{\beta\lambda}+\tfrac{1}{3}(\bar{\psi}\cdot\sigma)(\sigma\cdot\psi)] \tag{3.5}$$
$$-\tfrac{1}{4}m[\bar{\psi}^{\beta\lambda}\phi_{\beta\lambda}+\bar{\psi}^{\beta\lambda}\gamma_{\lambda}\gamma^{\kappa}\phi_{\kappa\beta}+\tfrac{1}{6}(\bar{\psi}\cdot\sigma)(\sigma\cdot\phi)]$$
$$-\tfrac{1}{4}m[\bar{\phi}^{\beta\lambda}\psi_{\beta\lambda}+\bar{\phi}^{\beta\lambda}\gamma_{\lambda}\gamma^{\kappa}\psi_{\kappa\beta}+\tfrac{1}{6}(\bar{\phi}\sigma)(\sigma\cdot\psi)]\}.$$

From this action we obtain the relation (2.10) and the system of the equations, from which the relations

$$\gamma_{\alpha}\psi^{\alpha\beta} = 2\gamma_{\alpha}\phi^{\alpha\beta}-i\gamma^{\beta}(\sigma\cdot\phi) \tag{3.6}$$

and

$$\sigma\cdot\psi = -2\sigma\cdot\phi \tag{3.7}$$

result. With the help of these relations one can reduce the system to two equations: (2.6) and (2.11). We note that due to Eqs. (3.6) and (3.7) the action (3.5) describes only one spin $\frac{3}{2}$.

Eliminating the Lagrange multipliers $\phi^{\alpha\beta}$ and $\bar{\phi}^{\alpha\beta}$ from the action (3.5) we obtain the description in terms of $\psi^{\alpha\beta}$ only. Putting $\psi_{\alpha\beta} = \varepsilon_{\lambda\alpha\beta\kappa}\gamma_5\gamma^\kappa\psi^\lambda$ we get the Rarita-Schwinger theory.

We finish with the conclusion that the field $\phi^{\alpha\beta}$ is not an independent variable. There exists no variational principle giving Eq. (2.6) only.

4. *Final remarks*

We have obtained the nonminimal description equivalent to the minimal one of Rarita and Schwinger. It is well known that theories equivalent for $m \neq 0$ need not to be equivalent in the $m = 0$ limit. The analysis of the zero mass limit of the nonminimal description obtained in the present paper will be given elsewhere.

I am very grateful to Professor J. Rembieliński and Drs S. Giler and P. Kosiński for discussions and remarks.

APPENDIX

The decomposition of the spin tensor $\phi^{\alpha\beta} = -\phi^{\beta\alpha}$ into the irreducible Lorentz parts (with determined parity)

$$[(\tfrac{3}{2}, 0) \oplus (0, \tfrac{3}{2})] \oplus [(1, \tfrac{1}{2}) \oplus (\tfrac{1}{2}, 1)] \oplus [(\tfrac{1}{2}, 0) + (0, \tfrac{1}{2})]$$

is

$$\phi^{\alpha\beta} = W^{\alpha\beta} + E^{\alpha\beta} + G^{\alpha\beta}, \tag{A1}$$

where

$$W^{\alpha\beta} = \phi^{\alpha\beta} - \tfrac{1}{2}(\gamma^\alpha\gamma_\sigma\phi^{\sigma\beta} - \gamma^\beta\gamma_\sigma\phi^{\sigma\alpha}) + \tfrac{1}{6}\sigma^{\alpha\beta}(\sigma\cdot\phi),$$

$$E^{\alpha\beta} = \tfrac{1}{2}(\gamma^\alpha\gamma_\sigma\phi^{\sigma\beta} - \gamma^\beta\gamma_\sigma\phi^{\sigma\alpha}) - \tfrac{1}{4}\sigma^{\alpha\beta}(\sigma\cdot\phi),$$

$$G^{\alpha\beta} = \tfrac{1}{12}\sigma^{\alpha\beta}(\sigma\cdot\phi).$$

The irreducible parts obey: $\gamma_\alpha W^{\alpha\beta} = 0$, $\sigma\cdot E = 0$. The dual properties of these parts are

$$\tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}W^{\alpha\beta} = -i\gamma_5 W_{\mu\nu},$$

$$\tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}E^{\alpha\beta} = i\gamma_5 E_{\mu\nu},$$

$$\tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}G^{\alpha\beta} = -i\gamma_5 G_{\mu\nu}.$$

REFERENCES

[1] W. Tybor, *Acta Phys. Pol.* **B17**, 887 (1986).
[2] W. Tybor, *Acta Phys. Pol.* **B18**, 69 (1987).
[3] W. Tybor, *Acta Phys. Pol.* **B18**, 369 (1987).
[4] S. Weinberg, *Phys. Rev.* **133B**, 1318 (1964).
[5] H. L. Baisya, *Nucl. Phys.* **B29**, 104 (1971).