CHIRAL SYMMETRY BREAKING IN QCD

BY A. TRZUPEK

Institute of Physics, Jagellonian University, Cracow*

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The effect of the Coulomb potential on the chiral symmetry breaking in a model of QCD is analyzed. The renormalized gap equation for massless quarks interacting through the Lorentz vector potential $V(\bar{r}) \sim \sigma r - \frac{\alpha_s}{r}$ is solved numerically. The chiral parameters are calculated for several values of α_s.

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1. Introduction

One of the basic questions of low energy Quantum Chromodynamics is whether the chiral symmetry is spontaneously broken and by what mechanism [1]. The large coupling constant in this energy region prevents us from using the perturbation theory, and we have to look for another methods. Some results on this subject have been obtained from the lattice calculations, however, the inclusion of the light fermions ($m \to 0$) on a lattice is still problematic. Suitable lattice methods have been proposed only recently [2]. In the continuum there are various approaches to that question. One of them [3], which is very often used, is based on writing down the gap equation in Landau gauge QCD. There was found large-Q^2 asymptotic behaviour of the dynamical quark mass. The other model [5], which we are using, relies on the analogy between strong interaction and the superconductivity [4].

In this approach the quark interaction is described by the Coulomb gauge QCD Hamiltonian, which was derived first by Finger et al. [5] and the QCD vacuum is approximated by a coherent superposition of the quark-antiquark pairs. The ground state is analyzed via the variational approach which leads to a gap equation. The starting point of the calculations, which were done in Ref. [5], consists of taking the normal ordering of the Hamiltonian in chirally symmetric vacuum. This renormalization procedure preserves the chiral invariance and produces the finite quark self-energy. Unfortunately, the prescription cannot be applied to confining potentials, since in this case the chiral invariant vacuum turns out

* Address: Instytut Fizyki UJ, Reymonta 4, 30-059 Kraków, Poland.

to be stable [6]. The renormalized gap equation for general potential was derived from the renormalized Dyson equations for the vector and axial-vector vertices in paper by Adler and Davis [7]. The gap equation for the linear potential was also solved there. However, the resulting chiral parameters ($\langle\bar{\psi}\psi\rangle$, dynamical quark mass, f_π) are several times smaller than the experimental ones. Similar large discrepancies were found in papers [8], where the model was applied to the system with finite baryonic number density. It was pointed out by several authors [6–8], that the addition of the short range attraction to the linear potential would improve the results. In this paper we solve the gap equation for potential which is composed from the Coulomb piece and the linear confining one.

In Chapter two we review the basic assumption of the model. The gap equation for the potential is derived in Chapter three. The numerical solutions of the equation and concluding remarks are presented in Chapters four and five.

2. The model

The chirally invariant Coulomb gauge QCD Hamiltonian, introduced first by Finger et al. [5], for quarks interacting through the fourth component of Lorentz vector instantaneous potential is

$$H = \sum_{\vec{x}} \psi^+(\vec{x})(-\vec{\alpha}\cdot\vec{\nabla})\psi(\vec{x}) + \tfrac{1}{2}\sum_{\vec{x},\vec{y},\beta} V(\vec{x}-\vec{y})\left(\psi^+(\vec{x})\frac{\lambda^\beta}{2}\psi(\vec{x})\right)$$
$$\times\left(\psi^+(\vec{y})\frac{\lambda^\beta}{2}\psi(\vec{y})\right) + (Z-1)\sum_{\vec{x}} \psi^+(\vec{x})(-\vec{\alpha}\cdot\vec{\nabla})\psi(\vec{x}). \tag{2.1}$$

The space integrations are discretized temporarily and do not play an important role in the model. The continuum limit will be taken at the end. $\vec{\alpha}$, λ^β, $\beta = 1 \ldots 8$ stand for Dirac and Gell-Mann matrices respectively and $\psi(\vec{x})$ is the coloured, single flavour, massless quark field. The generalization to the case of two flavours is straightforward.

The quark field ψ can be expanded in terms of free massless spinors v^0, u^0

$$\psi_\alpha(\vec{x}) = \frac{1}{n^{\frac{3}{2}}}\sum_{\vec{k},s}[u^0_{s\alpha}(\vec{k})b^0_{s\alpha}(\vec{k}) + v^0_{s\alpha}(\vec{k})d^{0+}_{s\alpha}(-\vec{k})]e^{i\vec{k}\cdot\vec{x}}, \tag{2.2}$$

where $b^0_{s\alpha}(\vec{k})$ and $d^0_{s\alpha}(\vec{k})$ are the annihilation operators for a massless quark and antiquark with colour index α, helicity s and momentum $\vec{k}$. $V(\vec{x})$ is equal to the sum of linear and Coulomb potentials[1]

$$V(\vec{r}) = -\left(\sigma r - \frac{\alpha_s}{r}\right), \tag{2.3}$$

[1] The minus sign in (2.3) is introduced in order to obtain the attractive force between the $q\bar{q}$ in a colour singlet state.

where $r = |\vec{r}|$. The last part of the sum in Eq. (2.1) is a counterterm, which must be included in order to obtain finite results [7].

Similarly to the BCS description of the superconductivity, the vacuum in the model is approximated by a coherent superposition of states of the colour singlet $q\bar{q}$ pairs

$$|\Omega\rangle = \frac{1}{N}\prod_{\vec{k},s,\alpha}(1 - s\tau\beta(k)b^{0+}_{s\alpha}(\vec{k})d^{0+}_{s\alpha}(\tau-\vec{k}))\,|0\rangle, \tag{2.4}$$

where $N = \prod_{\vec{k},s,\alpha}\sqrt{1+\tau\beta^2(k)}$ denotes the normalization factor, and τ is the volume element in the momentum space. The state $|\Omega\rangle$ is chirally asymmetric unless $\beta = 0$ — in this case $|\Omega\rangle$ is equivalent to the chirally symmetric, perturbative vacuum $|0\rangle$. The momentum space wave function of a quark-antiquark pair $\beta(\vec{k})$ is determined by the following gap equation

$$\frac{\delta\langle\Omega|H|\Omega\rangle}{\delta\beta(\vec{k})} = 0. \tag{2.5}$$

However, the equation (2.5) contains logarithmic divergences unless the appropriate renor malization is performed. This is done in the next Chapter.

3. *The gap equation*

In order to derive the gap equation we have to calculate the matrix element $\mathscr{E} = \langle\Omega|H|\Omega\rangle$. To this end, it is useful to introduce the Bogoliubov-Valatin (B-V) transformation, which consists of expanding the quark field ψ in terms of a new spinors basis v, u

$$\psi_\alpha(\vec{x}) = \frac{1}{n^{\frac{1}{2}}}\sum_{\vec{k},s}[u_{s\alpha}(\vec{k})b_{s\alpha}(\vec{k})+v_{s\alpha}(\vec{k})d^+_{s\alpha}(-\vec{k})]e^{i\vec{k}\cdot\vec{x}}, \tag{3.1}$$

with the new annihilation operators $b_{s\alpha}(\vec{k})$ and $d_{s\alpha}(\vec{k})$ defined by

$$b_{s\alpha}(\vec{k})\,|\Omega\rangle = 0 \quad \text{and} \quad d_{s\alpha}(\vec{k})\,|\Omega\rangle = 0. \tag{3.2}$$

With the aid of definitions (2.2) and (3.1) one can show that there exists the linear relation between the old (massless) and the new creation and annihilation operators

$$b_s(\vec{k}) = \cos\frac{\phi(k)}{2}\,b^0_s(\vec{k})+s\sin\frac{\phi(k)}{2}\,d^{0+}_s(-\vec{k}),$$

$$sd_s(\vec{k}) = -\sin\frac{\phi(k)}{2}\,b^{0+}_s(\vec{k})+s\cos\frac{\phi(k)}{2}\,d^0_s(-\vec{k}). \tag{3.3}$$

The rotation angle of the B-V transformation is defined by the following equations

$$\sin\frac{\phi(k)}{2} = \frac{\beta(k)}{\sqrt{1+\beta^2(k)}} \quad \text{and} \quad \cos\frac{\phi(k)}{2} = \frac{1}{\sqrt{1+\beta^2(k)}}. \tag{3.4}$$

After performing the B-V transformation, the vacuum energy can be calculated by means of the Wick's theorem. The latter implies that the Hamiltonian (2.1) can be rewritten in the form[2]

$$H = \mathscr{E} + :H_2: + :H_4:, \tag{3.5}$$

where

$$\mathscr{E} = 3Z \sum_{\vec{k}} \mathrm{Tr}\,[(\vec{\alpha}\cdot\vec{k})\Lambda_-(\vec{k})] + \frac{4}{(an)^3} \sum_{\vec{k},\vec{p}} \tfrac{1}{2}\tilde{V}(\vec{k}-\vec{p})\,\mathrm{Tr}\,[\Lambda_+(\vec{k})\Lambda_-(\vec{p})],$$

$$H_2 = \tfrac{2}{3}\frac{1}{n^3} \sum_{\vec{x},\vec{y},\vec{k}} V(\vec{x}-\vec{y})\,[\psi^+(\vec{x})\,(\Lambda_+(\vec{k})-\Lambda_-(\vec{k}))\psi(\vec{y})]e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$+Z\sum_{\vec{x}} \psi^+(\vec{x})\,(-i\vec{\alpha}\cdot\vec{\nabla})\psi(\vec{x}),$$

$$H_4 = \sum_{\vec{x},\vec{y},\beta} \tfrac{1}{2}V(\vec{x}-\vec{y})\left(\psi^+(\vec{x})\frac{\lambda^\beta}{2}\psi(\vec{x})\right)\left(\psi^+(\vec{y})\frac{\lambda^\beta}{2}\psi(\vec{y})\right). \tag{3.6}$$

$\Lambda_\pm(\vec{k})$ represents the projection operators ($\hat{k} = \vec{k}/|\vec{k}|$)

$$\Lambda_\pm(\vec{k}) = \tfrac{1}{2}(1 \pm \beta\sin\phi(k) \pm \vec{\alpha}\cdot\hat{k}\cos\phi(k)), \tag{3.7}$$

and $\tilde{V}(\vec{k})$ denotes the Fourier transform of the potential

$$\tilde{V}(\vec{k}) = a^3 \sum_{\vec{x}} V(\vec{x})e^{i\vec{k}\cdot\vec{x}} = V_L(\vec{k}) + V_c(\vec{k}), \tag{3.8}$$

where [7]

$$V_L(\vec{k}) = \frac{8\pi\sigma}{(\vec{k})^4} \quad \text{and} \quad V_c(\vec{k}) = \frac{4\pi\alpha_s}{(\vec{k})^2}. \tag{3.9}$$

Substituting Eqs. (3.6–7) into Eq. (2.5), after taking the continuum limit, we obtain the nonlinear integral equation

$$k\sin\phi(k) = (1-Z)k\sin\phi(k)$$

$$+\tfrac{2}{3}\int \frac{d^3p}{(2\pi)^3} V(\vec{p}-\vec{k})\,(\sin\phi(p)\cos\phi(k) - \hat{k}\hat{p}\sin\phi(k)\cos\phi(p)). \tag{3.10}$$

Inserting the Fourier transform of the potential (3.8) into Eq. (3.10) we find that the ultraviolet divergence in the integral in Eq. (3.10) can be eliminated by means of the momentum independent renormalization constant

$$Z-1 = \frac{1}{p}\tfrac{2}{3}\int \frac{d^3p}{(2\pi)^3} V_c(\vec{p}-\vec{k})\hat{k}\hat{p}. \tag{3.11}$$

[2] The normal ordering in the model was studied in papers [6, 11].

Substituting Eq. (3.11) into Eq. (3.10) and after performing integrations over angles we arrive at the ultraviolet finite, one-dimensional equation

$$k \sin \phi(k) = \frac{2}{3\pi} \int dp (I_2^{\mathrm{L}}(k, p) \sin \phi(p) \cos \phi(k) - I_2^{\mathrm{L}}(k, p) \sin \phi(k) \cos \phi(p))$$

$$+ \frac{2}{3\pi} \int dp \left(I_2^{\mathrm{c}}(k, p) \sin \phi(p) \cos \phi(k) + 2 I_2^{\mathrm{c}}(k, p) \sin \phi(k) \sin^2 \frac{\phi(p)}{2} \right), \tag{3.12}$$

where

$$I_2^{\mathrm{L}}(k, p) = \int_{-1}^{1} d(\hat{k}\hat{p}) V^{\mathrm{L}}(\vec{k} - \vec{p}) = \frac{4\sigma p^2}{(k^2 - p^2)^2},$$

$$I_1^{\mathrm{L}}(k, p) = \int_{-1}^{1} d(\hat{k}\hat{p}) \hat{k}\hat{p} V^{\mathrm{L}}(\vec{k} - \vec{p}) = \frac{k^2 + p^2}{2kp} I_2^{\mathrm{L}}(k, p) - \frac{\sigma}{p^2} \log \left| \frac{p+k}{p-k} \right|,$$

$$I_2^{\mathrm{c}}(k, p) = \int_{-1}^{1} d(\hat{k}\hat{p}) V^{\mathrm{c}}(\vec{k} - \vec{p}) = -\frac{\alpha_s p}{k} \log \left| \frac{p+k}{p-k} \right|,$$

$$I_1^{\mathrm{c}}(k, p) = \int_{-1}^{1} d(\hat{k}\hat{p}) \hat{k}\hat{p} V^{\mathrm{c}}(\vec{k} - \vec{p}) = \frac{k^2 + p^2}{2kp} I_2^{\mathrm{c}}(k, p) - \frac{\alpha_s p}{k^2}. \tag{3.13}$$

The infrared finiteness of the equation ($p = k$) was discussed in Ref. [7]. It is insured by using the colourless trial states. As a consequence one can add arbitrary constant to the potential without changing the results. The contribution from constant potential vanishes between colour singlets, since it is proportional to the Casimir operator of the colour symmetry group.

4. *Numerical solution*

The gap equation (3.12) can be solved numerically by using the over-relaxed Gauss--Siedel algorithm, which is described by Adler and Piran [9]. Before solving the equation, we have to single out the $p = k$ singularity apparent in the first integral (in $I^{\mathrm{L}}_{(1,2)}$). After discretizing, the Eq. (3.12) takes the form

$$C_1 \sin \phi(k_i) + C_2 \sin \phi(k_i) \cos \phi(k_i) + C_3 \cos \phi(k_i) = 0, \tag{4.1}$$

with $C_{(1,2,3)}$ depending on ϕ. The second term in (4.1), which comes from the p-integration over a small neighbourhood of k_i, is introduced in order to avoid the singularity.

We find $\phi(k)$ by successive approximations starting with ansatz obeying the boundary conditions

$$\phi(0) = \frac{\pi}{2} \quad \text{and} \quad \phi(k \to \infty) \to \frac{b}{k^3}, \tag{4.2}$$

which can be derived from (3.12).

For each i we solve Eq. (4.1) by Newton iterations. Then the coefficients $C_{(1,2,3)}$ are computed by replacing the old value $\phi(k_i)$ with the new one according to the Gauss-Siedel algorithm, and so on. Our solution for $\phi(k)$ is depicted in Fig. 1 for a few values of α_s. In the case of the linear potential ($\alpha_s = 0$) the momentum space wave function behaves like $\sim 1/k^5$ for large k. After the addition of the Coulomb potential the large-k asymptotic behaviour is replaced by (4.2).

We can calculate the dynamical quark mass m^* and the order parameter $\langle \bar{u}u \rangle$ for a quark flavour u once the solution of the gap equation is known. The effective quasiparticle mass in the momentum units $1 \Rightarrow \sqrt{\sigma}$ is given by the formula [7]

$$\left(\frac{d\phi}{dk}\right)_{k=0} = -\frac{1}{m^*}. \tag{4.3}$$

The calculation of the quark condensate is more involved. After introducing a temporary ultraviolet momentum cutoff Λ we compute the $\langle \bar{u}u \rangle$ using the definitions (2.4) and (3.1)

$$\langle \bar{u}u \rangle = \frac{1}{Z} \langle \Omega | \bar{\psi}\psi | \Omega \rangle, \tag{4.4}$$

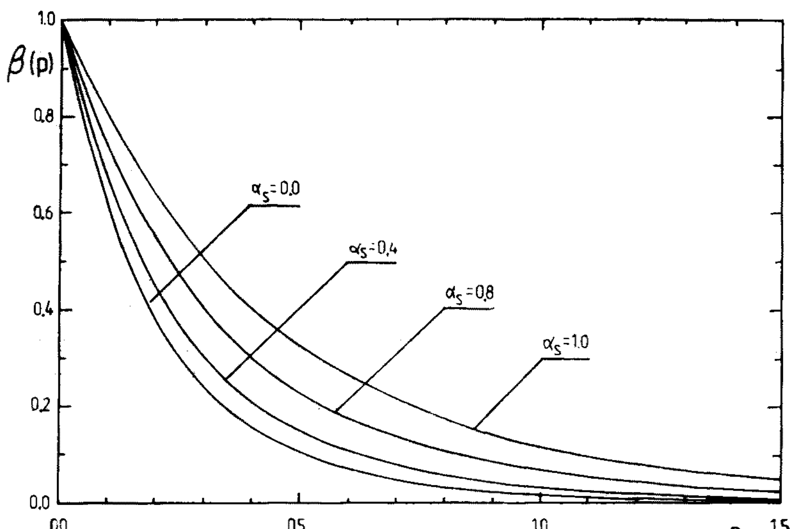

Fig. 1. The gap function $\beta(k) = \tan \dfrac{\phi(k)}{2}$ versus momentum k in units $1 = \sqrt{\sigma}$ for several values α_s

TABLE I

The quark condensate $\langle \bar{u}u \rangle$ and dynamical quark mass m^* for various α_s. At the bottom we put the experimental values taken from Ref. [12]

α_s	$\langle \bar{u}u \rangle$ (MeV^3)	m^* (MeV)
0.0	$-(95)^3$	70
0.4	$-(150)^3$	95
0.8	$-(210)^3$	120
1.0	$-(260)^3$	170
exp.	$-(225)^3$	300

then, taking the $\Lambda \to \infty$ we obtain

$$\langle \bar{u}u \rangle = -\frac{27}{8\pi}\frac{b}{\alpha_s}. \tag{4.5}$$

Applying the Eqs. (4.3), (4.4) we calculated $\langle \bar{u}u \rangle$ and m^* for some values of α_s — see Table I. In the calculations we chose $\sqrt{\sigma} = 350$ MeV.

5. *Discussion of the results*

From Table I we see that the chiral parameters increase reasonably after the addition of the Coulomb field. This is in agreement with earlier expectation [6–8]. In the first line of Table I we have displayed the results obtained by Adler and Davis [7] without V_c. The consistent growth with α_s may be easily understood. The inclusion of the Coulomb potential effectively increases the string tension and produces the higher values of the chiral parameters. Thus, the improvement is a result of the inclusion of the high momentum component to the potential $V(\vec{k})$. The strong influence of the Coulomb potential seems natural, since the main portion of the chiral symmetry comes from small Cooper pairs.

However, there are still discrepancies in this result, what is not surprising. The transverse gluons [7] and retardation effects have been neglected. The inclusion of the effects leads to a much more complicated equation which is difficult to study [10]. Also, taking into account the current quark masses should improve the results.

Since the Coulomb field makes the model more realistic we plan to solve the Bethe-Salpeter equation for the potential (3.8). Also it would be interesting to see how the Coulomb field influences restoration of the chiral symmetry.

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