FINITE-TEMPERATURE EFFECTIVE POTENTIAL FOR AN OPEN UNIVERSE*

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We investigate one-loop effects at finite temperatures for an open Robertson-Walker Universe, and we obtain formulae foı a scalar field Coleman-Weinberg potential. The meaning of non-zero temperature and negative curvature for the space-time topology $R^1 \times PS^3$ is discussed. Zero-point fluctuation of energy are determined.

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Classical Coleman-Weinberg potential, on which the inflationary scenario is based [1], has been determined for Minkowski spacetime. Hu and O'Connor argued [2] that such an approach is not fully consistent with the inflationary scenario. There are no reasons to think that it remains the same for Bianchi or Robertson-Walker models [3].

Hu and O'Connor [2] have shown that in Robertson-Walker cosmological models the potential contains curvature in such a way that the negative curvature causes the symmetry breakdown (suppressing inflation) whereas the positive curvature reproduces the symmetry (amplifies inflation).

From the point of view of inflationary scenario it is important to determine exact formulae describing the effective potential for cosmological models with non zero curvature and finite temperatures [1]. Maślanka [4] has determined such potential for the closed Robertson-Walker Universe. However, from cosmological point of view, both open and closed models are equally important.

In the present work we give exact formulae for the Coleman-Weinberg potential for Robertson-Walker spacetime with the negative curvature. According to expectations of the

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standard world model great densities of matter and radiation are characteristic for the early phase of evolution of the Universe. Thus quantum processes take place in thermal bath of the temperature T equal to the temperature of the Universe [1].

In our work we determine the effective action at a finite temperature in one-loop approximation by using the zeta-function regularization method proposed by Hawking [5]. Our work is a continuation of the research initiated by Parker and Fulling [6], and by Lee and Sakai [7] who invesigated quantum effects in the open Robertson-Walker Universe. After Wick transformation has been performed $R^1 \times PS^3$ spacetime possesses the topology of $S^1 \times PS^3$ where S^1 is the compactified Euclidean time and PS^3 is a pseudosphere with the scale factor a.

Radius β of the circle S^1 is equal to the inverse temperature of the Universe, $\beta = 1/T$ [8].

We performe calculations assuming the static approximation i.e. assuming that the timescale of expansion of the Universe ($H^{-1} = a/\dot{a}$) is much greater than the characteristic timescale of quantum effects. In other words, we assume that the thermodynamical equilibrium takes place at every moment.

We shall consider a scalar field ϕ with self-interaction potential $\lambda\phi^4$ coupled to the background metric of $R^1 \times PS^3$ space. In such a case, the one-loop correction to the effective potential is the following

$$V^{(1)}(\phi) = -\frac{1}{2\mathrm{Vol}\,(M)}\{\zeta'(0)+\ln\mu^2\zeta(0)\},$$

where $\zeta(s)$ is zeta-function of the operator

$$L = -\Box+\xi R+\tfrac{1}{2}\lambda\phi^2$$

and $R = -6/a^2$ is the curvature scalar for PS^3, ξ — a coupling parameter, $\Box$ — Laplace--Beltrami operator for $S^1 \times PS^3$ space, $\ln\mu^2$ — a renormalization term, Vol(M) is volume of the space $S^1 \times PS^3$ ($\mathrm{Vol}(M) = \beta\,\mathrm{Vol}\,PS^3$).

By using the spectrum and degeneracy of Laplace-Beltrami operator [6] the effective potential can be determined up to one-loop corrections.

Before the regularization the potential is

$$V(\phi) = \frac{\lambda}{4!}\phi^4+\frac{\delta\lambda}{4!}\phi^4-\frac{3\xi}{a^2}\phi^2$$
$$+\frac{1}{64\pi^2}\left(\frac{\lambda}{2}\phi^2+\kappa^2\right)^2\left[\ln\frac{\left(\frac{\lambda}{2}\phi^2+\kappa^2\right)}{\mu^2}-\frac{3}{2}\right]$$
$$-\frac{\left(\frac{\lambda}{2}\phi^2+\kappa^2\right)}{2(\pi\beta)^2}I_2\left(\beta\sqrt{\frac{\lambda}{2}\phi^2+\kappa^2}\right), \tag{1}$$

where

$$\kappa^2 = \frac{1}{a^2}(1-6\xi), \quad I_2(x) = \sum_{n=1}^{\infty} \frac{K_2(nx)}{n^2}.$$

$K_2(x)$ is the Mac Donald function [8]. Now, we shall renormaliee the potential $V(\phi)$. We shall do it only at the zero-temperature limit of (1). By using asymptotic properties of Mac Donald functions, we see that:

$$V(\phi) \xrightarrow{\beta\to\infty} V_\infty(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{\delta\lambda}{4!}\phi^4 - \frac{3\xi}{a^2}\phi^2$$

$$+\frac{1}{64\pi^2}\left(\frac{\lambda}{2}\phi^2+\kappa^2\right)^2\left[\ln\frac{\left(\frac{\lambda}{2}\phi^2+\kappa^2\right)}{\mu^2}-\tfrac{3}{2}\right]. \tag{2}$$

Renormalization conditions are the following:

$$\left.\frac{d^2V_\infty(\phi)}{d\phi^2}\right|_{\phi=0} = 0, \tag{3}$$

$$\left.\frac{d^4V_\infty(\phi)}{d\phi^4}\right|_{\phi=0} = \lambda. \tag{4}$$

From condition (3) we obtain:

$$\ln\frac{\kappa^2}{\mu^2} = 1 + \frac{192\pi\xi}{\lambda\kappa^2a^2}. \tag{5}$$

From (4) we have

$$\frac{\delta\lambda}{4!} = -\frac{\lambda^2}{256\pi^2}\ln\frac{\kappa^2}{\mu^2}. \tag{6}$$

If we use other renormalization conditions:

$$\left.\frac{d^2V_\infty}{d\phi^2}\right|_{\phi=M} = 0, \tag{3a}$$

$$\left.\frac{d^4V_\infty}{d\phi^4}\right|_{\phi=M} = \lambda, \tag{4a}$$

we obtain the following rather complicated formulae from (3a):

$$\frac{M^2}{2}(\lambda+\delta\lambda) - \frac{6\xi}{a^2} + \frac{1}{64\pi^2}(3\lambda^2M^2+2\lambda\kappa^2)\left[\ln\frac{\left(\frac{\lambda}{2}M^2+\kappa^2\right)}{\mu^2}-1\right]$$

$$+\frac{1}{128\pi^2}\frac{(\lambda^3M^3+2\lambda\kappa^2M)^2}{\left(\frac{\lambda^2M^4}{4}+\lambda M^2\kappa^2+\kappa^4\right)}=0, \tag{5a}$$

and from (4a):

$$\delta\lambda-\frac{12\lambda^2}{128\pi^2}+\frac{6\lambda^2}{64\pi^2}\ln\left(\frac{\frac{\lambda}{2}M^2+\kappa^2}{\mu^2}\right)$$

$$+\frac{3}{128\pi^2}\frac{(15\lambda^4M^4+24\lambda^3\kappa^2M^2+4\lambda^2\kappa^4)}{\left(\frac{\lambda}{2}M^2+\kappa^2\right)^2}$$

$$+\frac{M(\lambda^2M^2+2\lambda\kappa^2)}{128\pi^2}\left\{\frac{2(\lambda^2M^3+2\lambda\kappa^2M)^3+6\lambda^2M\left(\frac{\lambda}{2}M^2+\kappa^2\right)^4}{\left(\frac{\lambda}{2}M^2+\kappa^2\right)^6}\right.$$

$$\left.-\frac{12\lambda^4M^5+32\lambda^3\kappa^2M^3+16\lambda^2\kappa^4M}{\left(\frac{\lambda}{2}M^2+\kappa^2\right)^4}\right\}=0. \tag{6a}$$

By using (5) and (6) we obtain renormalized effective potential of a simpler form and with a lower number of arbitrary parameters:

$$V(\phi)=\frac{\lambda}{4!}\phi^4+\frac{1}{64\pi^2}\left(\frac{\lambda}{2}\phi^2+\kappa^2\right)^2\left[\ln\left(\frac{\frac{\lambda}{2}\phi^2+\kappa^2}{\kappa^2}\right)-\tfrac{3}{2}\right]$$

$$-\frac{\left(\frac{\lambda}{2}\phi^2+\kappa^2\right)}{2(\pi\beta)^2}I_2\left(\beta\sqrt{\frac{\lambda}{2}\phi^2+\kappa^2}\right)+\frac{3\kappa^2}{\lambda}\frac{\xi}{a^2}. \tag{7}$$

Formula (7) describes the renormalized one-loop potential for a massless scalar field at finite temperatures in the space $R^1\times PS^3$. Effective potential (7) allows one to determine the quantity $E_c=\lim_{\phi\to 0}V(\phi)$ which corresponds to the vacuum energy; to be more precise it is free-energy density

$$E_c=\frac{3\kappa^2}{\lambda}\frac{\xi}{a^2}-\frac{3\kappa^4}{128\pi^2}-\frac{\kappa^2}{2(\pi\beta)^2}I_2(\beta\kappa). \tag{8}$$

The case of the conformal coupling $\xi=1/6$ needs a special attention. In this case, we

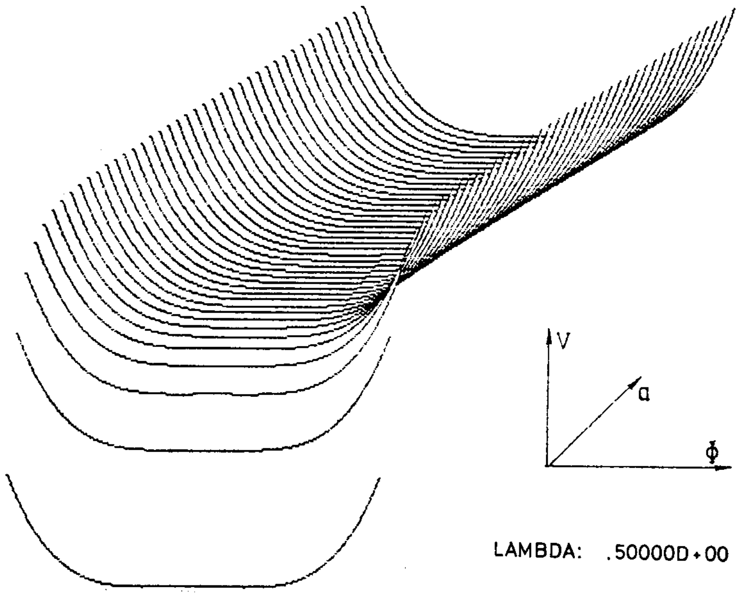


Fig. 1. Dependence $V(\phi, a)$, where a is the scale factor, for small a. If $a \to 0$ $V(\phi, a) \to -\infty$ (potential well are not shown); $\beta = \beta_0 = \text{const}$

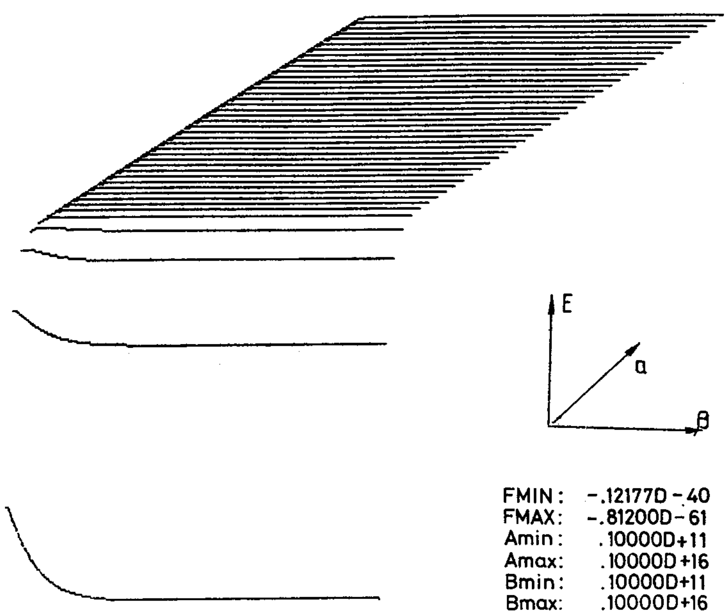


Fig. 2. Dependence of the Casimir energy $E_c(a, \beta)$. If $a \to 0$ dependence on β becomes manifest. In $E_c(a, \beta)$ dependence on a is dominated

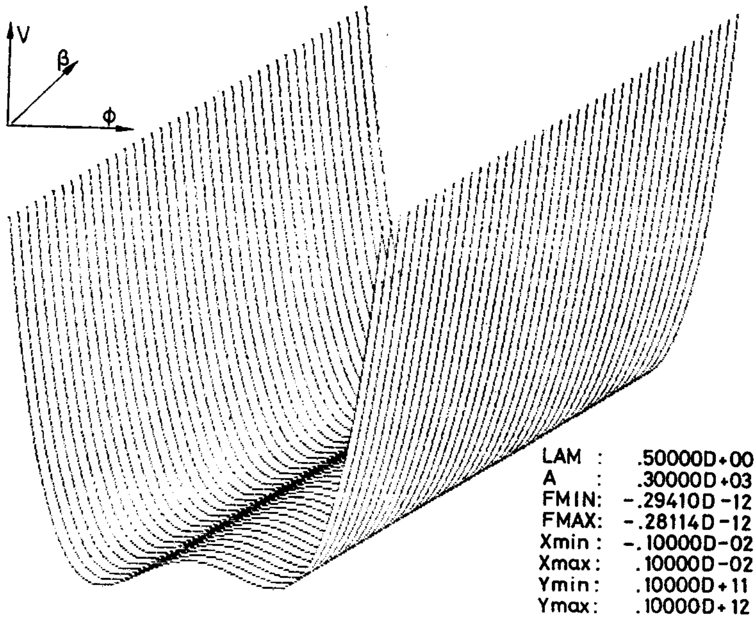


Fig. 3. Dependence $V(\phi, \beta)$ for a = const. Dubble potential well appears for a broad range of β

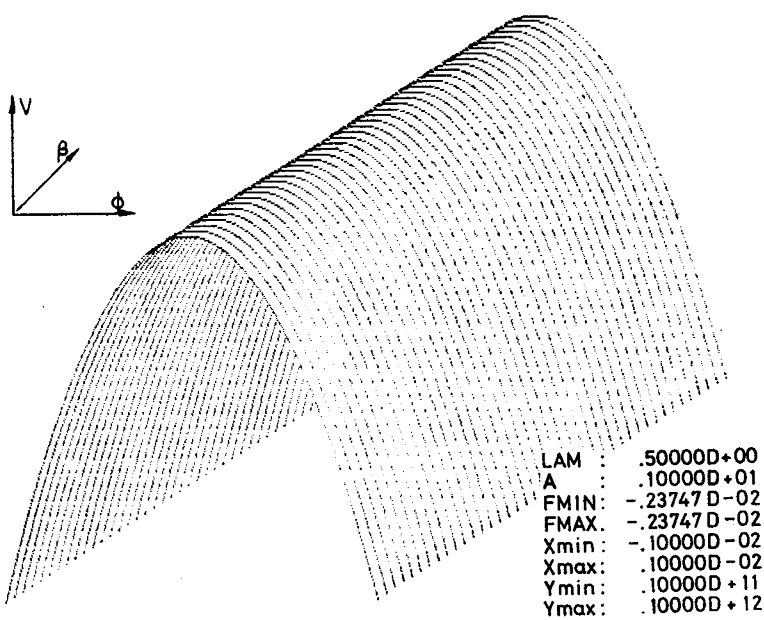
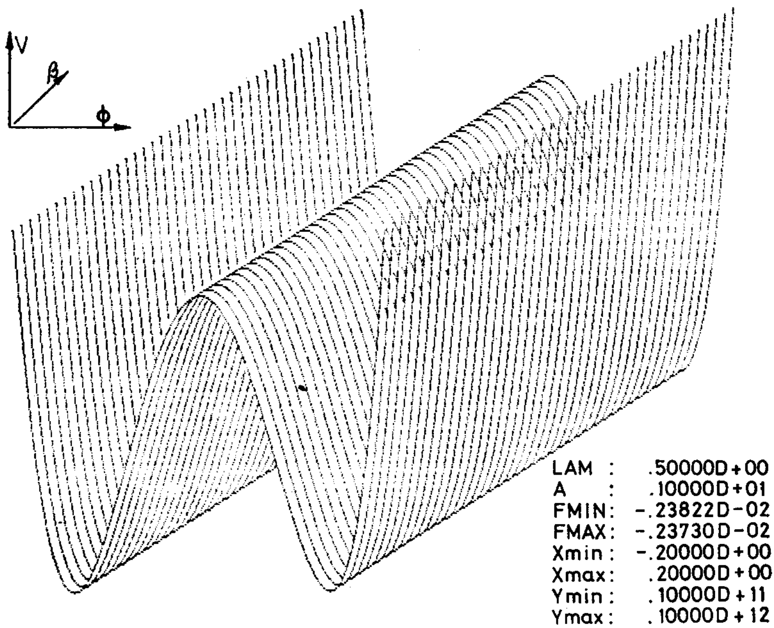


Fig. 4. Dependence $V(\phi, \beta)$ near the local maximum



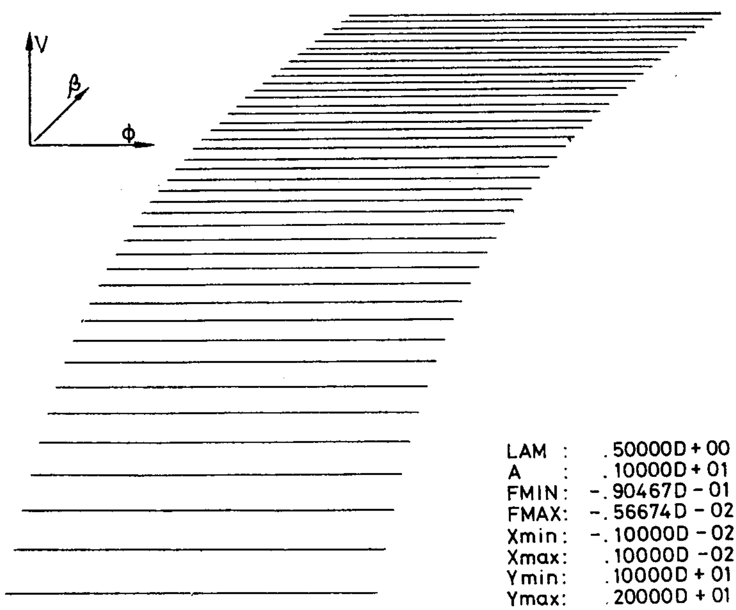


Fig. 5. Different potential shapes for different value a. For small β, $V(\phi, \beta) \to -\infty$ and $V(\phi, \beta)$ depends mainly on β

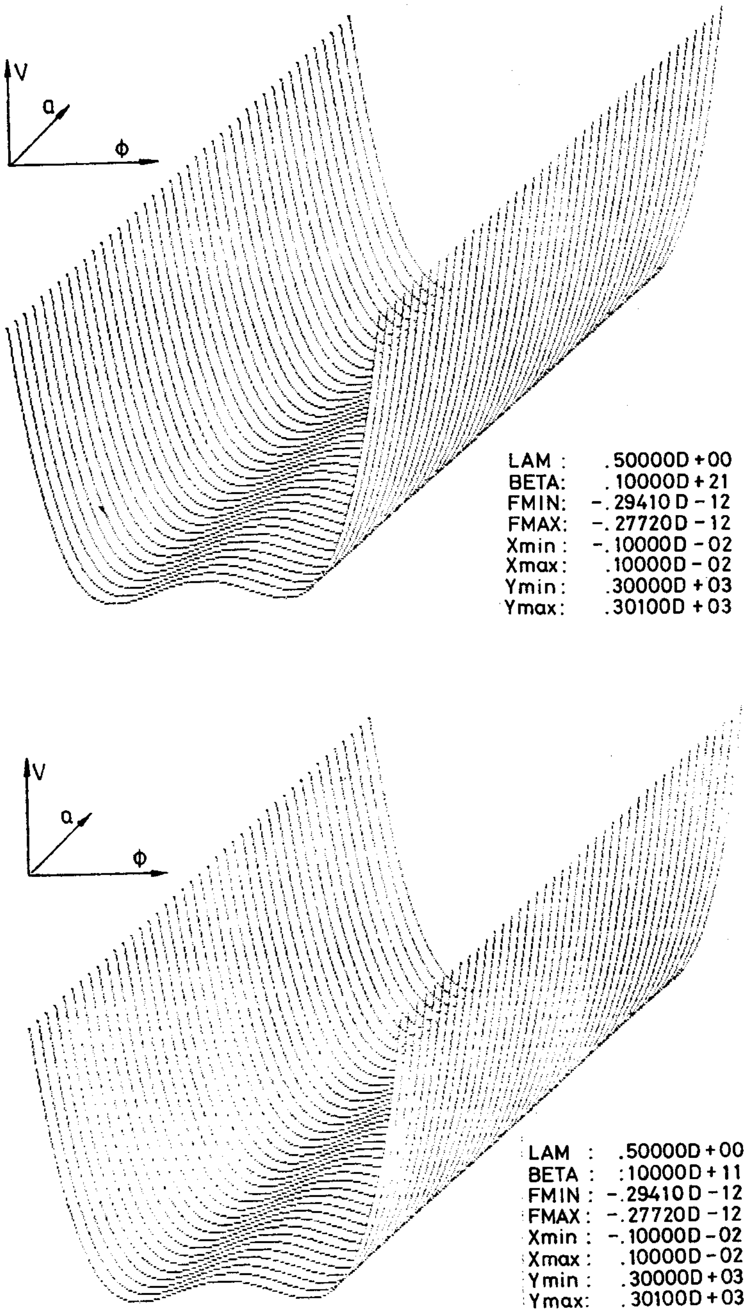


Fig. 6. Different potential shapes for β = const

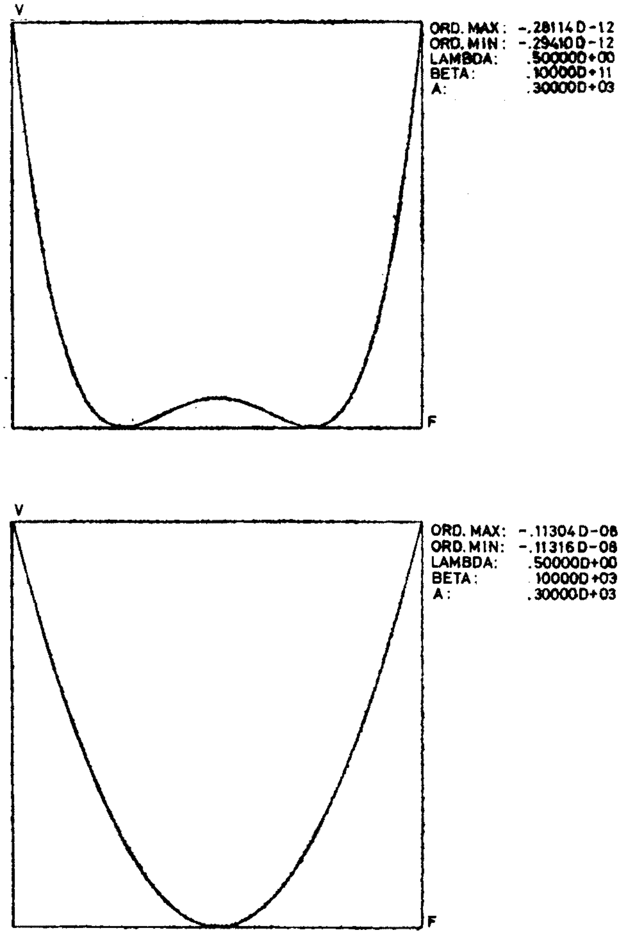


Fig. 7. Dependence $V(\phi)$ for $a, \beta = \text{const}$

cannot perform renormalization procedure at the point $\phi = 0$ because of infrared divergencies.

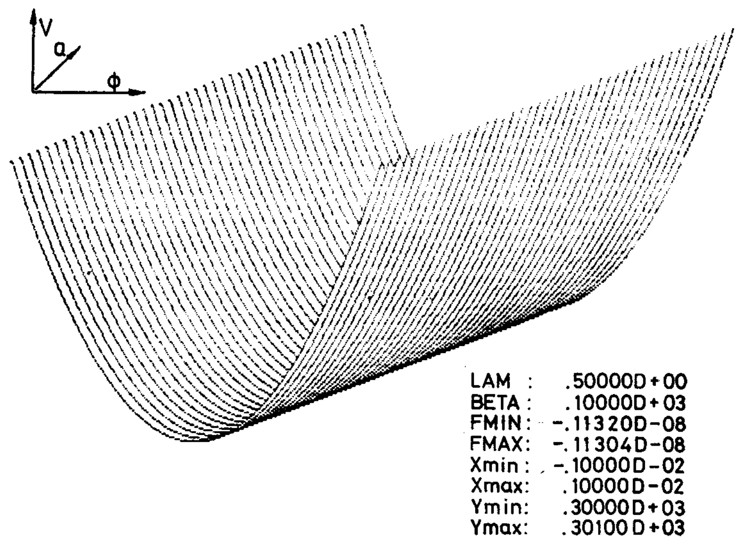
In the case of the conformal coupling, non renormalized potential is

$$V(\phi) = V_\infty(\phi) + V_\beta(\phi),$$

where

$$V_\infty(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{\delta\lambda}{4!}\phi^4 - \frac{1}{2a^2}\phi^2 + \frac{1}{64\pi^2}\left(\frac{\lambda}{2}\phi^2\right)^2\left[\ln\frac{\frac{\lambda}{2}\phi^2}{\mu^2} - \frac{3}{2}\right], \tag{9}$$

$$V_\beta(\phi) = -\frac{\frac{\lambda}{2}\phi^2}{2(\pi\beta)^2} I_2\left(\beta\sqrt{\frac{\lambda}{2}\phi^2}\right). \tag{10}$$



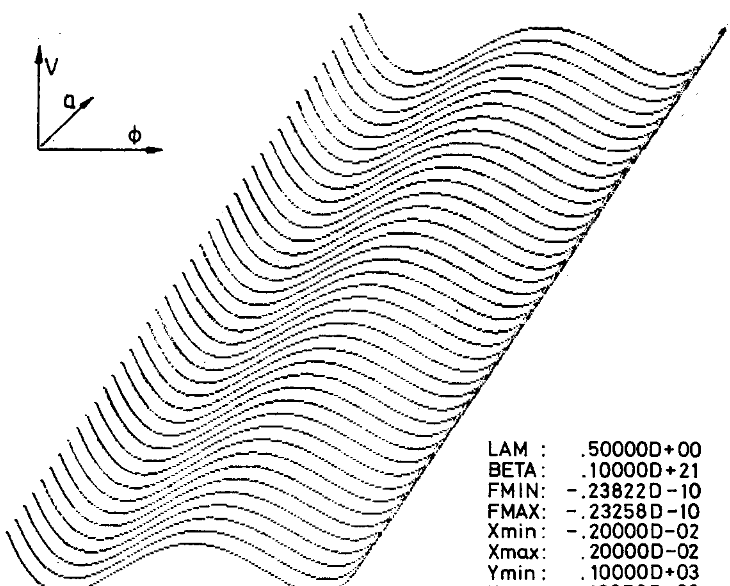


Fig. 8. Different potential shapes for small β; potential wells disappear, for large β; potential wells appear

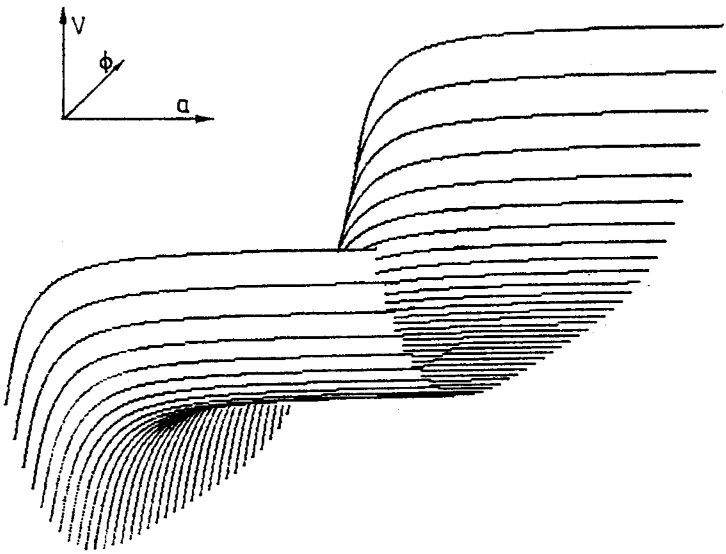


Fig. 9. Dependence $V(\phi, a)$ for β = const (potential wells are not shown). Dependence $V(\phi, \beta)$ is qualitatively the same

When renormalizing $V_\infty(\phi)$ we demand that:

$$\left.\frac{d^4 V_\infty(\phi)}{d\phi^4}\right|_{\phi=M} = \lambda.$$

Then

$$\frac{\delta\lambda}{4!} = -\frac{\lambda^2}{256\pi^2}\ln\left(\frac{\frac{\lambda}{2}M^2}{\mu^2}\right) - \frac{\lambda^2}{256\pi^2}\frac{8}{3}. \tag{11}$$

Finally, the renormalized one-loop potential for the conformal coupling is the following:

$$V(\phi) = \frac{\lambda}{4!}\phi^4 - \frac{1}{2a^2}\phi^2 + \frac{\lambda^2}{256\pi^2}\phi^2\left[\ln\left(\frac{\phi^2}{M^2}\right) - \frac{25}{6}\right] - \frac{\lambda\phi^2}{4\pi^2\beta^2}I_2\left(\beta\sqrt{\frac{\lambda}{2}\phi^2}\right). \tag{12}$$

Curvature has been taken into account only in the "mass term" — $(1/2a^2)\phi^2$. Qualitative dependence of the effective potential (7) and the free — energy density (8) on the parameters appearing in these quantities is shown in figures.

Numerical analysis shows that inflation can appear for a wide range of parameters a (scale factor) and β (inverse temperature). For small a and small β inflation vanishes. It is interesting that qualitative dependence $V(\phi, a)$ for a given β and $V(\phi, \beta)$ for a given a are similar. For large a (small curvatures) and large β (low temperatures) potentials $V(\phi, a)$ and $V(\phi, \beta)$ depend mainly on the field.

Discussion and conclusions

The finite-temperature field theory in a flat spacetime can be simply generalized to the case of static curved spacetime. The generalisation of the imaginary time formalism is then possible due to the fact that the background metric is time independent. However, there is no general method of determination of finite-temperature quantum effects in an arbitrary time-dependent curved spacetime. The analysis of thermodynamical properties of matter fields in an external gravitational field would be fully satisfactory if we could construct the thermal energy-momentum tensor of field being in the state of thermodynamical equilibrium (locally) due to sufficiently strong interactions. It is, however, a very difficult task. In order to determine the thermal energy-momentum tensor of matter fields one usually makes the following assumpions:

(*i*) The metric of the background spacetime is independent of time.

(*ii*) The thermodynamical equilibrium of matter fields is due to their interactions with the thermal bath.

In this paper we have used these assumptions. Therefore, we have used the so-called static approximation. In this approximation, one treats scale factor as independent of time while computing effective potential. In other words, we assume that the characteristic time of quantum processes is much smaller than the characteristic time of evolution of universe. The assumption of the static approximation is not obvious and requires further investigations.

The new inflationary model, elaborated by Linde, Albrecht and Steinhardt, has been criticized by Brandenberger and Kahn [10] as wrongly making use of the zero-temperature effective potential.

The effective potential in finite temperature for a universe with negative curvature obtained by us may be used in constructing the inflationary scenario [11], or Linde's scenario of chaotic inflation [12].

In this paper the exact formula for one-loop effective potential involving an arbitrary coupling to a gravitational field of type $\frac{1}{2}\xi R\phi^2$ is determined. We considered arbitrary values of the coupling parameter ξ, since the actual coupling is not known.

In the case of minimal coupling ($\xi = 0$) and massless scalar particles, due to infrared divergencies, the renormalization of the effective potential at $\phi = 0$ is not applicable. For that reason in the paper the renormalization procedure is performed twice: at the points $\phi = 0$ and $\phi = M$.

From numerical analysis of the potential obtained two main conclusions can be formulated:

— if a universe is of negative curvature, inflation can occur when the curvature is sufficiently small (there is some critical value of curvature, i.e. such $a_{\rm crit}$ that $V_{\rm eff}(a_{\rm crit}) = 0$);

— if a universe is of negative curvature, inflation can occur when the temperature is sufficiently low (there is some critical value of temperature $T_{\rm crit}$, such that $V_{\rm eff}(T_{\rm crit}) = 0$).

As from numerical analysis it follows that for big curvatures and high temperatures the effective potential goes to $-\infty$, it seems reasonable to include the chemical potential term in the expression for the effective potential [11] and [13].

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