POLYAKOV BOSONIC STRINGS*,**

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(*Received May 5, 1987; final version received January 19, 1988*)

Certain aspects of the pedagogical character of the Polyakov bosonic strings are discussed. In particular we use expansion in terms of the eigenfunctions of the operators ∇^2, P^+P and PP^+ to study: a) the ambiguity present in the definition of the path integral measure; b) Weyl invariance as the local property. Next, one-loop calculation is discussed and, as its new illustration, the dilaton tadpole amplitudes for open oriented and unoriented strings and the four-graviton amplitude for the closed oriented string are presented.

PACS numbers: 11.17.+y

The first three Sections of this paper contain a slightly different presentation than in Ref. [1–3] of a) the ambiguity present in the definition of the path integral measure; b) Weyl invariance as the local property. In Section 4 we calculate the 1-loop vacuum amplitude for closed and open bosonic strings recovering for the closed strings the result of Ref. [2]. In the last Section we present the explicit result for the 1-loop four-graviton amplitude in the closed oriented string theory and for one-dilaton tadpole for the open string (both oriented and unoriented). When completing this paper we became aware of Ref. [4, 5] where some of those points are also discussed in the way similar to ours.

1. *Polyakov string and gauge redundance*

The Polyakov path integral formulation of the bosonic open or closed string theory starts with the Euclidean path integral for the vacuum-to-vacuum transitions

$$W = \sum_{\rm top} DgDX^\mu e^{-S}/(\mathscr{V}_{\rm c}\mathscr{V}_{\rm D}). \tag{1}$$

* Based in part on a lecture given at IX Warsaw Symposium on Elementary Particle Physics, Kazimierz, May 1986.

** Partially supported by the Ministry of Science and Higher Education under the contract CPBP 01.03.

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Equation (1) deserves longer discussion. The integral is over the space of all metrics on the manifold M of a given topology and over all maps $X^\mu(\sigma_1, \sigma_2)$ to space-time from the two-dimensional manifold M. The action S is given by the classical string action and by the quantum counterterms

$$S = \frac{T}{2} \int_M d^2\sigma \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\mu + \text{counterterms} \tag{2}$$

which we will discuss later on. Classically, the first piece gives $g_{ab} = \gamma_{ab}$ where γ_{ab} is the induced metric,

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\mu}{\partial \sigma^b} \tag{3}$$

and it is equivalent to the Nambu-Goto expression for the string action as an integral over the area element of the world sheet of the string.

Classically, the string action has several symmetries. It is invariant under the group of diffeomorphisms i.e. differentiable one-to-one mappings of M into itself

$$\sigma^a \in M \to \sigma'^a \in M$$

such that

$$X'(\sigma') = X(\sigma), \quad g'_{ab}(\sigma') d\sigma'^a d\sigma'^b = g_{ab}(\sigma) d\sigma^a d\sigma^b \tag{4}$$

which generate the transformation

$$g_{ab}(\sigma) \to g'_{ab}(\sigma), \quad X^\mu(\sigma) \to X'^\mu(\sigma). \tag{5}$$

We take here the active view at the symmetry transformations i.e. keep the coordinates on the world sheet fixed and change the fields $g_{ab}(\sigma)$ and $X^\mu(\sigma)$. Classical action has also Weyl symmetry which means invariance under transformations

$$X'(\sigma) = X(\sigma), \quad g'_{ab}(\sigma) = e^{\phi(\sigma)} g_{ab}(\sigma). \tag{6}$$

It is our physical requirement that quantum theory preserves classical symmetries. This demand is quite natural as far as the world sheet reparametrization invariance is concerned and easy to satisfy at least for diffeomorphisms homotopic to identity. Weyl symmetry is more difficult to preserve: as it is well known it constrains the dimension of the space--time to be $D = 26$. The physical origin of Weyl symmetry can be understood as follows: since there are no "internal forces" (that keep the string together), expanding or contracting the string from the two-dimensional point of view should not give rise to any change in the lagrangian; otherwise strings would not be fundamental. From the D-dimensional point of view, expansion or contraction of strings needs some kinetic energy. Therefore, the D--dimensional metric has its scale fixed in terms of the string tension T.

With the two symmetries present at the quantum level we see that the integral over g_{ab} counts physically equivalent configurations related by the Weyl transformations and by the group of coordinate transformations. The volumes of the two groups can be factored out and divided out. This is the origin of the factors $\mathscr{V}_c$ and $\mathscr{V}_D$ in the denominator of Eq. (1). Nearly all the metric degrees of freedom are eliminated this way and only finite number of physically inequivalent degrees of freedom remains.

The full volume $\mathscr{V}_D$ should be discussed in some detail. Several examples given later will illustrate the discussion which follows. We have

$$\mathscr{V}_D = \mathscr{V}_{D_0}\mathscr{V}_M, \tag{7}$$

where $\mathscr{V}_{D_0}$ is the volume of the subgroup of diffeomorphisms which are homotopic to identity and $\mathscr{V}_M$ is the order of some discrete group of the remaining diffeomorphisms, called the group of modular transformations, which are global (disconnected) coordinate transformations. We can now consider the space of all metrics g_{ab} on the manifold and the ratio

$$\mathrm{G}/\mathrm{D}_0 \times \mathrm{Weyl} \tag{8}$$

which includes all the metrics inequivalent under Weyl and D_0 transformations. This subspace of G is called the Teichmüller space and it is finite dimensional. It also contains the metrics related by the discrete group $\hat{M}$ of modular transformations. If we introduce the Teichmüller variables τ_i, $i = 1, \ldots, t$, which enter the path integral through the gauge-fixed metric $e^{\phi(\sigma)}g_{ab}(\tau_i)$, then we conclude that some values of τ_i are related by the modular transformations and they are therefore physically equivalent. Physically equivalent values of τ_i must be eliminated by the proper choice of the range of integration over the Teichmüller parameters. This range is called the moduli space.

Final point about Eq. (1) is how to define the functional measures Dg_{ab}, DX^μ and also others which will appear in the subsequent discussion. There is no reference to Schrödinger equation like in ordinary quantum mechanics and one proceeds as follows [2]. First the space of small variations in the fields is converted into Hilbert space by defining the scalar products

$$||\delta g||^2 = \int d^2\sigma \sqrt{g}\,(G^{abcd} + Kg^{ab}g^{cd})\delta g_{ab}\delta g_{cd},$$
$$||\delta x||^2 = \int d^2\sigma \sqrt{g}\,\delta X^\mu \delta X^\mu, \tag{9}$$

where $G^{abcd} = \frac{1}{2}(g^{ac}g^{bd} + g^{ad}g^{bc} - g^{ab}g^{cd})$ is the projecting operator on the space of traceless symmetric tensors and K is an arbitrary constant which we set to zero in the following. Then the measure is defined implicitly in terms of the values of the Gaussian integrals

$$\int d\delta g e^{-\pi a_g||\delta g||^2} = 1, \quad \int d\delta X e^{-\pi a_x||\delta X||^2} = 1, \tag{10}$$

where a_g and a_x are arbitrary constants. (The measures in the tangent space $d\delta g$ and $d\delta X$ can be identified with the measures Dg and DX because the spaces are linear.) We will show later that the final results do not depend on the values of constants a_x and a_g. Here we only notice that a change $a_x \to a_x'$ changes the measure by

$$\sqrt{\det \frac{a_x}{a_x'}}$$

taken in some infinite dimensional space. We will show that this determinant, after regularization, can be absorbed into the counterterms present in Eq. (1). We also observe that the measures defined by (9) and (10) are not Weyl invariant.

To perform any concrete calculations we must divide out the volumes $\mathscr{V}_C$ and $\mathscr{V}_D$ in Eq. (1) explicitly. This requires certain care in change of variables of integration and in counting zero modes of various operators. In metric (9) the general variation δg_{ab} admits an orthogonal decomposition

$$\delta g_{ab} = \delta c g_{ab} + \delta h_{ab}, \tag{11}$$

where δh_{ab} is a traceless symmetric tensor. We can define separately

$$\|\delta c\|^2 = \int d^2\sigma \sqrt{g}\, \delta c^2, \qquad \|\delta h\|^2 = \int d^2\sigma \sqrt{g}\, g^{ac} g^{bd} \delta h_{ab} \delta h_{cd} \tag{12}$$

and the tensor δh admits further orthogonal decomposition

$$\delta h_{ab} = (P\delta V)_{ab} + \delta t_i (\Psi^i)_{ab} \tag{13}$$

into a part which can be generated by some vector δV and a part which cannot be generated this way. Explicitly we have

$$\begin{aligned}(PV)_{ab} &= (\nabla_a V_b + \nabla_b V_a - g_{ab} \nabla^c V_c) \\ &= (\nabla_a g_{bc} + \nabla_b g_{ac} - g_{ab} \nabla_c) V^c,\end{aligned} \tag{14}$$

where ∇_a is the covariant derivative. The tensors Ψ_i form an orthonormal basis (with respect to (12)) in the space orthogonal to PV. Since any infinitesimal diffeomorphism is generated by a contravariant vector

$$\sigma^a \to \sigma^a + \delta V^a,$$

we conclude that $P\delta V$ describes the effects of diffeomorphisms on δh and $\delta t_i V^i$ — the effects of varying the Teichmüller degrees of freedom. Since the adjoint operator (with respect to the norm

$$\|\delta V\|^2 = \int d^2\sigma \sqrt{g}\, g_{ab} \delta V^a \delta V^b. \tag{15}$$

transforms traceless tensors δh into vectors δV, the tensors Ψ^i are zero modes of P^+. We rewrite, therefore, relation (13) as

$$\delta h_{ab} = \delta h^\perp_{ab} + \delta t_i (\Psi^i)_{ab},$$

where $\delta h^\perp_{ab}$ denotes the space of tensors δh_{ab} orthogonal to the kernel of the operator P^+. Explicitly, for any symmetric traceless tensor h

$$(P^+ h)_a = -2\nabla^b h_{ba} = -2 g_a{}^c \nabla^b h_{bc} \tag{16}$$

and Ψ^i are solutions to the equations

$$g^{ab} h_{ab} = 0 \quad \text{and} \quad \nabla^a h_{ab} = 0. \tag{17}$$

For later use we also record the formulae[1]

$$(P^+PV)_a = -2\nabla^b(\nabla_a V_b+\nabla_b V_a-g_{ab}\nabla^c V_c)$$
$$= -2(\nabla^b\nabla_b+\tfrac{1}{2}R)V_a$$

or

$$(P^+P)_{ab}V^b = -2g_{ab}(g^{cd}\nabla_d\nabla_c+\tfrac{1}{2}R)V^b \tag{18}$$

and

$$(PP^+h)_{ab} = -2(\nabla^b\nabla_b-R)h_{ab}. \tag{19}$$

Using (11) and (13) the measure Dg can be easily expressed in terms of δc, $P\delta V$ and δt_i and it reads

$$Dg = DcD(PV)_{ab}\prod_i \delta t_i. \tag{20}$$

These variables are however, not convenient from the point of view of dividing by volumes of the Weyl group and "small" (i.e. homotopic to identity) diffeomorphism group which are generated by the Weyl variable $\phi(\sigma)$ ($g \to e^\phi g$) and by the vector δV_a, respectively. Also a more convenient form of the measure for the Teichmüller deformations can be obtained if we express them through the gauge-fixed metric $e^\phi \hat{g}(\tau_i)$. We consider then a change

$$\delta g_{ab} = \frac{\partial\hat{g}_{ab}}{\partial\tau_i}\delta\tau_i = \tfrac{1}{2}\,\delta\tau_i\hat{g}_{ab}\hat{g}^{cd}\frac{\partial\hat{g}_{cd}}{\partial\tau_i}+\left(\frac{\partial\hat{g}_{ab}}{\partial\tau_i}\delta\tau_i-\tfrac{1}{2}\,\delta\tau_i\hat{g}_{ab}\hat{g}^{cd}\frac{\partial\hat{g}_{cd}}{\partial\tau_i}\right) \tag{21}$$

which we have again written as a superposition of the traceless part and the remaining part. Since for the infinitesimal diffeomorphism we have

$$\delta g_{ab} = \nabla_a\delta V_b+\nabla_b\delta V_a = (\nabla^c\delta V_c)g_{ab}+(P\delta V)_{ab} \tag{22}$$

the full change δg_{ab} reads

$$\delta g_{ab} = \left(\delta\phi+\nabla^c\delta V_c+\tfrac{1}{2}\,\delta\tau_i\hat{g}_{ab}\hat{g}^{cd}\frac{\partial\hat{g}_{cd}}{\partial\tau_i}\right)\hat{g}_{ab}+(P\delta V)_{ab}$$
$$+\left(\frac{\partial\hat{g}_{ab}}{\partial\tau_i}\delta\tau_i-\tfrac{1}{2}\,\delta\tau_i\hat{g}_{ab}\hat{g}^{cd}\frac{\partial\hat{g}_{cd}}{\partial\tau_i}\right). \tag{23}$$

We are now ready to perform the change of variables

$$\delta c,\ \delta h^{\perp}_{ab},\ \delta t^i \to \delta\phi,\ \delta V^c,\ \delta\tau_i, \tag{24}$$

where we define

$$\|\delta\tau_i\|^2 = \int d^2\sigma\,\sqrt{g}\,d\tau_i d\tau_i$$

[1] $R^a{}_{bcd} = \partial_c\Gamma^a{}_{bd} - \ldots,\ R_{bd} = R^a{}_{bad},\ R = R^a{}_a.$

and therefore

$$\int \prod_i^t d\tau_i \exp\left(-\pi \sum \|\delta\tau_i\|^2\right) = \left(\int d^2\sigma \sqrt{g}\right)^{-t/2}. \tag{25}$$

The jacobian is given by the determinant of the transformation matrix

$$\begin{pmatrix} \delta c \\ \delta h^{\perp}_{ab} \\ \delta t_i \end{pmatrix} = \begin{pmatrix} 1, & \cdot & , & \cdot \\ 0, & P_{abc}, & & \cdot \\ 0, & 0 & , & \chi_{ij} \end{pmatrix} \begin{pmatrix} \delta\phi \\ \delta V^c \\ \delta\tau_i \end{pmatrix}, \tag{26}$$

where, taking Ψ^i to be orthonormal vectors in the metric $e^{\phi}\hat{g}$, we have

$$\chi_{ij} = \left\langle \Psi^i \left| \frac{\partial\hat{g}}{\delta\tau^j} - \tfrac{1}{2}\hat{g}^{cd}\frac{\partial\hat{g}_{cd}}{\partial\tau^j} \right.\right\rangle \tag{27}$$

and

$$|\det \chi_{ij}| = \sum_l \left|\det\left\langle \frac{\partial\hat{g}}{\partial\tau^i} - \tfrac{1}{2}\hat{g}^{cd}\frac{\partial\hat{g}_{cd}}{\partial\tau^i}\right| \Psi^l\right\rangle \left\langle \Psi^l \left| \frac{\partial\hat{g}}{\partial\tau^j} - \tfrac{1}{2}\hat{g}^{cd}\frac{\partial\hat{g}_{cd}}{\partial\tau^j}\right.\right\rangle\right|^{1/2}$$

$$= \left[\left(\frac{\partial\hat{g}_{ab}}{\partial\tau^i} - \tfrac{1}{2}\hat{g}_{ab}\hat{g}^{kl}\frac{\partial\hat{g}_{kl}}{\partial\tau^i}\right) G^{abcd}\left(\frac{\partial\hat{g}_{cd}}{\partial\tau^j} - \tfrac{1}{2}\hat{g}_{cd}\hat{g}^{kl}\frac{\partial\hat{g}_{kl}}{\delta\tau^j}\right)\right]^{1/2}$$

$$= \det{}^{1/2}\left[\hat{g}^{ac}\left(\frac{\partial\hat{g}_{ab}}{\partial\tau^i} - \tfrac{1}{2}\hat{g}_{ab}\hat{g}^{kl}\frac{\partial\hat{g}_{kl}}{\partial\tau^i}\right)\tilde{g}^{bd}\left(\frac{\partial\hat{g}_{cd}}{\partial\tau^j} - \tfrac{1}{2}\hat{g}_{cd}\hat{g}^{kl}\frac{\partial\hat{g}_{kl}}{\partial\tau^j}\right)\right]$$

$$\equiv \det{}^{1/2} T_{ij}. \tag{28}$$

We have used the fact that $\sum_l |\Psi^l\rangle\langle\Psi^l|$ is the projection operator for the h_{ab} onto the space of the Teichmüller deformations and in the space of the traceless tensors $\frac{\partial\hat{g}_{cd}}{\partial\tau^j} - \frac{1}{2}\hat{g}_{cd}\hat{g}^{kl}\frac{\partial\hat{g}_{kl}}{\partial\tau^j}$ it can be replaced by G^{abcd}.

Finally there remains the change of variables $\delta h^{\perp}_{ab} \to \delta V^c$ with, according to (26), the Jacobian given by

$$\det P = (\det P^+P)^{1/2},$$

where P^+P is a hermitean operator given by (18). The complication which arises here is due to the possible existence among the set of vectors V of the zero modes of the operator P, the so-called conformal Killing vectors V_{CK} which generate infinitesimal diffeomorphisms which are pure conformal transformations. Therefore we must split $\delta V \equiv \delta V^{\perp}\delta V^i_{CK}$ where $V^{\perp}$ are orthogonal to $\ker P$ and

$$\delta h^{\perp} = (\det{}' P^+P)^{1/2}\delta V^{\perp} = (\det{}' P^+P)^{1/2}\delta V/\delta V^i_{CK}. \tag{29}$$

2. *Path integral measure ambiguity*

A simple way to discuss the independence of the path integral of the choice of the arbitrary constants a_x, a_h, a_v, which are present in the definitions of the measures

$$\int d\delta X e^{-\pi a_x \|\delta X\|^2} = 1, \quad \int d\delta h e^{-\pi a_h \|\delta h\|^2} = 1, \quad \int d\delta V e^{-\pi a_v \|\delta V\|^2} = 1$$

is to rewrite the measures in terms of the coefficients of the expansion in eigenfunctions of the operators $-\nabla^2$, P^+P and PP^+. They are positive definite hermitean operators (we assume that this is assured by the appropriate choice of the boundary conditions as discussed by Alvarez [1]). We introduce the orthonormal set of eigenfunctions

$$\begin{array}{ll} X^i_\mu(\sigma),\ X^\mu_n(\sigma) & (-\nabla^2), \\ V^i_{\mathrm{CK}}(\sigma),\ V_n(\sigma) & (P^+P), \\ \Psi^i(\sigma),\ h^\perp_n(\sigma) & (PP^+), \end{array}$$

where the first columns are zero modes. Using the expansions (we work on a compact two-dimensional manifold)

$$X^\mu = \alpha^\mu_0 X^\mu_0 + \alpha^\mu_n X^\mu_n, \qquad V = \beta^i_0 V^i_{ck} + \beta_n V_n, \qquad h = t^i \Psi^i + \gamma_n h^\perp_n, \tag{30}$$

we can write the measures as

$$\begin{aligned} DX^\mu &= \sqrt{a_x}\, d\alpha^\mu_0 \prod_n \sqrt{a_x}\, d\alpha^\mu_n, \\ DV &= \prod_i \sqrt{a_v}\, d\beta^i_0 \prod_n \sqrt{a_v}\, d\beta_n, \\ Dh &= \prod_i \sqrt{a_h}\, dt^i \prod_n \sqrt{a_h}\, d\gamma_n. \end{aligned} \tag{31}$$

We observe first that the non-zero eigenvalues and eigenvectors of P^+P and PP^+ correspond one-to-one. Indeed

$$PP^+PV_n = \lambda^v_n PV_n$$

so that

$$h^\perp_n \sim PV_n.$$

Using

$$\langle V_n P | PV_n \rangle = \lambda^v_n,$$

we conclude that

$$h^\perp_n = \frac{1}{\sqrt{\lambda^v_n}} PV_n.$$

Therefore

$$h^{\perp} = \gamma_n \frac{1}{\sqrt{\lambda_n^v}} PV_n = \beta_n PV_n.$$

The last equality follows from $h^{\perp} = PV$, when we use the expansion for V. Thus

$$\gamma_n = \sqrt{\lambda_n^v}\, \beta_n \tag{32}$$

and we have

$$Dh = \prod_i \sqrt{a_h}\, dt^i \prod_n \sqrt{a_h} \sqrt{\lambda_n^v}\, d\beta_n$$

$$= \prod_i \sqrt{a_h}\, dt^i \prod_n \sqrt{\lambda_n^v a_h a_v / a_v}\, d\beta_n \prod_i \sqrt{a_v}\, d\beta_0^i / \prod_i \sqrt{a_v}\, d\beta_0^i$$

$$= DV \prod_n \sqrt{\lambda_n^v a_h / a_v} \prod_i \sqrt{a_h}\, dt^i / \prod_i \sqrt{a_v}\, d\beta_0^i \tag{33}$$

$$= DV \prod_n \sqrt{a_h / a_v}\, (\det{}' P^+ P)^{1/2} \prod_i \sqrt{a_h}\, dt^i / \prod_i \sqrt{a_v}\, d\beta_0^i.$$

If we put $a_h = a_v = 1$ we recover for the $Dh^{\perp}$ the formula (29). Let us study the factor

$$N = \prod_n \sqrt{a_h / a_v} \prod_i \sqrt{a_h} / \prod_i \sqrt{a_v}$$

$$= \sqrt{a_h / a_v}^{-\dim \ker P} (\det \sqrt{a_h / a_v}) \sqrt{a_h}^{\dim \ker P^+} / \sqrt{a_v}^{\dim \ker P}, \tag{34}$$

where we decide to calculate $\det \sqrt{a_h/a_v}$ e.g. in the basis of eigenvectors of P^+P. Since zero modes of P^+P are not zero modes of the constant we have the additional $\sqrt{a_h/a_v}^{-\dim \ker P}$ on the right hand side of (34). Thus

$$N = (\sqrt{a_h})^{\dim \ker P^+ - \dim \ker P} \det (C). \tag{35}$$

As it is shown in the Appendix $\det (C)$ can be absorbed into the counterterms and since

$$\dim \ker P - \dim \ker P^+ = 3\chi_E(M) = \begin{cases} \dfrac{3}{4\pi} \displaystyle\int d^2\sigma \sqrt{g}\, R \\ \dfrac{3}{4\pi} \displaystyle\int d^2\sigma \sqrt{g}\, R + \int_{\partial M} dsk \end{cases} \tag{36}$$

the same is true for the first factor (the counterterms must include therefore the terms on the rhs (of (36)). The dependence on a's amounts to the renormalization of the overall normalization constant which anyway has to be fixed by the unitarity arguments.

Similarly we have

$$DX^{\mu} = \sqrt{a_x}\, d\alpha_0^{\mu} \prod_n \sqrt{a_x}\, d\alpha_n^{\mu}. \tag{37}$$

Remember that

$$\langle X_n | X_m \rangle = \int d^2\sigma \sqrt{g}\, \bar{X}_n X_m = \delta_{nm} \tag{38}$$

and since the zero mode of the laplacian on a compact surface is a constant we take

$$X_0^2 = \frac{1}{\int d^2\sigma \sqrt{g}}. \tag{39}$$

Altogether

$$\int DX^\mu \exp\left\{-\frac{T}{2}\int d^2\sigma \sqrt{g}\, X^\mu \nabla^a \nabla_a X^\mu = \prod_{\mu=1}^{D} \sqrt{a_x} \int d\alpha_0^\mu \left[\prod_n \frac{T}{2\pi a_x} \hat{\lambda}_n^X\right]^{-D/2}\right.$$

$$= [\det{}'(-\nabla^2)]^{-D/2} \det{}^{-D/2}\left(\frac{T}{2\pi a_x}\right)\left(\frac{2\pi}{\int d^2\sigma \sqrt{g}}\right)^{-D/2} \prod_{\mu=1}^{D} \int \frac{d\alpha^\mu}{\sqrt{\int d^2\sigma \sqrt{g}}} \int DX_0^\mu. \tag{40}$$

Det (C) can again be absorbed into counterterms.

Taking into account (40), (33)–(36), (25), (28) we get at this point the following amplitude for the vacuum-to-vacuum transition

$$W = \sum_{\text{top}} \int \prod_i^t (d\tau_i \sqrt{\int d^2\sigma \sqrt{g}}) \det{}^{1/2} T_{ij} [\det{}'(-\nabla^2)]^{-D/2}$$

$$* \left(\frac{2\pi}{\int d^2\sigma \sqrt{g}}\right)^{-D/2} (\det(V_{\text{CK}}^i, V_{\text{CK}}^j))^{-1/2} (\det{}' P^+ P)^{1/2}$$

$$* \prod_{\mu=1}^{D} DX_0^\mu \frac{1}{\mathscr{V}_{\text{CK}} N}, \tag{41}$$

where the determinant of

$$(V_{\text{CK}}^i, V_{\text{CK}}^j) = \int d^2\sigma \sqrt{g}\, g_{ab} V_{\text{CK}}^{ia} V_{\text{CK}}^{jb} \tag{42}$$

takes into account the fact that the conformal Killing vectors may not be chosen to be orthonormal, and the last integration gives just the energy-momentum δ-function. The $\mathscr{V}_{\text{CK}}$ is the volume of the group generated by the conformal Killing vectors, N is the order of the subgroup of the modular group which preserves g_{ab} and the range of the Teichmüller parameters is understood to be restricted to the fundamental region.

3. Weyl invariance in $D = 26$ in the eigenfunction basis

Let us study the Weyl invariance of W. Since S_0 is Weyl invariant we have to study the effect of the transformation $\phi' = \phi + \delta\phi$ on the ratio (see (31))

$$\int d\alpha_0 \prod_n d\alpha_n \frac{\prod_i dt^i \prod_n d\gamma_n}{\prod_i d\beta_0^i \prod_n d\beta_n} (\text{metric } e^\phi \hat{g}) \tag{43}$$

i.e. compare it with

$$\int d\alpha_0^\phi \prod_n d\alpha_n^\phi \frac{\prod_i dt^{i\phi} \prod_n d\gamma_n^\phi}{\prod_i d\beta_0^{i\phi} \prod_n d\beta_n^\phi}, \tag{44}$$

where the superscript ϕ denotes the coefficients of the expansion of the conformally transformed DX, Dh, DV in terms of the operators $-\nabla^2, P^+P, PP^+$ in the metric $e^{\delta\phi}e^{\phi}\hat{g}$. X^μ do not change under Weyl transformation and working to order $\delta\phi$ we only need to know the diagonal term in the change of eigenfunctions X_n^μ: $X_n^\phi = (1+A_n^n)X_n + \sum_{m\neq n} A_m^n X_m$. It can be read off from the normalization condition

$$\langle X_m^\phi | X_n^\phi \rangle = \int d^2\sigma \sqrt{g}\, e^{\delta\phi} \overline{X}_m^\phi X_n^\phi = \int d^2\sigma \sqrt{g}\, \overline{X}_m X_n = \delta_{mn}.$$

Hence

$$X_m^\phi = (1 - \tfrac{1}{2} \int d^2\sigma \sqrt{g}\, \delta\phi \overline{X}_m X_n) X_m + \dots = (1 - \tfrac{1}{2} \langle \delta\phi \rangle_{X_m}) X_m$$

and since

$$X = \alpha_0 X_0 + \sum_n \alpha_n X_n = \alpha_0^\phi X_0^\phi + \sum_n \alpha_n^\phi X_n^\phi,$$

we get

$$\alpha_m^\phi = (1 + \tfrac{1}{2} \langle \delta\phi \rangle_{X_m}) \alpha_m.$$

So the Jacobian of the transformation $\alpha_m^\phi \to \alpha_m$ reads

$$\det \left| \frac{\partial \alpha_n^\phi}{\partial \alpha_m} \right| = 1 + \tfrac{1}{2} \sum_n \langle \delta\phi \rangle_{X_m}. \tag{45}$$

For the contravariant vectors we have

$$\delta V^a = \delta V_\phi^a.$$

The orthonormal basis changes according to

$$\int d^2\sigma \sqrt{g}\, g_{ab} V_n^a V_m^b = \int d^2\sigma \sqrt{g}\, e_{ab} e^{2\delta\phi} V_{n\phi}^a V_{m\phi}^b = \delta_{nm}$$

$$V_{n\phi}^a = (1 - \langle \delta\phi \rangle_{V_n}) V_n^a + \dots$$

and since

$$\sum_n \beta_n V_n = \sum_n \beta_n^\phi V_n^\phi,$$

we get

$$\beta_n^\phi = (1 + \langle \delta\phi \rangle_{V_n}) \beta_n + \dots .$$

Therefore

$$\det\left|\frac{\partial\beta_n^\phi}{\partial\beta_m}\right| = 1+\sum_m \langle\delta\phi\rangle_{V_m}. \tag{46}$$

Finally, from (14) and (23),

$$\delta h_{ab}^\phi = (1+\delta\phi)\delta h_{ab}$$

and the basis h_n (including zero modes of PP^+) transforms as (from (12))

$$h_n^\phi = (1+\tfrac{1}{2}\langle\delta\phi\rangle_{h_n})h_n+\ \dots .$$

From

$$\delta h_{ab} = \sum_n \gamma_n h_n$$

and

$$(1+\delta\phi)\delta h_{ab} = \sum_n \gamma_n^\phi h_n^\phi = \sum_n \gamma_n^\phi(1+\tfrac{1}{2}\langle\delta\phi\rangle_{h_n})h_n$$

we conclude that

$$\gamma_n^\phi = (1+\tfrac{1}{2}\langle\delta\phi\rangle_{h_n})\gamma_n+\ \dots$$

and

$$\det\left|\frac{\partial\gamma_n^\phi}{\partial\gamma_m}\right| = 1+\tfrac{1}{2}\sum_n \langle\delta\phi\rangle_{h_m}. \tag{47}$$

Combining everything together we get the following full Jacobian for the transition from (44) to (43):

$$1+\tfrac{1}{2}\sum_n\langle\delta\phi\rangle_{h_n}-\sum_n\langle\delta\phi\rangle_{V_n}+D\,\tfrac{1}{2}\sum_n\langle\delta\phi\rangle_{h_n},$$

where the sums include zero modes of the respective operators. The traces are evaluated in the Appendix. For $D = 26$ the result is Weyl invariant.

4. Vacuum 1-loop amplitudes

The two dimensional manifolds for the 1-loop closed oriented and open oriented and unoriented bosonic string amplitudes have the topology of the torus, the cylinder and the Möbius strip, respectively. In this Section we calculate in each case the vacuum amplitudes. For the closed string this is a summary of the Polchinski calculation. For the open string our result coincide with those of Ref. [4, 5] obtained independently.

The torus is topologically equivalent to a parallelogram with boundaries identified. We can take $0 \leqslant \sigma^1 \leqslant 1$, $0 \leqslant \sigma^2 \leqslant 1$ with X^μ and g_{ab} subject to periodic boundary conditions

$$X^\mu(\sigma^1,\sigma^2) = X^\mu(\sigma^1+1,\sigma^2) = X^\mu(\sigma^1,\sigma^2+1) = X^\mu(\sigma^1+1,\sigma^2+1).$$

The metric can be transformed into the form

$$g_{ab}d\sigma^a d\sigma^b = |d\sigma^1+\tau d\sigma^2|^2 = \begin{pmatrix} 1 & \tau_1 \\ \tau_1, & \tau_1^2+\tau_2^2 \end{pmatrix} d\sigma^a d\sigma^b, \tag{48}$$

where $\tau = \tau_1 + i\tau_2$ is the complex Teichmüller parameter. Hence

$$\sqrt{g} = \tau_2, \quad \int d^2\sigma \sqrt{g} = \tau_2$$

$$T_{ij} = \begin{pmatrix} \frac{2}{\tau_2^2} & \frac{\tau_1}{\tau_2^3} \\ 0 & \frac{2}{\tau_2^2} \end{pmatrix} \quad (T_{ij} \text{ is defined in Eq. (28)}). \tag{49}$$

(Equivalently we can keep the gauge $g_{ab} = \delta_{ab}$ and choose the parallelogram $0 \leqslant \sigma^1 \leqslant 1$, $0 \leqslant \sigma^2 \leqslant \tau$.)

Since, according to (18)

$$(P^+P)_{ab} = -2g_{ab}g^{cd}\partial_c\partial_d \quad (R = 0)$$

the conformal Killing vectors are zero modes of the laplacian i.e. constant vector fields. There are two independent solutions

$$V_1^a = \delta_1^a, \quad V_2^a = \delta_2^a$$

corresponding to two coordinate directions and generating transformations

$$\sigma^a \to \sigma^a + V^a,$$

where V^a is a constant vector with components in the interval [0, 1]. The $\mathscr{V}_{CK} = 1$ and the normalization of the conformal Killing vectors is

$$(V_i, V_j) = \int d^2\sigma \sqrt{g}\, g_{ab} V_i^a V_j^b = \tau_2 g_{ij}, \quad \det(V_i, V_j) = \tau_2^4. \tag{50}$$

The modular group consists of transformations $\tau' = (a\tau+b)/(c\tau+d)$ where a, b, c, d are integers satisfying $ad-bc = 1$. One can eliminate this gauge freedom by restricting $|\tau_1| < 1/2$, $\tau_2 > 0$, $|\tau| > 1$. There remains only the 2-fold discrete invariance $\sigma^1 \to 1-\sigma^1$, $\sigma^2 \to 1-\sigma^2$ so that the order $N = 2$.

Before we calculate the determinant of the laplacian let us summarize the analogous information for the 1-loop open oriented and unoriented strings. A manifold with the topology of the cylinder can be obtained from the square representing a torus

$$0 \leqslant \sigma^1 \leqslant 1, \quad 0 \leqslant \sigma^2 \leqslant 1$$

when we consider a mapping [6]

$$\mathrm{M(T \to C)} \begin{cases} \sigma^1 \to 1-\sigma^1 \\ \sigma^2 \to \sigma^2 \end{cases}$$

of the torus into itself and identify points related by the mapping. Then the cylinder is represented by the

$$0 \leqslant \sigma^1 \leqslant 1/2, \quad 0 \leqslant \sigma^2 \leqslant 1.$$

All the quantities of interest for a cylinder can be obtained from those for the torus by requiring the invariance under M(T → C). In particular invariance $X^\mu(\sigma^1) = X^\mu(1-\sigma^1)$ results in Neumann boundary conditions at $\sigma^1 = 0$ and $\sigma^1 = 1/2$. The Teichmüller parameters are obtained by demanding

$$\begin{aligned}&(d\sigma^1)^2+(\tau_1^2+\tau_2^2)(d\sigma^2)^2+2\tau_1 d\sigma^1 d\sigma^2\\ &= [d(1-\sigma^1)^2]+(\tau_1^2+\tau_2^2)(d\sigma^2)^2+2\tau_1 d(1-\sigma^1)d\sigma^2\end{aligned} \tag{51}$$

which gives $\tau_1 = 0$. For the torus we have two constant linearly independent vectors in the direction σ^1 and σ^2 as the conformal Killing vectors. Under the mapping M(T → C) only the one in the direction σ^2 remains invariant. Denoting $\tau_2 = \tau$ we get

$$\begin{aligned}&\sqrt{g} = \tau, \quad \int d^2\sigma \sqrt{g} = \tau/2, \quad \det T = 2/\tau^2,\\ &(V_{CK}, V_{CK}) = \int d^2\sigma \sqrt{g}\, g_{ab}\delta_2^a\delta_2^b = \tau^3/2\end{aligned} \tag{52}$$

and the volume $\mathscr{V}_{CK} = 1$ and $N = 2$.

The last topology to be considered before writing the final results for partition functions is the topology of a Möbius strip (open unoriented string). We consider the mapping from a cylinder into itself:

$$\mathrm{M(C \to MS)} \begin{cases}\sigma^1 \to 1/2-\sigma^1\\ \sigma^2 \to 1/2+\sigma^2\end{cases}$$

and identify points related by the mapping. The Möbius strip is represented by

$$0 \leqslant \sigma^1 \leqslant 1/2, \quad 0 \leqslant \sigma^2 \leqslant 1/2.$$

The Teichmüller parameter is the same as for the cylinder. We have now

$$\begin{aligned}&\sqrt{g} = \tau, \quad \int d^2\sigma \sqrt{g} = \tau/4, \quad \det T = 2/\tau^2,\\ &(V_{CK}, V_{CK}) = \int d^2\sigma \sqrt{g}\, g_{ab}\delta_2^a\delta_2^b = \tau^3/4\end{aligned} \tag{53}$$

and the volume $\mathscr{V}_{CK} = 1/2$ and $N = 2$. We proceed to calculate the determinant of the laplacian $\Delta = -g^{ab}\partial_a\partial_b$. Let us begin with the torus with the periodic boundary conditions $X^\mu(\sigma^1, \sigma^2) = X^\mu(\sigma^1+1, \sigma^2) = X^\mu(\sigma^1, \sigma^2+1) = X^\mu(\sigma^1+1, \sigma^2+1)$.

The orthonormal eigenfunctions of the operator

$$\Delta^{\mathrm{T}} = -\frac{1}{\tau_2^2}\left[(\tau_1^2+\tau_2^2)\left(\frac{\partial}{\partial\sigma^1}\right)^2 - 2\tau_1\left(\frac{\partial}{\partial\sigma^1}\right)\left(\frac{\partial}{\partial\sigma^2}\right) + \left(\frac{\partial}{\partial\sigma^2}\right)^2\right] \tag{54}$$

are

$$X^{\mathrm{T}}_{mn}(\sigma^1, \sigma^2) = \frac{1}{\sqrt{\tau_2}} e^{2\pi i m\sigma^1} e^{2\pi i n\sigma^2} \tag{55}$$

with the eigenvalues

$$\frac{4\pi^2}{\tau_2^2} [(\tau_1^2+\tau_2^2)m^2 - 2mn\tau_1 + n^2]. \tag{56}$$

For the cylinder we get

$$\Delta^{\mathrm{T}} = -\frac{1}{\tau^2}\left[\tau^2\left(\frac{\partial}{\partial\sigma^1}\right)^2 + \left(\frac{\partial}{\partial\sigma^2}\right)^2\right] \tag{57}$$

and imposing the Neumann boundary conditions

$$\left.\frac{\partial X^\mu}{\partial\sigma^1}\right|_{\sigma^1=0} = \left.\frac{\partial X^\mu}{\partial\sigma^1}\right|_{\sigma^1=1/2} = 0,$$

or equivalently the invariance under M(T → C) i.e. under $\sigma^1 \to 1-\sigma^1$, $\sigma^2 \to \sigma^2$ one has

$$X^{\mathrm{C}}_{mn}(\sigma^1, \sigma^2) = \frac{1}{\sqrt{2}}(X^{\mathrm{T}}_{mn}(\sigma^1, \sigma^2) + X^{\mathrm{T}}_{mn}(1-\sigma^1, \sigma^2))$$

$$= \frac{1}{\sqrt{2}} e^{2\pi i n\sigma^2}\sqrt{2}\cos(2\pi m\sigma^1) \tag{58}$$

with the eigenvalues

$$\frac{4\pi^2}{\tau^2}[\tau^2 m^2 + n^2].$$

We can now calculate the determinant of the laplacian in space of scalar functions e.g. for the open oriented string with Neumann boundary conditions (the prime omits the zero mode $n = m = 0$):

$$\ln \det{}' \Delta^{\mathrm{C}} = \sum_{m\geqslant 0,n} \ln\left[\frac{4\pi^2}{\tau^2}(\tau^2 m^2 + n^2)\right]$$

$$= \sum_{m,n}{}' \ln\left[\frac{4\pi^2}{\tau^2}(\tau^2 m^2 + n^2)\right] + \sum_{n}{}' \ln\left[\frac{4\pi^2}{\tau^2} n^2\right]$$

$$= \sum_{n=1}^{\infty} \ln|1 - e^{-2\pi n\tau}| - \frac{\pi}{6}\tau + \ln\tau^2. \tag{59}$$

The details of the calculation are given in Ref. [7]. Finally we get

$$\det{}' \Delta^{C} = \tau^2 e^{-\pi\tau/6} |f(e^{-2\pi n\tau})|^2, \quad f(e^{-2\pi n\tau}) = \prod_{n=1}^{\infty} (1-e^{-2\pi n\tau}). \tag{60}$$

The determinant of the P^+P is also readily obtained. One should, however, remember, that P^+P acts not on scalars, but on vectors. On a torus, the only requirement for both components of a vector is periodicity — in that case

$$\det{}' (P^+P)^{\mathrm{T}} = \det{}' (-2g_{ab}g^{-1/2}\partial_c(\sqrt{g}\, g^{cd}\partial_d)) = (\det{}' \Delta^{\mathrm{T}})^2 (\det{}' 2)^2 . \tag{61}$$

For a cylinder, the situation is different. For scalars (like X^μ) we impose Neumann boundary conditions. For vectors, first component satisfies Dirichlet and the second one satisfies Neumann boundary conditions. Hence

$$\det{}' (P^+P)^{\mathrm{c}} = (\det{}' \Delta_{\mathrm{N}}^{\mathrm{c}}) (\det \Delta_{\mathrm{D}}^{\mathrm{c}}) (\det{}'_{\mathrm{N}} 2) (\det{}_{\mathrm{D}} 2) \tag{62}$$

and

$$\ln \det{}' \Delta_{\mathrm{N}}^{\mathrm{c}} + \ln \det \Delta_{\mathrm{D}}^{\mathrm{c}} = \Bigg(\sideset{}{'}\sum_{\substack{m\geq 0, n \\ \text{even}}} + \sum_{\substack{m>0, n \\ \text{odd}}} \Bigg) \ln\left(\frac{4\pi^2}{\tau^2} (\tau^2 m^2 + n^2) \right)$$

$$= \sideset{}{'}\sum_{m,n} \ln\left[\frac{4\pi^2}{\tau^2} (\tau^2 m^2 + n^2) \right] = 4 \sum_{n=1}^{\infty} \ln(1-e^{-2\pi n\tau}) - \frac{\pi}{3}\tau + \ln \tau^2. \tag{63}$$

To calculate the determinant of a laplacian on the Möbius strip we need the set of eigenfunctions

$$\cos(2\pi n_1 \sigma^1) e^{2\pi i n_2 \sigma^2} \qquad n_1 \geq 0 \qquad \text{(Neumann)}$$

$$\sin(2\pi n_1 \sigma^1) e^{2\pi i n_2 \sigma^2} \qquad n_1 > 0 \qquad \text{(Dirichlet)}.$$

The symmetry $\sigma^1 \to 1/2 - \sigma^1$, $\sigma^2 \to 1/2 + \sigma^2$ requires that n_1, n_2 are both odd or both even. We have then

$$\ln \det{}' \Delta_{\mathrm{N}}^{\mathrm{MS}} = \Bigg(\sum_{\substack{n_1 \geq 0, n_2 \\ \text{even}}} + \sum_{\substack{n_1 > 0, n_2 \\ \text{odd}}} \Bigg) \ln\left(\frac{4\pi^2}{\tau^2} (\tau^2 n_1^2 + n_2^2) \right), \tag{64}$$

$$\ln \det \Delta_{\mathrm{D}}^{\mathrm{MS}} = \Bigg(\sum_{\substack{n_1 > 0\ n_2 \\ \text{odd}}} + \sum_{\substack{n_1 > 0, n_2 \\ \text{even}}} \Bigg) \ln\left(\frac{4\pi^2}{\tau^2} (\tau^2 n_1^2 + n_2^2) \right). \tag{65}$$

Using the techniques described in Ref. [7] we get

$$\det{}' \Delta_{\mathrm{N}}^{\mathrm{MS}} = \frac{\tau^2}{4} e^{-\pi\tau/12} \prod_{n=1}^{\infty} (1-(-1)^n e^{-\pi n\tau})^2, \tag{66}$$

$$\det \Delta_{\mathrm{D}}^{\mathrm{MS}} = e^{-\pi\tau/12} \prod_{n=1}^{\infty} (1-(-1)^n e^{-\pi n\tau})^2, \tag{67}$$

and

$$\det{}'(P^+P)^{\mathrm{MS}} = (\det{}' \Delta_{\mathrm{N}}^{\mathrm{MS}})(\det \Delta_{\mathrm{D}}^{\mathrm{MS}})(\det{}'_{\mathrm{N}} 2)(\det{}_{\mathrm{D}} 2)$$

$$= \frac{\tau^2}{4} e^{-\pi\tau/12} \prod_{n=1}^{\infty} (1-(-1)^n e^{-\pi n\tau})^4 (\det{}'_{\mathrm{N}} 2)(\det{}_{\mathrm{D}} 2). \tag{68}$$

For all three topologies, in the final result one should regularize the determinants of the constant. When we include the zero mode, for arbitrary value of n_1, n_2 runs from $-\infty$ to ∞ including 0.
Hence

$$\ln \det C = \sum_{n_1} \sum_{n_2=-\infty}^{\infty} \ln C = \sum_{n_1} (2\xi(0)+1) = \sum_{n_1} 0 = 0, \tag{69}$$

(we used zeta-function regularization). When we exclude the zero mode, for $n_1 = 0$, n_2 runs from $-\infty$ to ∞ excluding zero and therefore

$$\ln \det{}' C = \sum_{n_1 \neq 0} \sum_{n_2=-\infty}^{\infty} \ln C + 2 \sum_{n_2=1}^{\infty} \ln C = 2\xi(0) \ln C = -\ln C. \tag{70}$$

Therefore we use

$$\det C = 1 \qquad \det{}' C = 1/C.$$

Collecting the relevant factors for all three topologies, we get for W (Eq. 41) the following results

$$W_{\mathrm{torus}} = \int d\tau_1 d\tau_2 \tau_2 \frac{2}{\tau_2^2} \left[\tau_2^2 e^{-\pi\tau_2/3} \prod_{n=1}^{\infty} |1-e^{2\pi i n\tau}|^4 \right]^{-13}$$

$$* \left(\frac{2\pi}{T}\right)^{13} \tau_2^{13} \tau_2^{-2} \tfrac{1}{2} \tau_2^2 e^{-\pi\tau_2/3} \prod_{n=1}^{\infty} |1-e^{2\pi i n\tau}|^4 \frac{L^{26}}{2}$$

$$= \frac{L^{26}}{2(4\pi^2\alpha')^{13}} \int d\tau_1 d\tau_2 \tau_2^{-14} e^{4\pi\tau_2} \prod_{n=1}^{\infty} |1-e^{2\pi i n\tau}|^{-48}, \tag{71}$$

$$W_{\mathrm{cylinder}} = \int d\tau \sqrt{\frac{\tau}{2}} \frac{\sqrt{2}}{\tau} \left[\tau^2 e^{-\pi\tau/6} \prod_{n=1}^{\infty} (1-e^{-2\pi n\tau})^2 \right]^{-13}$$

$$* \left(\frac{2\pi}{T}\right)^{13} \left(\frac{\tau}{2}\right)^{13} \tau^{-3/2} \sqrt{2} \frac{\tau}{\sqrt{2}} e^{-\pi\tau/6} \prod_{n=1}^{\infty} (1-e^{-2\pi n\tau})^2 \frac{L^{26}}{2}$$

$$= \frac{L^{26}}{2(8\pi^2\alpha')^{13}} \int d\tau \tau^{-14} e^{2\pi\tau} \prod_{n=1}^{\infty} (1-e^{-2\pi n\tau})^{-24}, \tag{72}$$

$$W_{\text{Möbius}} = \int d\tau \sqrt{\frac{\tau}{4}} \frac{\sqrt{2}}{\tau}\left[\frac{\tau^2}{4} e^{-\pi\tau/12} \prod_{n=1}^{\infty} (1-(-1)^n e^{-\pi n\tau})^2\right]^{-13}$$

$$* \left(\frac{2\pi}{T}\right)^{13} \left(\frac{\tau}{4}\right)^{13} \tau^{-3/2} 2 \frac{\tau}{2\sqrt{2}} e^{-\pi\tau/12} \prod_{n=1}^{\infty} (1-(-1)^n e^{-\pi n\tau})^2 2 \frac{L^{26}}{2}$$

$$= \frac{L^{26}}{2(4\pi^2\alpha')^{13}} \int d\tau\tau^{-14} e^{\pi\tau} \prod_{n=1}^{\infty} (1-(-1)^n e^{-\pi n\tau})^{-24}. \tag{73}$$

5. One-loop dilaton tadpole and four-graviton amplitude

In this Section we give several examples of calculations in the path-integral formalism. First we calculate for the closed, oriented st ringone-dilaton tadpole on a torus (such a tadpole vanishes on a sphere); next we calculate one-dilaton tadpole in one loop for the open string (both oriented and unoriented topologies); at the end we calculate four-graviton one-loop amplitude for the closed, oriented string.

To calculate any amplitudes in the path-integral formalism one needs Green's functions. The Green's function on a torus is given in Ref. [2] ($\tau = \tau_1 + \tau_2 i$, $z = \sigma^1 + \tau\sigma^2$)

$$G^{\mathrm{T}}(z, z') = -\frac{(z-z'-\bar{z}+\bar{z}')^2}{8T\tau_2} - \frac{1}{2\pi T} \ln \left|\frac{\theta_1(z-z'|\tau)}{\theta_1'(0|\tau)}\right|. \tag{74}$$

G^{T} satisfies the equation

$$-T\Delta G^{\mathrm{T}}(z, z') = \frac{\delta^2(z-z')}{\tau_2} - \frac{1}{\tau_2}, \tag{75}$$

i.e. it is defined on a space of functions with $\int dz d\bar{z} f = 0$ (this requirement is satisfied in the physical amplitudes by the momentum conservation). Our conventions on the thetafunctions agree with Ref. [8]. For a cylinder we have the symmetry $\sigma^1 \to 1-\sigma^1$, $\sigma^2 \to \sigma^2$ i.e. $z \to 1-\bar{z}$. Hence the Green's function ($\tau = i\tau_2$) reads

$$G^{\mathrm{c}}(z, z') = -\frac{(z-z'-\bar{z}+\bar{z}')^2}{4T\tau_2} - \frac{1}{2\pi T} \ln \left|\frac{\theta_1(z-z'|\tau)}{\theta_1'(0|\tau)}\right|$$

$$- \frac{1}{2\pi T} \ln \left|\frac{\theta_1(\bar{z}+z'|\tau)}{\theta_1'(0|\tau)}\right|. \tag{76}$$

Because $-\Delta G^{\mathrm{c}} \propto \delta - 1$ we can impose $\frac{\partial G^{\mathrm{c}}}{\partial n} = 0$ on all boundaries (it is not possible for $-\Delta G^{\mathrm{c}} \propto \delta$). We recall that $0 \leqslant \operatorname{Re} z \leqslant 1/2$, $0 \leqslant \operatorname{Im} z \leqslant \tau_2$. For a Möbius strip we have the symmetry $\sigma^1 \to 1-\sigma^1$, $\sigma^2 \to 1/2+\sigma^2$ i.e. $z \to 1/2+i\tau_2/2-\bar{z}$. Using

$$\theta_1(1/2+i\tau_2/2-\nu|\tau) = e^{i\pi(\nu-\tau/4)}\theta_3(\nu|\tau), \tag{77}$$

and

$$\frac{\theta_1(\nu|\tau)\theta_3(\nu|\tau)}{\theta_1'(0|\tau)\theta_3(0|\tau)} = \frac{\theta_1(\nu|\tau/2+1/2)}{\theta_1'(0|\tau/2+1/2)}, \tag{78}$$

we get

$$\begin{aligned} G^{\mathrm{MS}}(z, z') = & -\frac{(z-z'-\bar{z}+\bar{z}')^2}{2T\tau_2} - \frac{1}{2\pi T}\ln\left|\frac{\theta_1(z-z'|\tau/2+1/2)}{\theta_1'(0|\tau/2+1/2)}\right| \\ & -\frac{1}{2\pi T}\ln\left|\frac{\theta_1(\bar{z}+z'|\tau/2+1/2)}{\theta_1'(0|\tau/2+1/2)}\right|. \end{aligned} \tag{79}$$

We recall that $0 \leqslant \mathrm{Re}\, z \leqslant 1/2$, $0 \leqslant \mathrm{Im}\, z \leqslant \tau_2/2$ for a Möbius strip. We proceed now to the actual calculations.

The dilaton vertex is given by (for a torus we do not have any other terms in the vertex)

$$V_{\mathrm{D}}(p) = \int d^2\sigma \, \frac{8\tau_2 g_{\mathrm{D}}}{\sqrt{24}\,\alpha'}\, \eta^{\perp}_{\mu\nu}\partial_z X^{\mu}\partial_{\bar{z}} X^{\nu} e^{ipX}, \tag{80}$$

where

$$\eta^{\perp}_{\mu\nu} = \eta_{\mu\nu} - n_\mu p_\nu - n_\nu p_\mu, \quad n^2 = p^2 = 0, \quad n\cdot p = 1.$$

Inserting this into (71) we get

$$\begin{aligned} A^{\mathrm{T}}_{\mathrm{D}}(p) = & (2\pi)^{26}\delta^{26}(p) T^{13}\left(\frac{8\tau_2 g_{\mathrm{D}}}{\sqrt{24}\,\alpha'}\right)\int d^2\sigma \int \frac{d^2\tau}{4\pi\tau_2^2}\, e^{4\pi\tau_2}(2\pi\tau_2)^{-12} \\ & \prod_{n=1}^{\infty}|1-e^{2\pi i n\tau}|^{-48} 12 \lim_{z\to z'}(\partial_z\partial_{\bar{z}'}+\partial_{\bar{z}}\partial_{z'})\langle X(z)X(z')\rangle \\ = & -(2\pi)^{26}\delta^{26}(p)\left(\frac{4\pi g_{\mathrm{D}}\sqrt{24}}{2(4\pi^2\alpha')^{13}}\right)\int \frac{d^2\tau}{\tau_2^{14}}\, e^{4\pi\tau_2}\prod_{n=1}^{\infty}|1-e^{2\pi i n\tau}|^{-48} \\ = & -4\pi g_{\mathrm{D}}\sqrt{24}\,(2\pi)^{26}\delta^{26}(p) W_{\mathrm{torus}}. \end{aligned} \tag{81}$$

In the above calculation we used (see Eq. (74))

$$\langle X^{\mu}(z)X^{\nu}(z')\rangle = \eta^{\mu\nu}G^{\mathrm{T}}(z, z'). \tag{82}$$

The regularization of the $\partial_z\partial_{\bar{z}'} \ln|z-z'|_{z\to z'}$, gives a term proportional to R [9] and hence vanishes for a torus.

Inserting the dilaton vertex (80) into (72) we get the dilaton tadpole for the open, oriented string in one-loop

$$A_{\rm D}^{\rm c} = \frac{(2\pi)^{26}\delta^{26}(p)}{2(8\pi^2\alpha')^{13}}\left(\frac{8\tau_2 g_{\rm D}}{\sqrt{24}\,\alpha'}\right)\int\frac{d\tau}{\tau^{14}}\,e^{2\pi\tau}\prod_{n=1}^{\infty}(1-e^{-2\pi n\tau})^{-24}$$

$$\int d^2\sigma 12\lim_{z\to z'}(\partial_z\partial_{\bar{z}'}+\partial_{\bar{z}}\partial_{z'})G^{\rm c}(z,z'). \tag{83}$$

It was shown in Ref. [9] that the regularized derivative of the term $\ln|z-z'|$ (in our case $\ln|\theta_1(z-z'|\tau)|$) is equal to $-\sqrt{g}R/(32\pi T)$. In another publication ([10]) we show that in case of a surface with boundaries

$$\int d^2\sigma\,\tfrac{1}{2}\lim_{z\to z'}(\partial_z\partial_{\bar{z}'}+\partial_{\bar{z}}\partial_{z'})G^{\rm c}(z,z')$$

$$= -\frac{1}{16\pi T}\left(\tfrac{1}{2}\int_C d^2\sigma\sqrt{g}\,R+\int_{\partial C}ds\,k\right)-\frac{1}{2T\tau_2}\int d^2\sigma = -\frac{\chi(C)}{8T}-\frac{1}{4T\tau_2}. \tag{84}$$

We see that the Fradkin-Tseytlin vertex [11] for the zero momentum dilaton coupling is always proportional to the Euler characteristic of a surface either with or without boundary. Hence we have ($\chi(C)=0$)

$$A_{\rm D}^{\rm c}(p) = -4\pi g_{\rm D}\sqrt{24}\,(2\pi)^{26}\delta^{26}(p)W_{\rm cylinder}. \tag{85}$$

Performing the analogous steps for the Möbius strip we get ($\chi({\rm MS})=0$)

$$A_{\rm MS}^{\rm c}(p) = -4\pi g_{\rm D}\sqrt{24}\,(2\pi)^{26}\delta^{26}(pW)_{\rm M\ddot{o}bius}. \tag{86}$$

To simplify these expressions we substitute $w=e^{-2\pi\tau}$ in $A_{\rm D}^{\rm c}$ and $w=e^{-\pi\tau}$ in $A_{\rm MS}^{\rm c}$ and use the relations

$$w^{-1}f(w)^{-24}(-2\pi/\ln w)^{12} = q^{-2}f(q^2)^{-24}$$

and

$$w^{-1}f(-w)^{-24}(-\pi/\ln w)^{12} = q^{-1/2}f(-\sqrt{q})^{-24},$$

where

$$\ln q = 2\pi^2/\ln w, \quad f(w) = \prod_{n=1}^{\infty}(1-w^n).$$

Hence we have

$$A_{\rm D}^{\rm c} = -\frac{(2\pi)^{26}\delta^{26}(p)}{2(8\pi^2\alpha')^{13}}\,4\pi g_{\rm D}\sqrt{24}\int_0^1\frac{dq}{\pi q^3}\prod_{n=1}^{\infty}(1-q^{2n})^{-24}, \tag{87}$$

$$A_{\rm MS}^{\rm c} = -\frac{(2\pi)^{26}\delta^{26}(p)}{2(4\pi^2\alpha')^{13}}\,4\pi g_{\rm D}\sqrt{24}\int_0^1\frac{2dq}{\pi q^3}\prod_{n=1}^{\infty}(1-(-q^2)^n)^{-24}. \tag{88}$$

Finally we discuss one-loop four-graviton amplitude for closed, oriented string. The graviton vertex reads

$$V(p_r, X) = \frac{8\kappa_0}{\alpha'} \xi^r_{\mu\nu} : \partial_w X^\mu \partial_{\bar{w}} X^\nu e^{ip_r X(w)}, \tag{89}$$

where $w = \sigma^1 + \tau\sigma^2$, or, in a more convenient form ($\xi^\mu_r \bar{\xi}^\nu_r : = \xi^{\mu\nu}_r$)

$$V(p_r, X) = \frac{8\kappa_0}{\alpha'} \exp\left\{i\left(p_r - i\xi_r \frac{\partial}{\partial w_r} - i\bar{\xi}_r \frac{\partial}{\partial \bar{w}_r}\right) X(w)\right\}. \tag{90}$$

With the vertex written in this form we get

$$A(p_1, \dots, p_M) = \frac{(2\pi)^{26}\delta^{26}(\sum p_i)}{2(4\pi^2\alpha')^{13}} \int \frac{d^2\tau}{\tau_2^{14}} e^{4\pi\tau_2} \prod_{n=1}^{\infty} |1 - e^{2\pi i n\tau}|^{-48}$$

$$\prod_{i=1}^{M} \left(\frac{8\kappa_0}{\alpha'} \int d^2\sigma_i\right) \exp\left\{-\tfrac{1}{2} \sum_{j,k}\right.$$

$$\left.\left(p^\mu_j - i\xi^\mu_j \frac{\partial}{\partial w_j} - i\bar{\xi}^\mu_j \frac{\partial}{\partial \bar{w}_j}\right)\left(p^\nu_k - i\xi^\nu_k \frac{\partial}{\partial w_k} - i\bar{\xi}^\nu_k \frac{\partial}{\partial \bar{w}_k}\right) \langle X_\mu(\sigma_j) X_\nu(\sigma_k)\rangle\right\}, \tag{91}$$

where the correlation function is given in Eq. (82) and (74). We will use the notation

$$\langle X_\mu(\sigma_j) X_\nu(\sigma_k)\rangle = \eta_{\mu\nu}\alpha' \langle j, k\rangle \tag{92}$$

and

$$C_{jk} = \frac{\partial^2}{\partial w_j \partial w_k} \langle j, k\rangle,$$

$$D_{jk} = \frac{\partial^2}{\partial \bar{w}_j \partial \bar{w}_k} \langle j, k\rangle = \bar{C}_{jk},$$

$$A_{\mu_i} = \sum_{j \neq i} p^j_{\mu_i} \frac{\partial}{\partial w_i} \langle i, j\rangle,$$

$$B_{\mu_i} = \sum_{j \neq i} p^j_{\mu_i} \frac{\partial}{\partial \bar{w}_i} \langle i, j\rangle = \bar{A}_{\mu_i}. \tag{93}$$

We will also use the fact that

$$\frac{\partial^2}{\partial w_j \partial \bar{w}_k} \langle j, k\rangle = \frac{\partial^2}{\partial \bar{w}_j \partial w_k} \langle j, k\rangle = \frac{-\pi}{2\tau_2}, \tag{94}$$

For M = 4 the exponent factor in (91) is equal to

$$\prod_{1 \leqslant k < j \leqslant 4} \exp\{-\alpha' p_j p_k \pi\tau_2(\sigma_j - \sigma_k)^2\} \left|\frac{\theta_1(w_j - w_k|\tau)}{\theta_1'(0|\tau)}\right|^{\alpha' p_j p_k} * K(\xi, p, \tau)$$

where

$$K = \xi_1^{\mu_1}\xi_2^{\mu_2}\xi_3^{\mu_3}\xi_4^{\mu_4}\xi_1^{\nu_1}\xi_2^{\nu_2}\xi_3^{\nu_3}\xi_4^{\nu_4}.$$

$$\Big\{\Big[(g_{\mu_1\mu_2}g_{\mu_3\mu_4}C_{12}C_{34}+2\text{ more})(g_{\nu_1\nu_2}g_{\nu_3\nu_4}D_{12}D_{34}+2\text{ more})$$

$$+\left(\frac{-\pi}{2\tau_2}\right)^2(g_{\mu_1\mu_2}g_{\nu_1\nu_3}g_{\mu_4\nu_2}g_{\mu_3\nu_4}C_{12}D_{13}+41\text{ more})$$

$$+\left(\frac{-\pi}{2\tau_2}\right)^4(g_{\mu_4\nu_3}g_{\mu_3\nu_2}g_{\mu_2\nu_1}g_{\mu_1\nu_4}+8\text{ more})\Big]$$

$$+\alpha'\Big[\left(\frac{-\pi}{2\tau_2}\right)(g_{\mu_1\mu_2}g_{\nu_1\nu_2}g_{\mu_4\nu_3}A_{\mu_3}B_{\nu_4}C_{12}D_{12}+107\text{ more})$$

$$+\left(\frac{-\pi}{2\tau_2}\right)^3(g_{\mu_4\nu_3}g_{\mu_3\nu_4}g_{\mu_2\nu_1}g_{\mu_3\nu_4}A_{\mu_1}B_{\nu_2}+43\text{ more})\Big]$$

$$+(\alpha')^2\Big[(g_{\mu_1\mu_2}g_{\mu_3\mu_4}C_{12}C_{34}+2\text{ more})B_{\nu_1}B_{\nu_2}B_{\nu_3}B_{\nu_4}$$

$$+(g_{\nu_1\nu_2}g_{\nu_3\nu_4}C_{12}C_{34}+2\text{ more})A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}$$

$$+(g_{\mu_1\mu_2}A_{\mu_3}A_{\mu_4}C_{12}+5\text{ more})(g_{\nu_1\nu_2}B_{\nu_3}B_{\nu_4}D_{12}+5\text{ more}).$$

$$\left(\frac{-\pi}{2\tau_2}\right)(g_{\mu_1\mu_2}g_{\mu_4\nu_3}A_{\mu_3}B_{\nu_1}B_{\nu_2}B_{\nu_4}C_{12}+35\text{ more})$$

$$+\left(\frac{-\pi}{2\tau_2}\right)(g_{\nu_1\nu_2}g_{\nu_4\mu_3}B_{\nu_3}A_{\mu_1}A_{\mu_2}A_{\mu_4}D_{12}+35\text{ more})$$

$$\left(\frac{-\pi}{2\tau_2}\right)^2(g_{\mu_4\nu_3}g_{\mu_3\nu_2}A_{\mu_1}A_{\mu_2}B_{\nu_1}B_{\nu_4}+41\text{ more})\Big]$$

$$+(\alpha')^3\Big[(g_{\mu_1\mu_2}A_{\mu_3}A_{\mu_4}C_{12}+5\text{ more})B_{\nu_1}B_{\nu_2}B_{\nu_3}B_{\nu_4}$$

$$+(g_{\nu_1\nu_2}B_{\nu_3}B_{\nu_4}C_{12}+5\text{ more})A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}$$

$$+\left(\frac{-\pi}{2\tau_2}\right)(g_{\mu_4\nu_3}A_{\mu_1}A_{\mu_2}A_{\mu_4}B_{\nu_1}B_{\nu_2}B_{\nu_4}+11\text{ more})\Big]$$

$$+(\alpha')^4[(A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}B_{\nu_1}B_{\nu_2}B_{\nu_3}B_{\nu_4})]\Big\}. \tag{95}$$

There are no $g_{\mu_i\nu_i}$ terms in the expansion. Also $g_{\mu_i\nu_j}$ and $g_{\nu_j\mu_i}$ are not distinguished in permutations (the same applies to $g_{\mu_i\mu_j}$ and $g_{\nu_i\nu_j}$). $g_{\mu_i\mu_j}$ comes with C_{ij}, $g_{\nu_i\nu_j}$ with D_{ij} and $g_{\mu_i\nu_j}$ with one power of $(-\pi/2\tau_2)$.

APPENDIX

In this Appendix we calculate the trace of a function f in the basis of the eigenfunctions of some hermitean operator B:

$$\operatorname{Tr} f = \lim_{t\to 0} \sum_n \langle \psi_n | f | \psi_n \rangle e^{-t\lambda_n} = \lim_{t\to 0} \operatorname{Tr}(f e^{-tB})$$

$$= \int d^2\sigma \sqrt{g} \langle \sigma | f e^{-tB} | \sigma \rangle \equiv \int d^2\sigma \sqrt{g} f h(\sigma, \sigma, t),$$

where $h(x, y, t)$ obeys the heat equation

$$B_x h(x, y, t) = -\partial_t h(x, y, t), \qquad h(x, y, t = 0) = \delta(x-y).$$

Using (4.17), (4.38) of Ref. [1]

$$\operatorname{Tr}(f e^{-tP^+P}) = \frac{1}{2\pi t} \int d^2\sigma \sqrt{g} f + \frac{4}{6\pi} \left[\int_{\partial M} dskf + \tfrac{1}{2} \int_M d^2\sigma \sqrt{g}\, Rf \right] + 0(\sqrt{t}),$$

$$\operatorname{Tr}(f e^{-tPP^+}) = \frac{1}{2\pi t} \int d^2\sigma \sqrt{g} f - \frac{5}{6\pi} \left[\int_{\partial M} dskf + \tfrac{1}{2} \int_M d^2\sigma \sqrt{g}\, Rf \right] + 0(\sqrt{t}),$$

$$\operatorname{Tr}(f e^{-t\Delta}) = \frac{1}{4\pi t} \int d^2\sigma \sqrt{g} f + \frac{8}{8\sqrt{\pi t}} \int_{\partial M} dsf + \frac{\beta}{8\pi} \int_{\partial M} dsn^a \partial_a f$$

$$+ \frac{1}{12\pi} \left[\int_{\partial M} dskf + \tfrac{1}{2} \int_M d^2\sigma \sqrt{g}\, Rf \right] + 0(\sqrt{t}),$$

($\beta = -1$ Dirichlet, $\beta = 1$ Neumann).

We see that

$$\tfrac{1}{2} \sum_n \langle \delta\phi \rangle_{h_n} - \sum_n \langle \delta\phi \rangle_{v_n} + \frac{D}{2} \sum_n \langle \delta\phi \rangle_{x_n}$$

$$= \frac{1}{2\pi t} \int d^2\sigma \sqrt{g}\, \delta\phi \left[\tfrac{1}{2} - 1 + \frac{D}{4} \right] + \frac{1}{6\pi} \left[\int_{\partial M} dsk\delta\phi + \tfrac{1}{2} \int_M d^2\sigma \sqrt{g}\, R\delta\phi \right] \left[-\tfrac{5}{2} - 4 + \frac{D}{4} \right].$$

The first term can be cancelled by a counterterm $\mu^2 \int d^2\sigma \sqrt{g}$ and the second one vanishes for $D = 26$.

For the traces of a constant we get

$$\operatorname{Tr} C = \lim \operatorname{Tr}'(C e^{-tP^+P}) + \sum_{\substack{\text{zero modes} \\ \text{of } P^+P}},$$

$$\tilde{\operatorname{Tr}}\, C = \lim \operatorname{Tr}'(C e^{-tPP^+}) + \sum_{\substack{\text{zero modes} \\ \text{of } PP^+}}.$$

Hence

$$\dim\ker P^{+}P - \dim\ker PP^{+} = -\tilde{\mathrm{Tr}}\,1 + \mathrm{Tr}\,1$$

$$= \frac{9}{6\pi}\left[\int_{\partial M} ds\,k + \tfrac{1}{2}\int_{M} d^2\sigma\,\sqrt{g}\,R\right] = 3\chi(M).$$

For the determinant of a constant C

$$\det C = e^{\mathrm{Tr}\ln C},$$

we get a combination of $\exp\,(\mu^2 \int d^2\sigma\,\sqrt{g})$ and $\exp\,(a\chi(M))$ terms which can be absorbed into the counterterms.

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