LARGE N_c LIMIT FOR PHYSICAL QUANTITIES IN SU(3) SKYRME MODEL*

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In SU(3)-flavor Skyrme model, three different ways of defining the $N_c \to \infty$ limit are proposed. Mass splittings and magnetic moments are calculated in this limit. The calculations show importance of $1/N_c$ corrections.

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In recent years the Skyrme model [1] has become a very popular description of low-energy hadron physics. It is based on the $1/N_c$ expansion [2, 3] and on the spontaneous breaking of the chiral symmetry. There have been made many predictions for quantities like baryon masses [4–6], magnetic moments [7], nucleon-nucleon potentials [8] and others. All quantities calculated in the model are generally within 20% of the experimental values.

The fact that N_c tends to infinity is the main assumption in "deriving" the Skyrme Lagrangian [9–11] from Quantum Chromodynamics (QCD). Only in this limit QCD reduces to the theory of Goldstone bosons and the model has solitonic solutions interpreted as baryons.

In the SU(3)-flavor Skyrme model for $N_c > 3$ the states belong to the high dimensional representations of SU(3) group, because of a constraint imposed on the Hilbert space [5a, 9, 12]. The fact that there is no such a constraint in the two flavor case enables to define smoothly large N_c limit for the SU(2) flavor model. For each N_c the lowest representations are spin-isospin $\frac{1}{2}$ and $\frac{3}{2}$ corresponding to nucleon and delta respectively [4]. In the SU(3) model in the large N_c limit we have no states with experimentally observed quantum numbers. The way to define this limit is to decide which states in large SU(3)-flavor representation correspond to the physical states (say nucleon or delta). The group theoretical considerations do not give the unique answer, the additional criterion can be phenomenology.

In this work we propose three ways of defining the $N_c \to \infty$ limit and discuss the resulting phenomenology.

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Now we shall describe the main points of the model which we shall use afterwards (for review see Ref. [13]). An important fact for our considerations is that in the SU(3) Skyrme model we have the constraint [5, 12] imposed on the states in the Hilbert space.

The wave function for the baryon $B = (Y, I, I_3)$ of spin (S, S_3) is given by [5]:

$$\psi_B = \mathcal{N} \left\langle Y, I, I_3 | D^{(\varrho)}(A) | \frac{N_c}{3} S, -S_3 \right\rangle, \tag{1}$$

where ϱ labels SU(3) flavor representations, $A(t)$ is the collective coordinate SU(3) matrix. The right hypercharge in Eq. (1) is equal to $N_c/3$, this arises from the presence of the constraint. Therefore ϱ has to include states with $Y = N_c/3$:

$$\varrho = (N_c, 0), (N_c-2, 1), \ldots, \left(1, \frac{N_c-1}{2}\right) \tag{2}$$

and higher, we use the notation $\varrho = (p, q)$ for an irreducible representation of SU(3).

In order to define the $N_c \to \infty$ limit we have to decide which representations from Eq. (2) are "physical". We shall give three different criterions for selecting representations and motivations for these choices.

(*i*) The first choice, following Ref. [10], is based on the assumption that the physical representations are those which have physical spin ($\frac{1}{2}$ and $\frac{3}{2}$) namely:

$$\text{“8”} = (1, n), \quad \text{“10”} = (3, n-1); \quad n = \frac{N_c-1}{2}. \tag{3}$$

Other representations are spurious. Note that according to Eq. (1) spin carried by the entire representation is equal to the isospin of the highest weight. The representations (3) contain many states. We select the octet- and decuplet-like structures in representations (3),

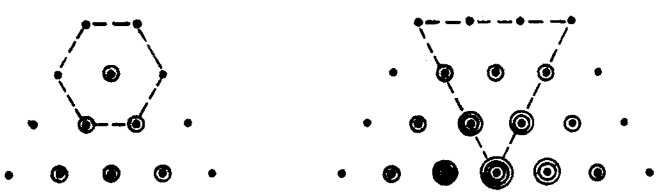

Fig. 1. Octet and decuplet structures in the weight diagrams of representations $(1, n)$ and $(3, n-1)$

requiring that the generalization of the physical state has the correct physical isospin (Fig. 1). All other states are spurious. In this case, the states corresponding to baryons have unphysical quantum numbers: hypercharge and charge (e.g. for "proton" $Y = \frac{1}{3}(2n+1)$, $Q = \frac{1}{3}(n+2)$).

(*ii*) The second possible choice of representations and states consists in selecting these representations which are obtained by the symmetrical extension of octet and decuplet weight diagrams:

$$\text{“8”} = (n, n), \quad \text{“10”} = (n+2, n-1); \quad n = \frac{N_c}{3}. \tag{4}$$

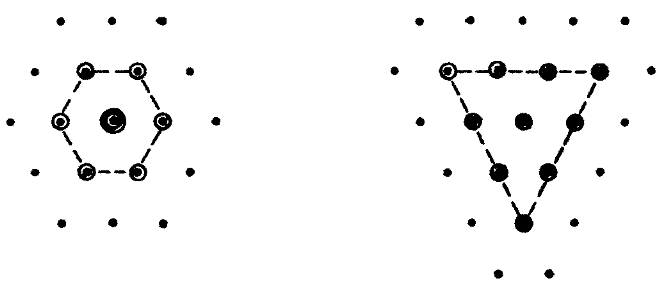

Fig. 2. Octet and decuplet structures in the weight diagrams of representations (n, n) and $(n+2, n-1)$

The main criterion in this case is the requirement that generalized octet ought to be the totally symmetrical SU(3) representation, like $\mathbf{8} = (1, 1)$. From this condition we can construct the generalized decuplet in the self-consistent way, requiring proton and delta to have identical hypercharge (Fig. 2). The states are selected to give correct physical isospin. For this choice we get unphysical spin $S = \frac{n}{2}$.

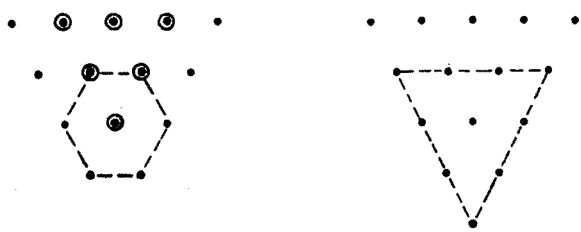

Fig. 3. Octet and decuplet structures in the weight diagrams of representations $(n, 1)$ and $(n+2, 0)$

(*iii*) The third choice of representations and states is based on the requirement that the generalized decuplet has to be a totally symmetric SU(3) representation and for generalized octet we require the proton and delta to have the same hypercharge (Fig. 3):

$$\text{“}\mathbf{8}\text{”} = (n, 1), \qquad \text{“}\mathbf{10}\text{”} = (n+2, 0); \qquad n = N_c - 2. \tag{5}$$

The selection rules for choosing states are analogous to the previous cases. This time, the baryon states have unphysical hypercharge, charge and spin.

The strangeness for representations (3)–(5) is defined as follows [10]: the states in the top line (with the highest hypercharge) have strangeness equal to zero, going to the lines below, each state has the strangeness greater by one then the states in the previous line. Accordingly the baryon states have the correct strangeness in representations (3) and uncorrect in representations (4), (5).

Now we may compute physical observables, our interest is to see how those quantities behave in the large N_c limit. We discuss here mass splittings and magnetic moments for the three mentioned types of representations. The baryon mass formula reads [5a, 12]:

$$M_B = N_c M_{cl} + M_{coll} + \frac{1}{N_c} M_s(S+1)S - N_c M_0 - N_c M_{BR} d_B, \tag{6}$$

where

$$d_B = \sum_{\pm} \begin{pmatrix} 8 & \varrho & \varrho_\pm \\ \Lambda^0 & B & B \end{pmatrix} \begin{pmatrix} 8 & \varrho & \varrho_\pm \\ \Lambda^0 & n & n \end{pmatrix}, \tag{7}$$

n refers to spin. To obtain Eq. (6) we have enriched the SU(3) symmetric Lagrangian by the breaking terms [6]: $\mathrm{Tr}(U+U^+-2)$ and $\mathrm{Tr}\,(\lambda_8(U+U^+-2))$, which after quantization lead to the operators:

$$N_c M_{BR} D^{(8)}_{88}, \tag{8}$$

$$N_c M_0, \tag{9}$$

where $D^{(8)}_{ab}$ is the matrix element of the adjoint SU(3) representation. Numbers d_B in Eqs. (6) and (7) are the matrix elements of the operator (8) between the baryon states (1).

For computation of magnetic moments we shall use formula from Ref. [7]:

$$\mu_B = N_c \xi h_B, \tag{10}$$

where

$$h_B = \sum_{\pm} \begin{pmatrix} 8 & \varrho & \varrho_\pm \\ \Sigma^0 & n & n \end{pmatrix} \left[\begin{pmatrix} 8 & \varrho & \varrho_\pm \\ \Sigma^0 & B & B \end{pmatrix} + \frac{1}{\sqrt{3}} \begin{pmatrix} 8 & \varrho & \varrho_\pm \\ \Lambda^0 & B & B \end{pmatrix} \right]. \tag{11}$$

Our aim is to calculate d_B and h_B for the specified representations and then to evaluate baryon masses and magnetic moments. The calculations are connected with finding the Clebsch-Gordan coefficients for arbitrary SU(3) representations [10, 11, 14], below we present some of them which have been used in the calculations of d_B and h_B (for $p, q \neq 0$):

$$\begin{pmatrix} (1,1) & (p,q) & (p,q)_\pm \\ \Lambda^0 & A_L & A_L \end{pmatrix} = -N_\pm \frac{q[2(p+q+1)-3L] \mp \alpha[p(p+q+1)-3L(p+1)]}{\sqrt{6}\,(p+q+1)}, \tag{12}$$

$$\begin{pmatrix} (1,1) & (p,q) & (p,q)_\pm \\ \Sigma^0 & A_L & A_L \end{pmatrix} = N_\pm \frac{1}{\sqrt{2}} \left\{ \pm \alpha p + \frac{L}{p+q+1} \left[q \mp \alpha(p+1) - \frac{2q(1\pm\alpha)}{p+2-L} \right] \right\}, \tag{13}$$

$$\begin{pmatrix} (1,1) & (p,q) & (p,q)_\pm \\ \Lambda^0 & \bar{A}_L & \bar{A}_L \end{pmatrix} = -N_\pm \frac{(p+q+1)(2q-3)+3L(1-q) \mp \alpha[p(p+q+1)-3L(p+2)]}{\sqrt{6}\,(p+q+1)}. \tag{14}$$

$$\begin{pmatrix} (1,1) & (p,q) & (p,q)_\pm \\ \Sigma^0 & \bar{A}_L & \bar{A}_L \end{pmatrix} = -N_\pm \frac{1}{\sqrt{2}} \left\{ 1 \mp \alpha p + \frac{L}{(p+q+1)(p+3-L)} \right.$$

$$\left. \times \left[(p+1-L)(1-q\pm a(p+2)) - 2(1-q\mp a(p+2q)) \right] \right\}, \tag{15}$$

$$A_L = (Y-L, I-\tfrac{1}{2}L, I-\tfrac{1}{2}L), \tag{16}$$

$$\bar{A}_L = (Y-(L+1), I-\tfrac{1}{2}(L-1), I-\tfrac{1}{2}(L-1)),$$

$$L = 0, \ldots, p. \tag{17}$$

$$\alpha^2 = \frac{q(6+3p+8q+2pq+2q^2)}{p(6+3q+8p+2qp+2p^2)}, \tag{18}$$

$$N_\pm^2 = \frac{3(1+p+q)}{2q[6+3p+8q+2pq+2q^2 \mp \alpha p(4+p+q)]}. \tag{19}$$

For $p, q \neq 0$ the direct product $(1, 1)\times(p, q)$ contains two different representations (p, q) differenciated by subscript $\pm$. For $q = 0$ there is only one representation $(p, 0)$ in $(1, 1) \times (p, 0)$ namely

$$(p, 0) = \frac{1}{\sqrt{2}}[(p, 0)_- - (p, 0)_+]. \tag{20}$$

Next we proceed to the discussion of the masses and magnetic moments for representations (3)–(5). In Refs. [10, 14] masses and magnetic moments were calculated for representations of type (3). Eq. (6) allows to separate the relevant linear dependence on N_c:

$$M_B = N_c(M_\varrho - M_{BR} d_B). \tag{21}$$

Fitting three parameters $M_{8,10}$ and M_{BR} for $N_c = 3$, we obtain the mass spectrum which agrees with the data with an accurancy of 6% [5]. In the limit of $N_c \to \infty$ all masses become infinite (all d_B tend to 1 for (3), to $\frac{1}{4}$ for (4) and to $-\frac{1}{2}$ for (5)), however the ratios of mass differences are finite and can be compared with the data. As an input we take the difference of masses $N-\Xi$ and masses of N and Σ^*. We want to emphasize that in the limit of $N_c \to \infty$ we have no information about the mean mass of the multiplet (there is no difference between octet and decuplet since the splitting is of the order $1/N_c$, see Eq. (6)). In fact the fitting procedure (21) differentiates betwen M_8 and M_{10} but this should be considered only as a convenient way to parametrize the splittings within the multiplets, and should not be considered as a prediction of the model for $N_c \to \infty$.

Masses for representations (3), (4) are identical in the limit of $N_c \to \infty$ and fulfill following formula:

$$\bar{M}_B = \bar{M}_\varrho - \bar{M}_{BR} Y_{SU(3)} \tag{22}$$

with $\bar{M}_{BR} = 189$ MeV, $\bar{M}_8 = 1129$ MeV and $\bar{M}_{10} = 1385$ MeV, $Y_{SU(3)} = Y_B - Y_\Lambda$. All experimental values lie between the predictions for $N_c = 3$ and $N_c = \infty$, however for $N_c = \infty$ there is no splitting between Λ and Σ, as it takes place for $N_c = 3$. In the case of representation (5) we have the splitting between Λ and Σ for $N_c \to \infty$.

The limits of the coefficients h_B for all proposed representations are summarized in Table I. We note here some properties of the magnetic moments. Magnetic moments for particles in representations (3) satisfy:

$$\sum_{I_3} \mu_B = 0 \tag{23}$$

TABLE I

Coefficients h_B in the $N_c \to \infty$ limit for representations (3)–(5)

Representation "8"

B	(3)	(4)	(5)
p	$-\frac{1}{3}$	0	$\frac{4}{9}$
n	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{5}{9}$
Σ^+	$-\frac{1}{3}$	$\frac{1}{4}$	$\frac{2}{3}$
Σ^0	0	$\frac{1}{4}$	$\frac{1}{2}$
Σ^-	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$
Λ^0	0	$\frac{1}{4}$	$\frac{1}{2}$
Ξ^0	$\frac{1}{9}$	$\frac{1}{2}$	$\frac{2}{3}$
Ξ^-	$-\frac{1}{9}$	0	$\frac{1}{3}$

Representation "10"

B	(3)	(4)	(5)
Δ^{++}	$-\frac{3}{5}$	$-\frac{1}{5}$	$\frac{1}{2}$
Δ^+	$-\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{2}$
Δ^0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{2}$
Δ^-	$\frac{3}{5}$	$\frac{7}{10}$	$\frac{1}{2}$
Σ^{*+}	$-\frac{1}{2}$	$-\frac{1}{8}$	$\frac{1}{2}$
Σ^{*0}	0	$\frac{1}{4}$	$\frac{1}{2}$
Σ^{*-}	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{1}{2}$
Ξ^{*0}	$-\frac{1}{3}$	0	$\frac{1}{2}$
Ξ^{*-}	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
Ω^-	0	$\frac{1}{4}$	$\frac{1}{2}$

for each isospin multiplet [14], this sum rule is not obeyed by the data. Let us point out that there is one N_c independent ratio:

$$\mu_p/\mu_{\Sigma^+} = 1\ (= 1.17 \text{ exp}). \tag{24}$$

Magnetic moments in representations (4), (5) satisfy following rule:

$$\sum_{I_3} \mu_B \sim (2I+1) \tag{25}$$

for each isospin multiplet. This sum rule is not obeyed by the data. We note that in representation (4) the proton magnetic moment is equal to 0 and that there are two N_c independent ratios:

$$\mu_{\Lambda^0}/\mu_n = \tfrac{1}{2}\ (= 0.32 \text{ exp}), \qquad \mu_{\Xi^0}/\mu_n = 1\ (= 0.65 \text{ exp}). \tag{26}$$

In the representation (5) exist several N_c independent ratios, for instance:

$$\mu_{\Sigma^-}/\mu_{\Lambda^0} = \tfrac{2}{3}\ (= 1.80 \text{ exp}), \qquad \mu_{\Sigma^-}/\mu_{\Xi^-} = 1\ (= 1.60 \text{ exp}). \tag{27}$$

To summarize: we have calculated the physical quantities in the Skyrme model in the limit of $N_c \to \infty$. We have proposed three different ways of defining this limit corresponding to three different choices of representations generalizing the $N_c = 3$ octet and decuplet. Since the ratios of mass splittings and magnetic moments for $N_c = \infty$ are quite far from the experimental values for all our choices, we conclude that $1/N_c$ corrections are important and should be taken into account (the $1/N_c$ corrections to magnetic moments provide a good example of improvement of theoretical results [7]).

Phenomenologically the choice (3) seems to be the best candidate to define the large

N_c limit. It is also the only possible choice which can be accepted if one insists on the validity of the semiclassical approximation, since for the choices (4) and (5) the third term in (6) is of the order of N_c.

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