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ON KERR-SCHILD EINSTEIN-MAXWELL FIELDS

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A general Kerr-Schild metric $g_{ik} = \eta_{ik} + H\xi_i\xi_k$ is considered in connection with Einstein-Maxwell field equations. The vector ξ_i generates a shear-free null geodetic congruence both in Minkowski space and in the Kerr-Schild space-times. In addition, we have assumed that ξ_i is hypersurface orthogonal. Two types of exact solutions are obtained. One is the solution for an accelerated charge given by Bonnor and Vaidya with a Λ-term. The other is a similar solution where a space-like curve plays the role of the time-like curve describing the world line of the accelerating charge. Taub's solutions describing high-frequency gravitational radiation in Kerr-Schild space-times are derived as particular cases.

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1. *Introduction*

In general relativity, the field equations are often simplified when we deal with null vector fields. One important and noteworthy example is the Kerr-Schild (1965) space-times. The metric tensor of such space-times can be expressed in the form,

$$g_{ik} = \eta_{ik} + H\xi_i\xi_k, \tag{1}$$

where η_{ik} is the flat metric tensor in the Cartesian co-ordinate system, H is a function of co-ordinates and ξ_i is a null geodetic vector field. Kerr-Schild space-times have been investigated by many relativists and a vast amount of literature is available on these space-times. One of the reasons for the importance of these space-times is that the familiar solutions like the Schwarzschild solution, Reissner-Nordstrom solution. Kerr (1963) solution, Vaidya (1943) radiating-star solution etc. are of Kerr-Schild form.

Kerr and Schild (1965) have considered the space-time with metric tensor (1) and obtained exact solutions of the vacuum field equations. These include the Kerr solution, from which the Schwarzschild solution follows as a particular case. Misra (1970) has given a unified treatment of Kerr and Vaidya solutions. Kowalczyński and Plebański (1977) have studied Kerr-Schild type solutions of $R_{ik} = \Lambda g_{ik}$ where Λ is the cosmological constant.

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Bonnor and Vaidya (1970, 1972), Debney, Kerr and Schild (1969), Białas (1963), Debney (1974), Patel and Misra (1975) and Patel and Koppar (1986) have discussed various aspects of Einstein-Maxwell fields in terms of Kerr-Schild type metrics.

Kerr-Schild metrics with energy momentum tensor of pure radiation are investigated by Vaidya (1972, 1973, 1974), Vaidya and Patel (1973), Kinnersley (1969), Urbantke (1975) and Herlt (1980). Taub (1976) has considered the high-frequency gravitational radiation in Kerr-Schild space-times.

The principal aim of the present investigation is to obtain the electromagnetic generalizations of the solutions derived by Taub (1976).

In the present paper we give such generalizations in the form of exact solutions of Einstein-Maxwell equations

$$R_{ik} = -8\pi[E_{ik}+\sigma\xi_i\xi_k]+\Lambda g_{ik}, \qquad \xi_i\xi^i = 0, \tag{2}$$

$$F^{ik}{}_{;k} = 4\pi J^i \tag{3}$$

in terms of the metric (1) where ξ_i is a null shear-free, geodetic and hypersurface orthogonal vector field. Here the semicolon indicates covariant derivative. F_{ik} is the electromagnetic field tensor, J^i is the 4-current vector, E_{ik} is the electromagnetic energy tensor, Λ is the cosmological constant and $\sigma\xi_i\xi_k$ is the tensor arising out of following null radiation.

We shall be freely using the method of real tetrad introduced by Vaidya (1972, 1974). The next Section is devoted to a very brief description of this method and other related results.

2. *The real tetrad method*

Consider Minkowskian space-time with signature -2 and assume that it is pervaded by a null geodetic and shear-free congruence ξ_i so that

$$\xi_i\xi^i = 0, \; \xi_{i,k}\xi^k = 0, \qquad (\xi^i_{,k}+\xi_{k,l}\eta^{li})\xi^k_{,i}-(\xi^i_{,i})^2 = 0 \tag{4}$$

a comma indicating ordinary derivative.

We use the geometrical framework developed by Vaidya (1972) to obtain a real tetrad system in the Minkowskian space-time appropriate to the congruence ξ_i. In such a space-time we can always obtain four uniform vector fields such that (*i*) any two of them are mutually orthogonal and (*ii*) one of them is time-like and the other three are space-like. Let λ^i be the unit tangent to the time-like vector field through a point P (co-ordinates x^i) and A^i, B^i, C^i be the unit tangents to the three space-like vector fields. We raise or lower indices with the aid of η^{ik} or η_{ik}. These four uniform vector fields give rise to a Euclidean reference frame with co-ordinates x, y, z, t for P such that

$$x_{,i} = A_i, \quad y_{,i} = B_i, \quad z_{,i} = C_i, \quad t_{,i} = \lambda_i.$$

Let us denote by S the 3-flat at right angles to λ^i at the point P. If l_i is the projection of ξ_i on S at P, we shall take $\xi_i = \lambda_i+l_i$. In the 3-flat S let l_i have the spherical angles α and β with respect to the triad A_i, B_i, C_i.

We can now define an orthonormal triad l_i, $\bar{l}_i$, $\bar{m}_i$ as follows:

$$l_i = \sin\alpha\, m_i + \cos\alpha\, C_i, \qquad \bar{l}_i = \cos\alpha\, m_i - \sin\alpha\, C_i,$$

$$m_i = \cos\beta\, A_i + \sin\beta\, B_i, \qquad \bar{m}_i = -\sin\beta\, A_i + \cos\beta\, B_i.$$

The derivatives of these vectors are listed in the Appendix B. We take α and β as functions of x^i. The conditions (4) after some simplification will lead to

$$\alpha_{,i}\xi^i = 0, \qquad \beta_{,i}\xi^i = 0 \tag{5}$$

and

$$\bar{l}^i\alpha_{,i} - m^i \sin\alpha\, \beta_{,i} = 0, \qquad \bar{m}^i\alpha_{,i} + \bar{l}^i \sin\alpha\, \beta_{,i} = 0. \tag{6}$$

It can be verified that if we define

$$u = x \sin\alpha \cos\beta + y \sin\alpha \sin\beta + z \cos\alpha + t,$$

$$V = x \cos\alpha \cos\beta + y \cos\alpha \sin\beta - z \sin\alpha,$$

$$W = x \sin\beta - y \cos\beta,$$

then $u_{,i}\xi^i = 0$, $V_{,i}\xi^i = 0$, $W_{,i}\xi^i = 0$.

Therefore, the conditions (5) can be integrated and exhibited in the form

$$V = V(u, \alpha, \beta), \qquad W = W(u, \alpha, \beta).$$

In terms of the variables V and W, the conditions (6) may be written as

$$VV_u - WW_u + V_\alpha - V\cot\alpha + W_\beta \operatorname{cosec}\alpha = 0, \tag{7}$$

$$WV_u + VW_u + W_\alpha - W\cot\alpha - V_\beta \operatorname{cosec}\alpha = 0, \tag{8}$$

where $V_u = \dfrac{\partial V}{\partial u}$ etc. We take θ and Ω, the expansion and rotation parameters for the congruence ξ_i as

$$\theta = \xi^i_{,i}, \qquad \Omega^2 = (\xi_{i,k} - \xi_{k,i})\xi^i_{,l}\eta^{kl}.$$

The principal results of this real tetrad method which are used in the present paper are reproduced in the Appendix B for ready reference.

Let us now assume further that ξ_i is hypersurface orthogonal. Taub (1976) has shown that this condition is equivalent to $\Omega = 0$. Using the results of Appendix B, it can be seen that $\Omega = 0$ implies

$$VW_u + W_\alpha = 0. \tag{9}$$

We shall limit our discussion to the solutions of equations (7), (8) and (9) to the situation when $V_\beta = 0$. Taub (1976) has shown that this gives rise to two cases: case (a): $W = 0$

and case (b): $V_u = \cot\alpha$. He has integrated these equations for the two above-mentioned cases.
For case (a)

$$V = \sin\alpha\, h(g), \tag{10}$$

where h is an arbitrary non constant function and

$$g = u + V\cot\alpha = u + h(g)\cos\alpha. \tag{11}$$

For case (b),

$$U = K(w)\sin\alpha\sin(w-\beta),$$

$$V = K(w)\cos\alpha\sin(w-\beta),$$

$$W = K(w)\cos(w-\beta), \tag{12}$$

where K is an arbitrary function of w and the function w satisfies $w_{,i}\xi^i = 0$.

The remaining part of the paper will be devoted to the derivation of the solutions of (2) and (3) in terms of the metric (1) for the two above-mentioned cases.

Here, it should be noted that the solutions discussed by Vaidya and Patel (1973), Herlt (1980), Patel and Misra (1975) and Patel and Koppar (1986) do not satisfy the hyper-surface orthogonal condition.

3. *The Maxwell field*

We now take the Maxwell equations in a Riemannian space-time described by the metric (1) in which ξ_i is null, geodetic, shear-free and hypersurface orthogonal. It is easy to see that

$$g^{ik} = \eta^{ik} - H\xi^i\xi^k, \qquad g = |g_{ik}| = -1. \tag{13}$$

The results (3), (13) and $\Omega = 0$ will be frequently used without mention.

Here it should be noted that the congruence ξ_i continues to be null, geodetic, shear-free and hypersurface orthogonal in the Kerr-Schild space-time described by (1) (see Vaidya, 1974). We choose the electromagnetic 4-potential ϕ_i as

$$\phi_i = P\theta\xi_i, \qquad P_{,a}\xi^a = 0. \tag{14}$$

A simple calculation using (14) leads to the following expression for the electromagnetic field tensor

$$F_{ik} = \phi_{i,k} - \phi_{k,i}:$$

$$F_{ik} = (P\theta)_{,k}\xi_i - (P\theta)_{,i}\xi_k + P\theta(\xi_{i,k} - \xi_{k,i}).$$

The electromagnetic energy tensor can be easily computed. It is given by

$$E_{ik} = [\tfrac{1}{8}HP^2\theta^2 - \eta^{ab}(P\theta)_{,a}(P\theta)_{,b}]\xi_i\xi_k$$

$$-\tfrac{1}{2}P\theta^2[(P\theta)_{,i}\xi_k+(P\theta)_{,k}\xi_i]$$
$$-P\theta\eta^{ab}(P\theta)_{,a}[\xi_k(\xi_{i,b}-\xi_{b,i})+\xi_i(\xi_{k,b}-\xi_{b,k})]$$
$$-P^2\theta^2\eta^{ab}(\xi_{i,a}-\xi_{a,i})(\xi_{k,b}-\xi_{b,k})-\tfrac{1}{8}P^2\theta^4\eta_{ik}. \tag{15}$$

Let us now turn our attention to the Maxwell equations (2). Patel and Koppar (1986) have shown that $F^{ik}{}_{,k}\xi_i = 0$. They have also proved that the results $F^{ik}{}_{;k}l_i = 0$ and $F^{ik}{}_{,k}\bar{m}_i = 0$ give the following two differential equations for the function P:

$$P_\alpha+VP_u+2PV_u = 0 \tag{16}$$

and

$$WP_u+2PW_u-P_\beta \operatorname{cosec}\alpha = 0. \tag{17}$$

We shall now try to integrate these two equations for case (a) and case (b).

Case (a)

Using (10) and (11), it is easy to see that $Vg_u+g_\alpha = 0$. With the aid of this result along with (10) and (11), it is easy to integrate (16) and (17). The solution can be expressed as

$$2P = e(g)\,[1-h'(g)\cos\alpha]^{-2}, \tag{18}$$

where the prime indicates the derivative of $h(g)$ with respect to its argument and $e(g)$ is an arbitrary differentiable function of g. For this case it is easy to see that

$$V_u = h'\sin\alpha(1-h'\cos\alpha)^{-1}, \quad \theta = -\frac{2}{r}.$$

Here, and in what follows, r is defined by $r = g-t$. Using the above results and the results of Appendix B, the 4-current vector J^i can be evaluated. It is given by

$$4\pi J^i = -\frac{1}{r^2}\{e'(1-h'\cos\alpha)+2h''e\}(1-h'\cos\alpha)^{-4}\xi^i, \tag{19}$$

where again $e' = \dfrac{de}{dg}$.

Case (b)

From the results (12), it is easy to check that

$$w_u = \varrho^{-1}, \quad w_\alpha = -V\varrho^{-1}, \quad w_\beta \operatorname{cosec}\alpha = W\varrho^{-1}, \tag{20}$$

where

$$\varrho = \sin\alpha\,[K'\sin(w-\beta)+K\cos(w-\beta)] \tag{21}$$

and K' denotes the derivative of $K(w)$ with respect to its argument. In view of the results (12), (20) and (21) Eqs. (16) and (17) admit the solution

$$2P = e(w)\varrho^{-2}, \tag{22}$$

where ϱ is given by (21) and $e(w)$ is an arbitrary differentiable function of w. In this case the expansion parameter θ is found to be $\theta = -\dfrac{2}{t}$. The 4-current vector J^i in this case can be expressed as

$$4\pi J^i = 2(2eA-e')t^{-2}\varrho^{-3}\xi^i, \quad e' = \frac{de}{dw}, \tag{23}$$

where ϱ is given by (21). Here, and in what follows, the symbol A is defined by

$$A = \sin\alpha\{(K''-K)\sin(w-\beta)+2K'\cos(w-\beta)\}. \tag{24}$$

In both the cases the 4 current vector J^i is a null vector.

In the next Section we shall direct our attention to the Einstein-Maxwell field.

4. *The Einstein-Maxwell field*

Vaidya (1974) has computed the components R_{ik} of Ricci tensor for the metric (1). They are listed in Appendix A for the reference. From these expressions for R_{ik} it is easy to see that the curvature scalar R is given by the amazingly simple expression

$$R = g^{ik}R_{ik} = (H\theta+h)_{,a}\xi^a+(H\theta+h), \tag{25}$$

where $h = H_{,a}\xi^a$.

We first note that the field equation (2) implies $R = 4\Lambda$. This equation can be easily integrated. The solution can be expressed as

$$H = L\theta+N\theta^2-\frac{4\Lambda}{3\theta^2}, \tag{26}$$

where L and N are arbitrary function of u, α and β to be determined from the field equations (2).

From the field equations (2) one can see that the vectors ξ^i, $\bar{l}^i$, $\overline{m}^i$ are eigenvectors of $(R_{ik}+8\pi E_{ik}-\Lambda g_{ik})$, the corresponding eigenvalues being zero. Substituting E_{ik} from (15) and R_{ik} from Appendix A in the equation $(R_{ik}+8\pi E_{ik}-\Lambda g_{ik})\xi^i = 0$, we get the following relation between the functions P and N:

$$N = 4\pi P^2. \tag{27}$$

Using the results of Appendix B, the equations $(R_{ik}+8\pi E_{ik}-\Lambda g_{ik})\bar{l}^i = 0$, $(R_{ik}+8\pi E_{ik}-\Lambda g_{ik})\overline{m}^i = 0$ are considerably simplified. A straightforward calculation will lead to the following two differential equations for the function L:

$$L_\alpha+VL_u+3LV_u = 0, \tag{28}$$

$$WL_u+3LW_u-L_\beta \operatorname{cosec}\alpha = 0. \tag{29}$$

Eqs. (28) and (29) are similar to Eqs. (16) and (17). These equations can be easily integrated for both the cases. For case (a), it is found that

$$L = m(g)\,[1-h'(g)\cos\alpha]^{-3}, \tag{30}$$

where $m(g)$ is an arbitrary differentiable function of g. For case (b) L is given by

$$L = m(w)\varrho^{-3}, \tag{31}$$

where $m(w)$ is an arbitrary differentiable function of w and ϱ is given by (21).

Using the relevant results of this section and those given in Appendix B, in the equation

$$(R_{ik}+8\pi E_{ik}-\Lambda g_{ik})\lambda^i\lambda^k = -8\pi\sigma,$$

one can determine the radiation density σ for both the cases.

The value of σ for the case (a) is given by

$$\begin{aligned}8\pi\sigma = {} & 2r^{-3}(1-h'\cos\alpha)^{-5}[r\{m'(1-h'\cos\alpha)+3h''m\}\\ & +4\pi e\{e'(1-h'\cos\alpha)+2h''e\}],\end{aligned} \tag{32}$$

where

$$m' = \frac{dm}{dg} \quad \text{and} \quad e' = \frac{de}{dg}.$$

The value of σ for the case (b) is given by

$$8\pi\sigma = 2t^{-3}\varrho^{-6}\{t\varrho(\varrho m'-3mA)+4\pi e(\varrho e'-2eA)\}, \tag{33}$$

where

$$m' = \frac{dm}{dw}, \quad e' = \frac{de}{dw},$$

and ϱ and A are given by (21) and (24), respectively.

When the electromagnetic field is switched off (i.e. when $e(g) = 0$ for case (a) and $e(w) = 0$ for the case (b)) the expressions (32) and (33) agree with those given by Taub (1976). This accomplishes the task of solving the Einstein-Maxwell equations (2).

5. *Discussion*

(i) Case (a)

For this case it can be easily seen that

$$H = -\frac{2m(g)}{r}(1-h'\cos\alpha)^{-3}+\frac{4\pi e^2(g)}{r^2}(1-h'\cos\alpha)^{-2}-\tfrac{1}{3}\Lambda r^2. \tag{34}$$

In order to write down the Kerr-Schild metric in an explicit form we use (g, α, β, r) as co-ordinates of an event in the space-time. Making use of (34), the line-element for case (a) can be expressed in the form:

$$ds^2 = -r^2(d\alpha^2+\sin^2\alpha d\beta^2)-2(1-h'\cos\alpha)dg\,dr-2rh'\sin\alpha dg\,d\alpha$$
$$+\left[1-h'^2-\frac{2m(g)}{r(1-h'\cos\alpha)}+\frac{4\pi e^2(g)}{r^2}-\tfrac{1}{3}\Lambda r^2(1-h'\cos\alpha)^2\right]dg^2. \tag{35}$$

The solution described by the metric (35) is the Λ-term generalization of the accelerated charge solution of Bonnor and Vaidya (1972).

When $h' = 0$, $V_u = 0$ and consequently we have $V = k\sin\alpha$ where k is an integration constant which by the Lorentz transformation can be made to vanish. So, $V = 0$ and the metric (35) reduces to

$$ds^2 = -r^2(d\alpha^2+\sin^2\alpha d\beta^2)-2du\,dr+\left[1-\frac{2m(u)}{r}+\frac{4\pi e^2(u)}{r^2}-\tfrac{1}{3}\Lambda r^2\right]du^2. \tag{36}$$

The metric (36) describes the Λ-term generalization of the solution discussed by Bonnor and Vaidya (1970) in connection with the spherically symmetric radiation of charge. When $e(g) = 0$ in (35), we get the metric describing the Λ-term generalization of the accelerated point mass solution of Kinnersley (1969). If $e(u) = 0$ in (36), we recover Vaidya's shining--star metric. Further, if m and e are constants, the radiation density σ and 4-current vector J^i become zero, and we get the Reissner-Nordstrom solution with a cosmological constant. If $m = e = 0$ in (36) we get the well-known de Sitter metric.

(*ii*) Case (b)

For this case it can be seen that

$$H = 2m(w)t^{-1}\varrho^{-3}+4\pi e^2(w)t^{-2}\varrho^{-4}-\tfrac{1}{3}\Lambda t^2, \qquad \xi_i dx^i = dw, \tag{37}$$

where ϱ is given by (21).

One can use w, α, β and t as the co-ordinates and write down the metric explicitly in terms of these coordinates. The metric for case (b) can be expressed as

$$ds^2 = -t^2(d\alpha^2+\sin^2\alpha d\beta^2)+2\varrho dwdt-2t\varrho dw(\cot\alpha d\alpha-W_u\sin\alpha d\beta)$$
$$-\varrho^2 dw^2\left[1+\cot^2\alpha+W_u^2-\frac{2m(w)}{t\varrho^3}-\frac{4\pi e^2(w)}{t^2\varrho^4}+\tfrac{1}{3}\Lambda t^2\right], \tag{38}$$

where $\varrho W_u = K'\cos(w-\beta)-K\sin(w-\beta)$ and ϱ is given by (21). The metric (38) describes the geometry of electromagnetic generalization of the solution discussed by Taub (1976). When $e(w) = 0$, the metric (38) represents Taub's solution with a non-zero cosmological constant.

In the absence of electromagnetic field, Taub (1976) has shown that a space-like curve (in the geometry of case (b)) plays the role of the time-like curve describing the world line of the accelerated mass in case (a). The same conclusion is true in the presence of electromagnetic field also. The arguments for arriving at this conclusion are the same as those made by Taub (1976). Hence, we shall not repeat them here.

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APPENDIX A

$$\begin{aligned}-2R_{ik} = {} & (H\theta+h)_{,i}\xi_k+(H\theta+h)_{,k}\xi_i+(H\theta+h)(\xi_{i,k}+\xi_{k,i}) \\ & -2H\eta^{ab}\xi_{i,a}\xi_{k,b} \\ & -\eta^{ab}[H(\xi_{i,ab}\xi_k+\xi_{k,ab}\xi_i) \\ & -2H_{,b}(\xi_{i,a}\xi_k+\xi_{k,a}\xi_i)] \\ & +[Hh\theta+Hh_{,a}\xi^a-\eta^{ab}H_{,ab}+H^2\Omega^2]\xi_i\xi_k.\end{aligned}$$

APPENDIX B

$$\xi_{i,k} = l_{i,k} = \alpha_{,k}\bar{l}_i+\sin\alpha\beta_{,k}\bar{m}_i,$$

$$\bar{l}_{i,k} = -l_i\alpha_{,k}+\cot\alpha\sin\alpha\beta_{,k}\bar{m}_i,$$

$$m_{i,k} = -\beta_{,k}\bar{m}_i, \qquad \bar{m}_{i,k} = -\beta_{,k}m_i,$$

$$\theta(\theta^2+\Omega^2)^{-1} = -\tfrac{1}{2}(VV_u+V_\alpha+u-t),$$

$$\Omega(\theta^2+\Omega^2)^{-1} = \tfrac{1}{2}(VW_u+W_\alpha),$$

$$\alpha_{,i} = -\frac{\theta}{2}(\bar{l}_i-V_u\xi_i)-\frac{\Omega}{2}(\bar{m}_i+W_u\xi_i),$$

$$\sin\alpha\beta_{,i} = \frac{\Omega}{2}(\bar{l}_i-V_u\xi_i)-\frac{\theta}{2}(\bar{m}_i+W_u\xi_i),$$

$$\theta_{,i}\xi^i = \tfrac{1}{2}(\Omega^2-\theta^2), \qquad \Omega_{,i}\xi^i = -\theta\Omega,$$

$$\theta_{,k}\bar{l}^k+\Omega_{,k}\bar{m}^k = \Omega(\Omega V_u+\theta W_u),$$

$$\theta_{,k}\bar{m}^k-\Omega_{,k}\bar{l}^k = \Omega(\theta V_u-\Omega W_u),$$

$$\eta^{ab}\alpha_{,ab}+\tfrac{1}{4}(\theta^2+\Omega^2)\cot\alpha = 0,$$

$$\eta^{ik}(\sin\alpha\beta_{,i})_{,k} = 0, \qquad \eta^{ik}\xi^a{}_{,ik} = \tfrac{1}{2}(\theta^2+\Omega^2)l^a.$$

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