CHARGED PARTICLE RADIATION ALONG A FINITE TRAJECTORY IN A MEDIUM

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The expressions for the energy emitted by a charged particle moving along a straight line finite trajectory in a transparent medium have been analysed. It has been shown that the dependence of the irradiated energy on the particle velocity lacks that peculiarity, which may be treated as a threshold. A possibility of dividing the radiation into two parts caused by different mechanisms of the particle-medium interaction has been considered.

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1. Introduction

The theory of Vavilov-Cerenkov radiation (VCR) was first formulated for the case of an infinite straight-line uniform motion of a charged particle in a medium without boundaries [1]. Then the irradiated energy can be calculated from the formula:

$$W = \frac{e^2L}{c^2} \int\limits_{\beta n > 1} \omega d\omega \left(1 - \frac{1}{\beta^2 n^2}\right) \tag{1}$$

where e is the electron charge, c — the velocity of light in vacuum, βn — the particle velocity, ω — the emitted light frequency. To avoid the infinity in (1) the authors of this theory had to divide the energy W by the trajectory length L.

The authors have noticed that the same result can be obtained, if one performs calculation for the particle moving along a finite trajectory. The results of this approach were published[1] in [2]. But in this case the formula (1) describes only a part of radiation emitted by the particle and it is valid only if the trajectory length is much longer than the emitted wavelength. This condition becomes the more rigorous (i.e. L becomes infinite) the nearer is the velocity of the particle to the light phase velocity.

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[1] According to Prof. I. M. Frank [3] he used that method several times.

Thus the widely known properties of VCR refer, in fact, to the particular case of an infinite particle trajectory. One of these properties is the VCR threshold, i.e. the phenomenon, which occurs when the velocity of the particle exceeds the phase velocity of light in the medium the particle is moving in. As it is easily seen from formula (1) at

$$\beta = \frac{1}{n}. \tag{2}$$

$W = 0$ and the particle with the velocity $\beta < 1/n$ should not radiate at all.

The condition (2) may be obtained directly from the expression describing the characteristic angle of VCR:

$$\cos\theta = \frac{1}{\beta n}. \tag{3}$$

All these simple considerations (1–3) seem to be convincing, but one should not forget that they were derived from Maxwell equations for the charged particle moving infinitely in the medium. Based on the expressions (1–3) there were constructed many different types of Cerenkov detectors. Their successful operation made it a strong belief that the above theory was applicable for any real detector.

The first report on the experimental test of the threshold condition (2) was very short [4]. There was no information about the thickness of the mica radiator it was performed with. Probably no attention was paid to a connection between the thickness of the radiator and the threshold. It should be noted here that the possibility of a deviation from a theoretical description (3) due to a finite thickness of the radiator was mentioned [5] many years before the first Cerenkov detector had been used. Unfortunately, it was left without attention. The problem was considered again in [6] and a satisfactory discussion was given in [7]. Under the experimental conditions of [6] and [7] the VCR does not have a threshold (like a δ-function), so the expression (2) determines the respective velocity β only approximately. There exists quite definite angular distribution of the radiation. This feature has been known since 1939 (see Ref. [2]), but the authors of Ref. [7] showed that the main maximum will disappear when

$$\beta = \frac{1}{n + \lambda/L}, \tag{4}$$

where λ is the wavelength of emitted light in vacuum. Therefore, the threshold velocity is smaller than that obtained from (2).

Paper [7] refers to experiments with a target of the thickness of an order of several wavelength where the above formula was confirmed. In practice Cerenkov detectors are much longer, but even in a 1.5 m long detector the radiation has been observed undre threshold (in the sense of (2)) and this under threshold radiation had to be taken into account to determine particle energy more accurately in an experiment [8]. In Refs. [9–12] there was investigated the Cerenkov gas detector yield as a function of the pressure inside the detector (up to several dozens of milimeters long) and no peculiarities were found which could be interpreted as a threshold.

The problem of the VCR threshold existence is interesting not only from an empirical point of view. A particular mechanism of the interaction with the medium may exist when the charged particle velocity is greater than the phase velocity of light. The purpose of this work is the detailed study of the VCR energetic yield in the case of a finite particle trajectory. To avoid referring too often to [2], the main formulas and their interpretation given by I. E. Tamm will be quoted below.

2. *Tamm's results*

Let a charged particle move in a homogeneous transparent medium with a velocity v in the time interval $(-t_0, t_0)$ and stay at rest outside this interval. Then the density of the current may be written as:

$$j_z = ev\delta(x)\delta(y)\delta(z-vt) \quad \text{for} \quad -vt_0 < z < vt_0$$

$$j_z = 0 \quad \text{for} \quad |z| > vt_0 \tag{5}$$

and the Fourier integral expansion is as follows:

$$j_z(\omega) = \frac{e}{2\pi}\delta(x)\delta(y)e^{-\frac{i\omega z}{v}} \quad \text{for} \quad |z| < vt_0 \tag{6}$$

$$j_z(\omega) = 0 \quad \text{for} \quad |z| > vt_0.$$

The retarded potential, describing the radiation field, has the form:

$$A_\omega(x, y, z) = \frac{1}{c}\int \frac{j_\omega(x', y', z')}{R} e^{-\frac{i\omega nR}{c}} dx'dy'dz', \tag{7}$$

where

$$R = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}.$$

Now one can calculate the electric and magnetic fields and, after integrating over time interval, the Poynting vector, and obtain the following formula for the emitted energy:

$$W = \frac{2e^2\beta^2 n}{\pi c}\int_0^\infty J(\omega)d\omega, \tag{8}$$

where

$$J(\omega) = \int_0^\pi \frac{\sin^2[\omega t_0(1-\beta n\cos\theta)]}{(1-\beta n\cos\theta)^2}\sin^3\theta d\theta. \tag{9}$$

Having integrated (9) under the condition:

$$\omega t_0 \gg 1 \tag{10}$$

and neglecting fast oscillating terms such as $\sin[\omega t_0(1 \pm \beta n)]$ one can write:

$$J = J_1 = \frac{1}{\beta^3 n^3}\left(\ln\frac{1+\beta n}{|1-\beta n|} - 2\beta n\right) \quad \text{for} \quad \beta n < 1 \tag{11}$$

$$J = J_1 + \frac{\pi\omega t_0}{\beta n}\left(1 - \frac{1}{\beta^2 n^2}\right) \quad \text{for} \quad \beta n > 1. \tag{12}$$

It is clear that if $\beta n \to 1$, the condition (10) is not sufficient and the formulas (11) and (12) cannot be employed. The more rigorous condition is:

$$\omega t_0 |1-\beta n| \gg 1. \tag{13}$$

The expression (13) excludes the area where the function $J(\beta)$ reveals discontinuity, while from Eq. (9) one can directly derive the simple formula for $\beta n = 1$:

$$J = \ln(4\gamma\omega t_0) - 1, \tag{14}$$

where $\gamma = 1.781$. The VCR was singled out by Tamm from the expression (12) (i.e. for $\beta n > 1$) as $J - J_1$. This difference together with (8) results in the formula (1). The remaining part of the expression (12) is connected (according to Tamm) with an instantaneous change of particle velocity from 0 to v at the moments $\pm t_0$.

Thus, the formula (1) has been obtained in another way. Further on we shall consider only the peculiarities due to a finite trajectory of the particle.

3. Complete expressions for the yield of irradiated energy

When integrating the initial formula (9) with respect to a new variable

$$x = \omega t_0(1 - \beta n \cos\theta) \tag{15}$$

one can obtain the following sum of three integrals:

$$J_\omega = \left(1 - \frac{1}{\beta^2 n^2}\right)\frac{\omega t_0}{\beta n}\int_{x_1}^{x_2}\frac{\sin^2 x}{x^2}dx + \frac{1}{\beta^3 n^3}\int_{x_1}^{x_2}\frac{\sin^2 x}{x}dx - \frac{1}{\beta^3 n^3 \omega t_0}\int_{x_1}^{x_2}\sin^2 x dx, \tag{16}$$

where

$$x_1 = \omega t_0(1 - \beta n), \quad x_2 = \omega t_0(1 + \beta n).$$

The last integral may be calculated immediately:

$$\int_{x_1}^{x_2}\sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4}\bigg|_{x_1}^{x_2}. \tag{17}$$

The second one was, in Ref. [2], expressed by the sum of the cosine integral and a logarithm, but it leads, as it will be shown below, to an infinity of the function $J(\beta)$ for $\beta = 1/n$. That integral is commonly denoted by $S1$ and its values are tabulated, e.g. in [13].

The first integral in the sum (16) may be expressed by the sine integral and elementary functions:

$$\int_{x_1}^{x_2} \frac{\sin^2 x}{x^2}\, dx = Si(2x) + \frac{\cos 2x}{2x} - \frac{1}{2x}\Bigg|_{x_1}^{x_2}. \tag{18}$$

Then, the complete formula for the energy irradiated by the particle with a frequency ω has the form:

$$W = \frac{2e^2\beta^2 n}{\pi c} J_\omega \tag{19}$$

$$\begin{aligned} J_\omega = {} & \left(1 - \frac{1}{\beta^2 n^2}\right)\frac{\omega t_0}{\beta n}\{Si[2\omega t_0(1+\beta n) - Si[2\omega t_0(1-\beta n)]\} \\ & + \frac{1}{\beta^3 n^3}\{S1[\omega t_0(1+\beta n)] - S1[\omega t_0(1-\beta n)]\} \\ & - \frac{1}{4\omega t_0 \beta^3 n^3}\{\sin[2\omega t_0(1+\beta n)] - \sin[2\omega t_0(1-\beta n)]\} \\ & + \frac{\beta n - 1}{2\beta^3 n^3}\cos[2\omega t_0(1+\beta n)] + \frac{\beta n + 1}{2\beta^3 n^3}\cos[2\omega t_0(1-\beta n)] - \frac{2}{\beta^2 n^2}. \end{aligned} \tag{20}$$

It is easily seen that the terms with sine integral at greater values of the argument and under the condition that $\beta n > 1$, tend to π and become identical to the formula (1). For this reason they were identified by Tamm with VCR. It is important to note that these terms do not turn into zero for $\beta n < 1$ and, at the same time, do not occur in the formula (11). On the other hand, there is no logarithmic term in the expression (20), which is responsible for discontinuity of the function $J(\beta)$. Furthermore, some terms vanish at $\beta n = 1$ and one can rewrite the complete formula in a simpler form:

$$J = S1(2\omega t_0) + \frac{\sin(4\omega t_0)}{4t\omega_0} - 1. \tag{21}$$

4. *Example*

All the terms in the formula (20) remain real for any particle velocity. To realize what is the contribution of each term into the radiation, the particular case reported in [7] will be analysed below with parameters shown in Fig. 1.

In Table I it is shown to what degree the condition (13) is satisfied for the above case.

TABLE I

E keV	$\omega t_0(1-\beta n)$
40	10.6479
140	0.3313
149	0.0004
160	0.3609
200	1.3811

The results are shown in Fig. 1. The energy of the radiation is given in eV per cm per electron, since the calculation is performed for the unit interval $d\lambda$. As it is seen in Fig. 1 the sum of the terms containing sine integral (denoted as Si) oscillates for $\beta n < 1$, vanishes at $\beta n = 1$ and increases practically linearly with energy for $\beta n > 1$. The sum of the terms containing the function S1 reaches its maximum value near $\beta n = 1$; its contribution decreases at high electron energy, but it is predominant for $\beta n < 1$. The sum of terms with cosine oscillates with the amplitude comparable with that of Si (the part for $\beta n < 1$), but with an opposite sign. The terms with sine function are too small to be seen on the figure. The term $-\dfrac{2}{\beta^2 n^2}$ is represented by "constant" on Fig. 1. The sum of all terms is posi-

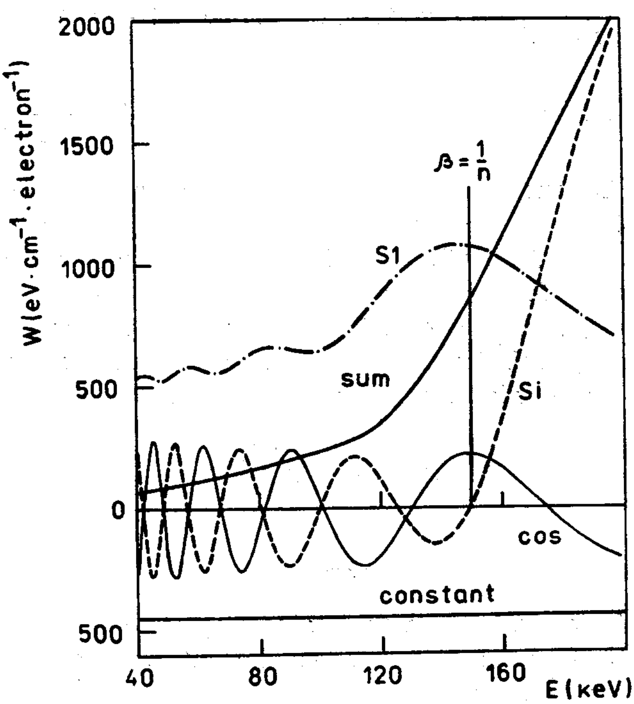

Fig. 1. The contribution of individual terms of the formula (20) into the yield of VCR initial parameters: L = 1240 nm, n = 1.58; λ = 400 nm

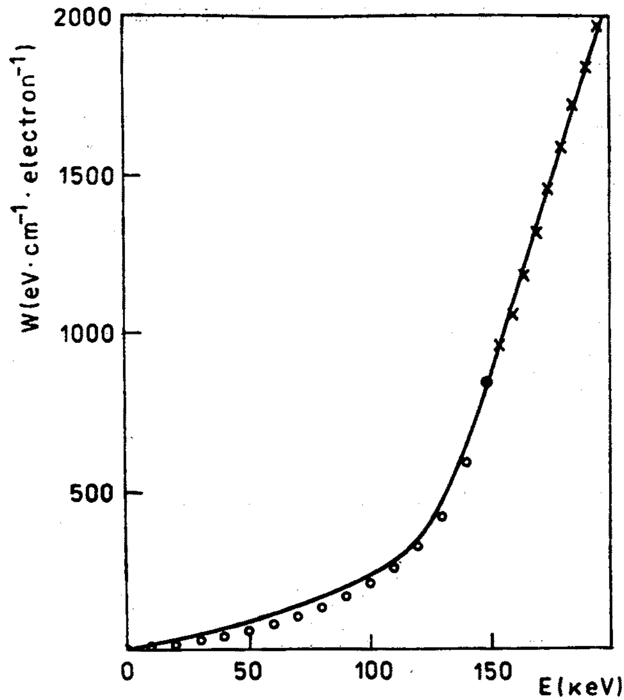

Fig. 2. The approximation of the formula (20) for the radiation intensity (continuous line) by expressions: (11) — circles, (12) — crosses, (14) — full circle

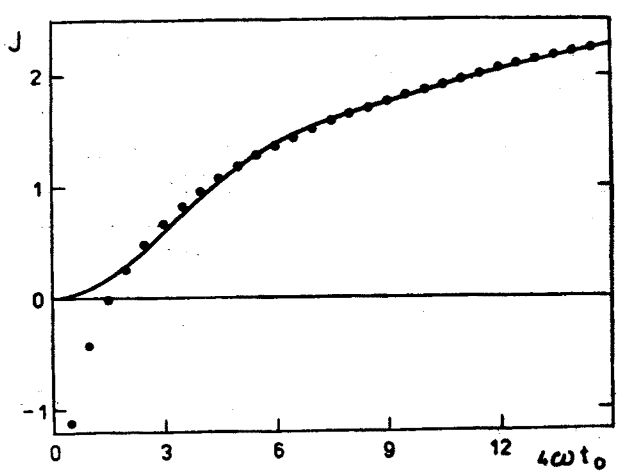

Fig. 3. The approximation of the formula (21) — solid line — by the expression (14) — points

tive, it increases with energy and is represented by a smooth line having no discontinuity at the threshold $\left(\beta = \frac{1}{n}\right)$.

Fig. 2 shows an agreement between the approximate formulas (11), (12) and (14) and the exact formula (20) for the case analysed in this section. In the energy region $\beta > 1/n$, the agreement is good because the formula (12) contains the whole set of terms (in the sense of (20)) which contribute significantly into radiation. The situation does not change considerably near the threshold, where the condition (13) is not satisfied any longer, but, at the same time, the influence of the logarithmic term is still small. Some discrepancy at lower energies may be connected with the neglecting of the terms containing cosine and integral sine — the two terms of opposite contributions, although not cancelling each other totally.

At $\beta n = 1$ the formulas (14) and (21) agree well, because the logarithmic argument is sufficiently large. When the argument becomes smaller, the approximation of function $S1$ by a logarithm fails. In Fig. 3 one can see that the logarithmic term in the formula (14) tends to minus infinity as $\omega t_0 \to 0$, while the function $S1$ tends to zero, as it should be expected. Let us return to Eqs. (11) and (12). It should be noted here that the infinity of the logarithm appears at any value of the parameter ωt_0. It is the result of the approximation of the function $S1$ by a logarithm. The area where this approximation is not correct, in paper [2] is excluded with the help of Eq. (13).

5. Energy irradiated in the main maximum

The basic work [1] treated the fact that VCR may be observed only in the direction of the characteristic angle (3) as one of the main features of VCR and that the destructive interference takes place in all other directions. The radiation from real detectors should be described by the angular distribution as in the formula (9). This distribution has the main maximum at an angle approximately equal to that given by (3), [6] and some additional secondary peaks [14].

There are two reasons for investigating the dependence of the radiation intensity in the main maximum on the particle energy. The first one is the connection of the main maximum with the VCR in the sense of [1] as it was mentioned above. The second one is the Tamm description, i.e. his division of the radiation into two parts, each of them caused by the different mechanism of the particle-medium interaction. The main maximum has the limits clearly seen: the numerator of the formula (9) equals zero at a sine argument of $\pm\pi$.

After integrating (9) within these limits, one can obtain

$$W = 0.904\frac{e^2}{c^2}L\omega\left(1-\frac{1}{\beta^2 n^2}\right) - \frac{4e^2}{\omega L n^2}\,. \tag{22}$$

The formula may be used only when both limits appear at the real angles, i.e. when the following condition is satisfied:

$$\beta \geqslant \frac{1}{n-\lambda/L}\,. \tag{23}$$

The first term of the formula (22) except for the coefficient 0.9, corresponds to the formula (1), i.e. to the VCR in the sense of [1]. The second term is small. Within the application range of the formula (22) and for the data from the previous section, the second term value is 0.6% of the main maximum. It is worth noting that the term understood as VCR is strictly connected with the second term, representing the bremsstrahlung. Similar expressions can be obtained for other maxima with the only difference in the constant coefficient values.

6. *Conclusions and remarks*

It should be noted that:

1. The complete expression for the irradiated energy (20) does not reveal a discontinuity at the energy $\beta = 1/n$. The intensity of the radiation changes smoothly with energy.
2. The terms containing sine integral in the expression (20) are continuous functions of energy as well.
3. The radiation in both main and secondary maxima is described by the two terms connected with different mechanisms of the particle-medium interaction.

The discussion allows one to formulate the following conclusions:

1. The formulas (11) and (12) (under the condition (13)) and formula (14) approximate well Eq. (20), which does not reveal any discontinuity near $\beta n = 1$ and lacks any peculiarity which might be treated as a threshold.
2. The division of the radiation into two parts caused by two different particle-medium interaction mechanisms seems rather conventional and is quite impossible outside the limits set by formula (13).

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