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CONSTRAINT OF THE LIGHT-CONE TYPE IN CONSTRAINT DYNAMICS OF WORLD LINES

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For a particle constrained in the internal variables by relations of the light cone type, we derive generators of motion within the framework of the Sudarshan, Mukunda and Goldberg formalism as an extended form of the Dirac generator procedure for constrained Hamiltonian dynamics. It is shown that the realization of the Poincaré group obtained for this particle is compatible with the Lie algebra of the de Sitter group SO(2, 1), proper to rotator models. Characteristic features of a relativistic rotator are then demonstrated on evolution equations for dynamical variables.

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1. Introduction

One of approaches examining the space-time structure of composite particles is based on the construction of representations of the Poincaré group. A general method of incorporating relativistic symmetry in the Hamiltonian form has been given by Dirac [1]. In the Dirac formalism the central idea is to realize the ten generators of the Poincaré group in terms of a set of dynamical variables defined on a hypersurface of the Minkowski space. We are usually interested in the *transitive* realizations i.e. such realizations of the Poincaré group, when its elements map every point of the hypersurface onto any other point of this hypersurface. The transitive realizations are determined by the invariants of the group, which in the case of the Poincaré group are formed by the two quantities P^2 and w^2, P^μ being the momentum of the given system and w^μ its Pauli-Lubański vector. The latter is defined as

$$w_\mu = (1/2)\varepsilon_{\mu\nu\varrho\sigma}P^\nu M^{\varrho\sigma}, \tag{1.0}$$

where $\varepsilon_{\mu\nu\varrho\sigma}$ is the totally antisymmetric unit tensor and $M^{\mu\nu}$ is the angular momentum. The Poisson brackets (PB) of these invariants with all generators of the Poincaré group can be shown to vanish.

A familiar transitive realization of the Poincaré group is given by the generators

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu, \tag{1.1}$$

$$P^\mu = p^\mu, \tag{1.2}$$

with the canonical coordinates x^μ and p^μ obeying the PB

$$\{x^\mu, p^\nu\} = g^{\mu\nu}, \quad \{x^\mu, x^\nu\} = \{p^\mu, p^\nu\} = 0. \tag{1.3}$$

This realization is special in this sense that it corresponds to the case when $w^2 = 0$; this means that it describes a free massive point-like particle with the vanishing internal angular momentum. If one takes a more general ansatz with $w^2 \neq 0$, for instance that introduced by Wigner [2] in connection with the unitary irreducible representations of the Poincaré group (for illumination we add that the Wigner program based on the little groups even if formulated on the quantum level, can be without troubles translated into the language of the classical theory because of intertheory correspondence mediated by the Hamiltonian formalism)

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + \xi^\mu \eta^\nu - \xi^\nu \eta^\mu \tag{1.4}$$

$$P^\mu = p^\mu, \tag{1.5}$$

it is quite possible that the appropriate realization ceases to be transitive. In (1.4) ξ and η are the internal canonical variables involving as the only nonzero PB$\{\xi^\mu, \eta^\nu\}$, namely

$$\{\xi^\mu, \eta^\nu\} = g^{\mu\nu}, \quad \{\xi^\mu, \xi^\nu\} = \{\eta^\mu, \eta^\nu\} = 0 \tag{1.6}$$

and the squares ξ^2 and η^2 are subject to some constraints. Actually, if the constraints are chosen to be $\xi^2 = 0, \eta^2 = 0$ for any P^2, they become inconsistent with the PB ($\{\xi^2, \eta^2\} \neq 0$) and so the realization is not transitive (in Wigner [2] $\eta^2 = 0$, $\xi^2 = 0$ if $p^2 < 0$, $\xi^2 = \text{const} < 0$ if $p^2 = 0$).

In the present paper we want to examine consequences of the violation of this property, assuming the existence of the constraints

$$\hat{1} \equiv \xi^2 = 0, \quad \hat{2} \equiv \eta^2 = 0,$$

$$\xi^0 = |\vec{\xi}|, \quad \eta^0 = -|\vec{\eta}|. \tag{1.7}$$

The constraints (1.7) remind the light-cone conditions and hence they can be denoted as the *Light-Cone Constraints* (LCC). It is obvious that the object described by the relations (1.3)–(1.6) and restricted by the LCC (1.7) *is not pointlike*. It represents a composite particle since the appropriate realization is intransitive.

To obtain both realization of the Poincaré Lie algebra and equations of motion for such a *LCC particle* we shall use the eleven-generator formalism by Sudarshan Mukunda and Goldberg [3]. It amounts to an extended form of the Dirac [4] generator formalism for constrained Hamiltonian dynamics, admitting ten of generators to realize the Poincaré algebra and one to yield the equations of motion. One of characteristic features of this treatment is that all the eleven generators obey a condition concerning objective reality of world lines, called the *world-line condition*, as a necessary condition for any form of relativistic dynamics to escape the *no-interaction theorem* (Currie et al. [5]). In such a form of relativistic Hamiltonian dynamics if the evolution parameter is chosen dynamically, all the generators are independent, if however kinematically, one obtains only ten independent generators and the formalism is properly reduced to Dirac's instant form of dynamics. We shall follow closely Sudarshan et al. [3] and for more details the reader is referred to this work.

In Sect. 2 we derive within the accepted formalism the evolution equations and the ultimate form of the generators of motion for the LCC particle. We shall see that even if the original phase space is 16 dimensional and the pair of the constraints (1.7) is considered to be weak, suitable transformations and two other constraints reduce the system to the appropriate number (12) of the variables, which determine its states. One of two latter constraints allows to select the evolution parameter. In the present article we prefer the more familiar instant form of dynamics and choose this parameter in a kinematic way. A brief discussion is made also of the alternative dynamical choice.

In Sect. 3 we shall show explicitly that the examined particle is not pointlike. Using the set of generators, deduced in the previous Section, we construct the classical realization of the symmetry group for this particle. We find that it corresponds to the group SO(2, 1). This de Sitter group appears also in the kinematical structure of rotator models that were extensively studied by Mukunda et al. [6], Aldinger et al. [7], Bohm et al. [8] and Corben [9]. The LCC particle belongs thus to the category of classical relativistic rotator models.

The fact that the evolution equations of the particle exhibit features characteristic to the relativistic rotators is confirmed in Sect. 4 by giving solutions of these equations.

Our problem is particular in this kind of dynamics. The constraints (1.7) are primary and moreover represented by two independent functions. Therefore, a mapping of the manifold onto itself in the appropriate reduced phase space must be limited to canonical transformations utilizing the Dirac Brackets (DB). The constraints are now applied in the strong sense and the mapping becomes one-to-one and bicontinous.

2. *Form of the eleven generators of motion*

2.1. Equations of world sheets

We start with the set of basic canonical variables x, p, ξ, η which fulfil the algebra of the PB given in Sect. 1 (Eqs (1.3) and (1.6)). In constructing the canonical realization of the considered particle we begin with the generators expressed by relations (1.4) and (1.5). Now, we shall assume that for this particle the two independent LCC constraints (1.7) are initially satisfied in the weak sense ($\xi^2 \approx 0$, $\eta^2 \approx 0$). Both the functions define a 14 dimensional constraint hypersurface Σ in the original 16 dimensional phase space Γ. The conditions (1.7) are not consistent with the PB (1.3) and (1.6), since $\{\xi^2, \eta^2\} \neq 0$.

To use these constraints in the strong sense, we must pass to the DB $\{f, g\}^*$, which for two dynamical variables f and g, or their functions, are defined by the formula

$$\{f, g\}^* = \{f, g\} + \{1, 2\}^{-1}(\{f, 1\}\{2, g\} - \{f, 2\}\{1, g\}), \tag{2.1}$$

where and in what follows we omit the operator label over the constraint symbol $\hat{1}$–$\hat{4}$. The corresponding DB for the basic independent variables of our phase space have then the form

$$\{x^\mu, p^\nu\}^* = g^{\mu\nu}; \quad \{\xi^\mu, \eta^\nu\}^* = g^{\mu\nu} - \eta^\mu \xi^\nu (\xi.\eta)^{-1}, \tag{2.2}$$

remaining DB being equal to zero. It may be verified that the DB of the generators of the Poincaré group are given by the relations

$$\{M^{\mu\nu}, M^{\varrho\sigma}\}^* = g^{\mu\varrho}M^{\nu\sigma} + g^{\nu\sigma}M^{\mu\varrho} - g^{\mu\sigma}M^{\nu\varrho} - g^{\nu\varrho}M^{\mu\sigma},$$

$$\{P^\mu, M^{\nu\varrho}\}^* = g^{\mu\varrho}p^\nu - g^{\mu\nu}p^\varrho, \tag{2.3}$$

i.e. they have the same form as the PB of these quantities for the free particle. Since

$$\{M^{\mu\nu}, i\}^* = 0, \quad \{P^\nu, i\}^* = 0, \quad i = 1,2 \tag{2.4}$$

the hypersurface Σ is invariant under the canonical transformations of the given variables. Hence, the generated realization of the Poincaré group in terms of the transformations R^* can play now the role of the canonical transformations, replacing the familiar transformations R of the standard model of the single free particle. The new canonical transformations map Σ onto itself. The hypersurface Σ as the union of two-dimensional "orbits"-sheets is generated by the set of these transformations applied to any point (x, p, ξ, η) of the reduced phase space. To construct the sheets, it is sufficient, for the basic dynamical variables, symbolized by Ω, to solve a system of the differential equations which clearly respect the presence of the constraints (1.7)

$$\frac{d\Omega(\sigma)}{d\sigma} \approx v_i\{\Omega(\sigma), i\}^*, \tag{2.5}$$

where v_i $(i = 1, 2)$ are arbitrary factors and $\Omega^\mu(0) = \Omega^\mu$. From Eqs. (1.3), (1.6), (2.1) and (2.5) one has

$$\frac{dx^\mu(\sigma)}{d\sigma} = 0, \quad \frac{dp^\mu(\sigma)}{d\sigma} = 0$$

$$\frac{d\xi^\mu}{d\sigma} = v_2\{\xi^\mu, \eta^2\} = 2v_2\eta^\mu(\sigma); \quad \frac{d\eta^\mu}{d\sigma} = v_1\{\eta^\mu, \xi^2\} = -2v_1\xi^\mu(\sigma). \tag{2.6}$$

Eqs (2.6) characterize the world sheets of the LCC particle. The concrete form of these sheets depends, obviously, on the specification of the factors v_i.

2.2. Next reduction of the phase space dimensions

A world line on each sheet will be determined only if one chooses suitably new constraints. These new constraints — label them 3 and 4 — must, of course, respect the two previous ones. The dynamical constraint 3 will help to specify a world line as a curve C which is a function of the only parameter σ. It can be deduced with still an unspecificated parameter σ from the set of the differential equations for each of the basic variables

$$\frac{d\Omega^{\mu}(\sigma)}{d\sigma} \approx \frac{\partial\Omega^{\mu}(\sigma)}{\partial\sigma} + v\{\Omega^{\mu}(\sigma), 3\}^{*}, \tag{2.7}$$

where $\Omega^{\mu}(0) = \Omega^{\mu}$ and v is an arbitrary constant.

The sense of the constraint 4 is then to adjoin to each point (x, p, ξ, η) on C a concrete value of the evolution parameter σ. If we choose 4 in such a form that the PB with 3 are different from zero, the rigorous physical variables and their mutual relations are reproduced by way of the new DB that for 2 phase variables f and g become

$$\begin{aligned}\{f, g\}^{**} = \{f, g\} &+ (\{1, 2\}\{3, 4\})^{-1}[\{1, 2\}(\{f, 3\}\{4, g\} - \{f, 4\}\{3, g\}) \\ &+ \{1, 3\}(\{f, 4\}\{2, g\} - \{f, 2\}\{4, g\}) + \{2, 3\}(\{f, 1\}\{4, g\} \\ &- \{f, 4\}\{1, g\}) + \{3, 4\}(\{f, 1\}\{2, g\} - \{f, 2\}\{1, g\})].\end{aligned} \tag{2.8}$$

(2.8) defines the DB for the twelve dimensional phase space corresponding to the four chosen constraints of the LCC particle. If 3 possesses yet the property $\{M^{\mu\nu}, 3\} \approx 0$ and $\{P^{\nu}, 3\} \approx 0$, one finds that the two-star DB between M and P reproduce precisely the same Lie algebra of the Poincaré group as their one-star DB (2.3). Hence, the new realization of the Poincaré group in terms of the transformations $R^{*}(\Lambda, a)$ provides again canonical relations with respect to the two-star DB. The new canonical transformation $R^{**}(\Lambda, a)$, moreover, preserves σ.

If one assumes that 4 depends explicitly on σ, the arbitrariness of the factor v in Eq. (2.7) may be suppressed and then there is the possibility to specify the evolution parameter using just this equation. It implies

$$v \approx \frac{\partial 4}{\partial\sigma}(\{3, 4\}^{*})^{-1}. \tag{2.9}$$

We find that the equation of motion for a function $f(x, p, \xi, \eta)$, constrained on Σ, is an equation of the type

$$\frac{df}{d\sigma} \approx \frac{\partial f}{\partial\sigma} + \frac{\partial 4}{\partial\sigma}\,\frac{\{3, f\}^{*}}{\{3, 4\}^{*}}. \tag{2.10}$$

It turns out that it is possible always to find an appropriate operator of dynamic evolution — the Hamiltonian — to such a type of the equation [3]. Then

$$\frac{df}{d\sigma} \approx \frac{\partial' f}{\partial\sigma} + \{\mathscr{H}, f\}^{**} \tag{2.11}$$

and $\frac{\partial' f}{\partial \sigma}$ is dependent on the prescribed constraints 1–3 as well as on another new constraint, f being a function of the 12 new variables and σ.

2.3. World line conditions

We must accept yet the requirement that the canonical transformations reproduce the geometrical transformation of world points, this means we need to formulate conditions of *objective reality* for the world lines, thus the world line conditions of the LCC particle. Let the geneiator G

$$G = \tfrac{1}{2}\,\omega^{\mu\nu} M_{\mu\nu} - a^{\nu} P_{\nu}, \tag{2.12}$$

constructed from the generators (1.4) and (1.5), be the generator corresponding to two infinitesimal transformations of the Poincaré group. Let the point $x(\sigma)$, $p(\sigma)$, $\xi(\sigma)$, $\eta(\sigma)$ on the curve C of the phase space lead to the world points with the coordinates $x^{\mu}(\sigma)$, $\xi^{\mu}(\sigma)$ in the inertial frame O. By a transformation $R^{**}(\Lambda, a)$ applied to x, p, ξ, η we obtain a point x', p', ξ', η' on a curve C' and to it there correspond world points with the coordinates $x'^{\mu}(\sigma)$ and $\xi'^{\mu}(\sigma)$ in O'. If one combines Eq. (2.11) with two other equations expressing both the reconstruction of the appropriate world line in terms of the space--time vectors as well as the generator G and the Lorentz transform of these vectors [3], one finds

$$\{G, x^{\mu}\}^{**} \approx \omega^{\mu}_{\nu} x^{\nu} + a^{\mu} + \left(\frac{\partial' x^{\mu}}{\partial \sigma} + \{\mathscr{H}, x^{\mu}\}^{**}\right)\delta_1 \sigma,$$

$$\{G, \xi^{\mu}\}^{**} \approx \omega^{\mu}_{\nu} \xi^{\nu} + a^{\mu} + \left(\frac{\partial' \xi^{\mu}}{\partial \sigma} + \{\mathscr{H}, \xi^{\mu}\}^{**}\right)\delta_2 \sigma. \tag{2.13}$$

Eqs (2.13) define the world line conditions for the LCC object. They are an expression of the requirement for $\delta_1\sigma$ and $\delta_2\sigma$ to be linear in ω and a. There remains the equestion whether $\delta_i\sigma$ ($i = 1, 2$) do exist, and if they do, then what difference is between them.

To obtain the expressions for $\delta_i\sigma$ in an explicit manner, one needs to specify the constraints 3 and 4. Let the constraint 3 be chosen, as an invariant under $R(\Lambda, a)$, in the following way

$$3 \equiv p^2 - m^2 + (\xi - \eta)\cdot p. \tag{2.14}$$

The most straightforward manner of the choice for the constraint 4, which leads to introducing the laboratory time and to adopting the familiar Dirac instant form, is to pose

$$4 \equiv x_0 - \sigma. \tag{2.15}$$

First we must express in Eqs (2.13) the final DB in terms of the PB, using Eqs (2.8) and (2.10)–(2.11). The left hand side of the first of Eqs (2.13) becomes

$$\omega^{\mu\nu} x_{\nu} + a^{\mu} - \{G, 4\}\,\frac{\{3, x^{\mu}\}}{\{3, 4\}}, \tag{2.16}$$

since the PB of $\{1, x^\mu\}$, $\{2, x^\mu\}$, $\{4, x^\mu\}$ and $\{G, 3\}$ are equal to zero and $\{G, x^\mu\} = \omega^{\mu\nu}x_\nu + a^\mu$. On the other hand, the right hand side of the above equation is equal to

$$\omega^{\mu\nu}x_\nu + a^\mu - \frac{\partial 4}{\partial \sigma}(\{3, 4\}^*)^{-1}\{3, x^\mu\}^*\delta_1\sigma, \tag{2.17}$$

because Eqs (2.10) and (2.11) lead to the last term of Eq. (2.17). However, owing to the relations $\{3, x^\mu\}^* = \{3, x^\mu\}$ and $\{3, 4\}^* = \{3, 4\}$ and taking into account Eqs (2.16) and (2.17), one finds

$$\delta_1\sigma = \left(\frac{\partial 4}{\partial \sigma}\right)^{-1}\{G, 4\}, \tag{2.18}$$

and this is formally the same relation that was obtained for the standard free pointlike particle $\left(\text{compare with [3]; here the formula (37) has to be expressed as } \delta\tau = \{G, \chi\}\left(\frac{\partial \chi}{\partial \tau}\right)^{-1}\right)$.
A similar analysis, performed on the second of Eqs (2.13), provides exactly the same result (2.18), i.e. $\delta_1\sigma = \delta_2\sigma = \delta\sigma$. Consequently, the infinitesimal changes of the evolution parameter, linear in ω and a, are equal for each of both sets of variables. A world line of the LCC particle involves then information on evolution of the whole particle including its internal behaviour. The world line condition (2.13) as a requirement that there exist expressions $\delta_i\sigma$ is thus obeyed with the choice (2.15). Hence, due to this kinematic constraint 4, the "*no go*" *theorem* of Curie et al. [5] does not enter the dynamics of the particle.

2.4. Generators of motion

Now we can determine the eleven generators of motion of the particle, following the procedure introduced in [3]. We easily find using Eqs (2.12), (2.15) and (2.18) as well as (1.4) and (1.5) the relation

$$\delta\sigma = \{G, 4\} = -\omega^{0j}x_j - a^0. \tag{2.19}$$

From (2.19) one sees that $\delta\sigma$ does not have to contain terms with ω^{jk} and a^j relating to the purely Euclidean group. The part of the world line condition (2.13), linking to this group, is satisfied trivially. The remaining components of this condition are connected exclusively with the Lorentz transformations. It holds

$$\{M_{0j}, x^k\}^{**} = -\sigma\delta_j^k + x_j\{\mathscr{H}, x^k\}^{**}, \tag{2.20}$$

because (in metrics $1, -1, -1, -1$)

$$\{M_{0j}, x^k\} = -\sigma\delta_j^k, \quad \{M_{0j}, 4\} = x_j, \quad \{3, x^k\} = 2p^k + \xi^k - \eta^k$$

$$\{\mathscr{H}, x^k\}^{**} = \{p^0, x^k\}^{**} = (2p^0 + \xi^0 - \eta^0)^{-1}(2p^k + \xi^k - \eta^k). \tag{2.21}$$

Similarly we have for the components ξ^k

$$\{M_{0j}, \xi^k\}^{**} = \xi_0\delta_j^k + x_j\{\mathscr{H}, \xi^k\}^{**}, \tag{2.22}$$

since

$$\{M_{0j}, \xi^k\} = \xi_0\delta_j^k, \quad \{M_{0j}, 1\} = 0, \quad \{M_{0j}, 4\} = x_j \tag{2.23}$$

$$\{\mathscr{H}, \xi^k\}^{**} = -(\{1,2\}\{3,4\})^{-1}(\{1,2\}\{3,\xi^k\}-\{1,3\}\{2,\xi^k\}),$$

or explicitly

$$\{\mathscr{H}, \xi^k\}^{**} = (2p^0+\xi^0-\eta^0)^{-1}\left(\frac{\xi \cdot p}{\xi \cdot \eta}\eta^k - p^k\right). \tag{2.24}$$

Now one can readily find the ultimate form of equations of motion. They may be derived either from Eqs (2.7) and (2.9), alternatively from Eq. (2.10), or in terms of the two-star DB from the equation

$$\frac{df}{d\sigma} = \frac{\partial f}{\partial \sigma} + \{\mathscr{H}, f\}^{**}, \tag{2.25}$$

the expressions $\{\mathscr{H}, x^k\}^{**}$ and $\{\mathscr{H}, \xi^k\}^{**}$ being evaluated above (see Eqs (2.21) and (2.24)). If we accept the constraint 4 as given by (2.15) and require for the resulting evolution equations to have a *manifestly covariant form* with the natural choice

$$d\sigma \approx dx^0 = m^{-1}(2p^0+\xi^0-\eta^0)d\tau, \tag{2.26}$$

where τ is another evolution parameter, we deduce directly:

$$\dot{x}^k = m^{-1}(\xi^k-\eta^k+2p^k), \quad \dot{p}^k = 0,$$

$$\dot{\xi}^k = (p^2)^{-1/2}[(\xi \cdot \eta)^{-1}\xi \cdot p\eta^k - p^k], \quad \dot{\eta}^k = (p^2)^{-1/2}[(\xi \cdot \eta)^{-1}\eta \cdot p\xi^k - p^k]. \tag{2.27}$$

The equations of this type for the set of variables x, p, ξ, η have appeared first in [10]. Their solution will be given in Sect. 4.

The constraints 3 and 4, introduced by Eqs (2.14) and (2.15), modify the relation between the laboratory time and the path parameter, as demonstrated by Eq. (2.26). On the other hand, this relation has an impact on the ultimate form of the generator M, fixing in it the coordinate x^0. Let us summarize now all the 11 generators valid for the particle, constrained by the conditions (1.7), (2.14) and (2.15):

$$M_{jk} = x_jp_k - x_kp_j + \xi_j\eta_k - \xi_k\eta_j,$$

$$M_{0j} = m^{-1}\tau(2p^0+\xi^0-\eta^0)p_j - x_j(p^2+m^2)^{1/2} + \xi_0\eta_j - \xi_j\eta_0,$$

$$P_j = p_j, \quad P_0 = p_0 = (p^2+m^2)^{1/2}, \quad \mathscr{H} = p_0 \approx (p^2+m^2)^{1/2}. \tag{2.28}$$

We see that the object described by the generators (2.28) preserves features of a structural entity. The mentioned constraints exert an influence upon the character of equations of motion and, naturally, on x^0. The behaviour of this entity will be discussed in more detail in Sect. 3–4. If internal motion is excluded ($\xi \to 0, \eta \to 0$), then from (2.28) we arrive immediately at the generators describing the single free particle in the Dirac instant form (compare again with results of [3]; here the limit demands some fixation of p_0, adequate to the elementary limit of a composite model). In this form of dynamics only the generators P^μ and $M^{\mu\nu}$ are independent. The equations of motion (2.27) are dependent on these quantities and can be derived in a self-consistent way from them, as it is apparent, if one applies simultaneously the conditions (1.7) and (2.26).

Now we adopt the *dynamical* variant of the constraint 4, choosing

$$4 \equiv P \cdot x - m\sigma. \tag{2.29}$$

From a similar analysis of the world line condition (2.13), as made in the case of the kinematic choice, one deduces that again $\delta_1\sigma = \delta_2\sigma = (\partial 4/\partial\sigma)^{-1}\{G, 4\} = \delta\sigma$. Then using (2.13) and (2.29), we have $\delta\sigma = -m^{-1}p \cdot a$. Thus, contrary to the previous case, we find that there is no dependence $\delta\sigma$ on $\omega_{\mu\nu}$.

The evolution equations can be established either in terms of the relation (2.10) with the one-star DB, or on the basis of Eq. (2.25) containing the two-star DB with an appropriate Hamiltonian. It can be straightforwardly verified that if $\mathscr{H}$ is of the form

$$\mathscr{H} = -m \ln (p_0/m), \qquad p_0 = (\vec{p}^{\,2} + m^2)^{1/2}, \tag{2.30}$$

identical with that of the free-particle case [3], and if one chooses the infinitesimal change $d\sigma$ to be

$$d\sigma \approx m^{-1} p \cdot dx = m^{-2} p_\mu (2p^\mu + \xi^\mu - \eta^\mu) d\tau, \tag{2.31}$$

with a new evolution parameter τ, the equations of motion (2.27) remain conserved. In the set of new generators there appears the difference only in the definition of M_{0j} and $\mathscr{H}$. The Hamiltonian has the form as above (Eq. (2.30)) and M_{0j} is given by

$$M_{0j} = p_0^{-1}[m^{-1}\tau(2p_\mu + \xi_\mu - \eta_\mu)p^\mu + \vec{p} \cdot \vec{x}]p_j - x_j p_0. \tag{2.32}$$

If we compare (2.32) in the limit $\xi \to 0$, $\eta \to 0$ with the outcome for the ordinary free particle, we decude that the spatial position variable x_j coincides with the corresponding x_j only in the centre of mass frame (as seen by comparing with [3]). The description of the LCC particle is, however, different than in the preceding case. It turns out that it does not belong to any of the Dirac forms. All the 11 generators are now fully independent.

3. *LCC particle and the SO(2, 1) realization of Poincare's group*

Now we shall attempt to construct the canonical realization of the Poincaré group generated by the expressions (1.4) and (1.5) for the particle constrained by the relations (1.7). We shall use Wigner's treatment of the construction of representations of the Poincaré group via the *little* Lorentz group (see [2]). The standard procedure is to find the state vectors, which are diagonal in the quantities P^2 and w^2, having the meaning of analogons of the Casimir operators. The Pauli-Lubański vector w_μ given by (1.0) has the following components

$$w^0 = \vec{p} \cdot \vec{M}, \qquad \vec{w} = p^0 \vec{M} - \vec{p} \times \vec{N}, \tag{3.1}$$

where $\vec{M} \equiv (M^{23}, M^{31}, M^{12})$ and $\vec{N} \equiv (M^{01}, M^{02}, M^{03})$.

The appropriate algebra can be built up without troubles, because its fundamental relations have been discussed in the preceding sections. As emphasized there, the constraints (1.7) do not fulfil the condition $\{\xi^2, \eta^2\} = 0$ and therefore it has been necessary to pass to the DB. So we have obtained the Lie algebra of the Poincaré group, in which the DB of the basic dynamical variables are defined by relations (2.2) and the DB of the generators M and P retain the standard form valid for the free pointlike particle (2.3).

We now construct the quantities

$$J_0 = \tfrac{1}{2}\lambda^{-1} p\cdot\eta\xi\cdot\eta + \lambda p\cdot\xi,$$

$$J_1 = \tfrac{1}{2}\lambda^{-1} p\cdot\eta\xi\cdot\eta - \lambda p\cdot\xi, \quad J_2 = -\xi\cdot\eta, \tag{3.2}$$

where λ is a constant. Taking into account Eqs (2.2), one can immediately prove the relations

$$\{p\cdot\xi, \xi\cdot\eta\}^* = p\cdot\xi, \quad \{\xi\cdot\eta, p\cdot\eta\xi\cdot\eta\}^* = p\cdot\eta\xi\cdot\eta,$$

$$\{p\cdot\xi, p\cdot\eta\xi\cdot\eta\}^* = p^2\xi\cdot\eta. \tag{3.3}$$

Finally, from Eqs (2.2), (3.2) and (3.4) we derive

$$\{J_0, J_2\}^* = J_1, \quad \{J_0, J_1\}^* = -m^2 J_2, \quad \{J_1, J_2\}^* = J_0. \tag{3.4}$$

We have the result: the algebra represented by the relations (2.3) and (3.4) corresponds to the Lie algebra of the *SO(2, 1) group*. The appropriate Casimir operator has the form

$$J_0^2 - J_1^2 - m^2 J_2^2 = -m^2\xi\cdot\eta\left(\xi\cdot\eta - \frac{2}{m^2}p\cdot\xi p\cdot\eta\right). \tag{3.5}$$

We must find yet the expression for the Pauli-Lubański vector, to complete calculations concerning the little group. We need first simplify the DB $\{\xi^\mu, \eta^\nu\}^*$ involved in (2.2). It is convenient to introduce the new variable

$$\vec{\eta}' = \vec{\eta} - \frac{\eta_0}{\xi_0}\vec{\xi}, \tag{3.6}$$

ξ and η being constrained hereafter by the conditions (1.7). The inverse relation to Eq. (3.6) may be immediately computed and reads

$$\vec{\eta} = \vec{\eta}' - \frac{\vec{\eta}'^2}{2\vec{\xi}\cdot\vec{\eta}'}\vec{\xi}, \quad \eta^0 = -\frac{\vec{\eta}'^2}{2\vec{\xi}\cdot\vec{\eta}'}\xi^0. \tag{3.7}$$

Then the DB for the new variables $\vec{\xi}$ and $\vec{\eta}'$ become

$$\{\eta'_k, \xi_i\}^* = \delta_{ik}, \quad \{\xi_k, \xi_i\}^* = 0, \quad \{\eta'_k, \eta'_i\}^* = 0. \tag{3.8}$$

Likewise, the generators of the little group, defined by (2.2), will have the very simple form, namely

$$\vec{M} = \vec{x}\times\vec{p}+\vec{\xi}\times\vec{\eta}', \quad \vec{N} = x^0\vec{p}-p^0\vec{x}+\xi^0\vec{\eta}', \tag{3.9}$$

in terms of $\vec{\eta}'$. The pair of the dynamic variables $\vec{\xi}$ and $\vec{\eta}'$ determines that part of the angular momentum which corresponds to the particle *spin*, as it is seen from Eq. (3.9), since $\vec{s} \propto \vec{\xi}\times\vec{\eta}'$. Thus we have found that appropriate canonical realization of the LCC particle with the adequate description of its spin is compatible with the representation of the de Sitter group SO(2, 1). The same kind of groups emerges also in the kinematic structure of the Mukunda et al. [6] model of the classical rotator. Thus, the LCC particle belongs to the family of *rotator* particle models discussed in the papers [6–9].

4. *Rotator footing of the equations of motion*

Now we shall describe motion of the LCC object. Eqs (2.27), derived in Sect. 2, are the equations of constraint Hamilton dynamics with the rotator physical content that now has to be demonstrated explicitly. To show that these equations lead to rotator motion, one does not have to look directly for their solutions. Eqs (2.27) are identical (disregarding multiplication constants) with the set of equations derived by Petras [10] for a model of the composite lepton, if a specific choice of interaction of its components is accepted. The model has not been fully successful, however the equations of motion of the given composite system appear to be applicable in a wider framework. It is sufficient to specify only the factors in these equations and we can have in hand the complete solutions. They read [10]

$$\begin{aligned}
\xi^k &= A^k \cos\omega\tau + B^k \sin\omega\tau - \tfrac{1}{2}\, p^k, \\
\eta^k &= -A^k \sin\omega\tau + B^k \cos\omega\tau + \tfrac{1}{2}\, p^k, \\
x^k &= \tfrac{1}{2}\,[(A^k - B^k)\sin\omega\tau - (A^k + B^k)\cos\omega\tau] + 2p^k,
\end{aligned} \tag{4.1}$$

where the integration constants A and B must generally obey the conditions

$$p\cdot A = p\cdot B = A\cdot B = 0, \quad A^2 = B^2 = -1 \tag{4.2}$$

and the Lorentz scalars are chosen consistently with the equations of motion to yield

$$\eta\cdot p = -\xi\cdot p = 2, \quad \xi\cdot\eta = -1; \quad \omega = 2. \tag{4.3}$$

We see that the motion of our particle consists of rectilinear *uniform* motion in the direction of the velocity $\vec{p}/m$ and the *rotation* motion with the frequency $\omega = 2$ (in the appropriate units) according to (4.3). It is clear from (4.1) that in the rest frame we have the pure rotation motion

$$\vec{x} = \tfrac{1}{2}\,[(\vec{A}-\vec{B})\sin\omega t - (\vec{A}+\vec{B})\cos\omega t], \tag{4.4}$$

with $|\vec{A}| = |\vec{B}| = 1 = R$ and $\vec{A}\cdot\vec{B} = 0$, the elementary lenght R of the rotator being given by the magnitude of the vectors $\vec{A}$ and $\vec{B}$.

5. *Conclusion*

In this paper an attempt has been made to support revived interest in theories of composite models of particles such as those of [11–14, 6–9] and others (a complete bibliography up to 1968 see Ref. [15]). The present article gives a nontrivial application of the Sudarshan et al. [3] formalism reproducing simultaneously the current generators of the Poincaré group and the equations of motion, if the *transitivity* of this group is *violated*. In our case the violation is due to the *light cone* condition (1.7), imposed upon the internal variables ξ and η. It is worthwhile to note that these conditions acted in our theory as the first-class constraints, both being independent of each other. Therefore an ambiguity of the original Hamiltonian of the free particle, evoked by the addition of a vanishing linear combination of ξ^2 and η^2, implies the equality to zero of corresponding coefficients. Nevertheless, the established Lie algebra with $\{\ \}^*$ secured, as we have seen, canonical transformations mapping of the hypersurface Σ onto itself.

By accepting two other constraints: invariant under $R(\Lambda, a)$ (2.14) and kinematic (dynamic) (2.15) ((2.29)), we were able to generate states characterized by the SO(2, 1) realization of the Poincaré group and thereby to show that the examined particle exhibits features of the classical relativistic *rotator*. The case of the LCC particle shows that the constraint dynamics of world lines [3] is an efficient tool regardless whether all the eleven generators are independent or not.

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