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LAGRANGIAN AND HAMILTONIAN FORMULATIONS OF DYNAMICS OF CLASSICAL PARTICLES WITH SPIN AND COLOUR*

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Lagrangian and Hamiltonian descriptions of classical particles with spin and colour are presented. We take into account a new classical observable which is due to a possible coupling between spin and colour degrees of freedom. In a configurational space of the particle we find a topologically non-trivial gauge structure related to a generalized Hopf fibration $S^7 \to CP^3$.

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1. Introduction

Motivation for this work is twofold. First, it has been pointed out in the book [1] that Lagrangian description of a simple classical system, e.g. a non-relativistic spinning particle, leads to a beautiful, topologically non-trivial gauge structure in a configurational space Q of the particle. In particular, it turns out that a global Lagrangian description requires an explicit introduction in Q of unphysical degrees of freedom, and that Lagrangians containing only physical degrees of freedom can be constructed only locally, in some map on the configurational space. In this paper we would like to present a new example of such a non-trivial gauge structure in classical mechanics. Its topological non-triviality is due to the fact that a generalized Hopf fibration $S^7 \to CP^3$ is non-trivial. Examples considered in [1] involve only the standard Hopf fibration $S^3 \to S^2$.

The second part of our motivation is the following. In the paper [2] we have noticed that a complete set of dynamical variables for the so-called classical particle with spin and colour moving in an external Yang-Mills field should contain a new dynamical variable in addition to the more common classical spin and colour. The presence of the new variable is due to a possibility of a coupling between spin and colour degrees of freedom on a more

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fundamental level of quantum mechanics which underlies this classical model. In the particular case of absence of such a coupling the new variable reduces to a product of components of the classical spin and colour. However, in the general case the new variable is an independent dynamical variable. Classical equations of motion for the particle with spin and colour were obtained in [2] as a classical limit of a Dirac equation. They contain 15 internal dynamical variables, namely the classical spin (S^k), the classical colour (I^a) and the new variables (J^{ak}), where $a, k = 1, 2, 3$. These internal dynamical variables are constrained by 9 independent constraint relations. Thus, the internal configurational space of the particle is a non-trivial 6-dimensional algebraic submanifold of R^{15}. Therefore, the resulting dynamical system is by no means simple. In particular, it is not a trivial superposition of the well-known classical mechanics of a spinning particle [3], and of the classical mechanics of a coloured particle [4]. For this reason one may expect that it is rather a difficult task to formulate Lagrangian and Hamiltonian descriptions of such a particle with spin and colour. However, it turns out that we can construct the Lagrangian and Hamiltonian descriptions once we adopt ideas presented in Ref. [1]. Especially inspiring is a Lagrangian description of the spinning particle given in Chapter 3 of Ref. [1].

The main step consists of regarding the 15 dynamical variables S^k, I^a, J^{ak} as composite, secondary quantities which are built out of a smaller set of primordial dynamical variables $w^{\alpha\eta}$, $\alpha, \eta = 1, 2$. The variables $w^{\alpha\eta}$ are chosen in such a way that the nine constraint relations are satisfied automatically. Such interpretation of the classical variables S^k, I^a, J^{ak} looks a little bit unnatural on the grounds of classical mechanics. However, it is perfectly suited for our case because the variables S^k, I^a, J^{ak} have been defined in [2] just as some secondary quantities, namely as expectation values of certain quantum mechanical operators. Let us quote here the relevant formulae:

$$I^a = \tfrac{1}{2}\,\mathrm{Tr}\,(\hat{w}\sigma_a^{\mathrm{T}}\hat{w}^{\dagger})/\mathrm{Tr}\,(\hat{w}^{\dagger}\hat{w}), \tag{1}$$

$$S^k = \tfrac{1}{2}\,\mathrm{Tr}\,(\hat{w}^{\dagger}\sigma_k\hat{w})/\mathrm{Tr}\,(\hat{w}^{\dagger}\hat{w}), \tag{2}$$

$$J^{ak} = \tfrac{1}{4}\,\mathrm{Tr}\,(\hat{w}_k^{\dagger}\sigma_k\hat{w}\sigma_a^{\mathrm{T}})/\mathrm{Tr}\,(\hat{w}^{\dagger}\hat{w}), \tag{3}$$

where σ_k, σ_a are Pauli matrices, a, $k = 1, 2, 3$, T denotes the matrix transposition, 2 by 2 matrix $\hat{w} = (w^{\alpha\eta})$, $\alpha, \eta = 1, 2$, is a time-dependent and $\vec{x}$-independent spinor. The index α (η) refers to the spin (colour) degrees of freedom. Here we do not assume that $\hat{w}$ is normalized, i.e. $\mathrm{Tr}\,\hat{w}^{\dagger}\hat{w} \neq 1$, in general. The obvious choice for the primordial dynamical variables is the spinor $\hat{w} = (w^{\alpha\eta})$. It is clear that the new primordial variables can be interpreted as coordinates of a point of $R^8\backslash\{0\}$. Zero is excluded because of the denominators in formulae (1)–(3).

It follows from (1)–(3) that an overall time-dependent phase factor, as well as a normalization factor of $\hat{w}$ do not contribute to the expectation values. Thus, two of the eight primordial variables ($w^{\alpha\eta}$) seem to be superfluous. This impression turns out to be true in the case of the normalization factor and false in the case of the phase factor. The phase factor cannot be globally eliminated essentially because of the mathematical fact that the generalized Hopf fibration $S^7 \to CP^3$ is non-trivial, [5]. In other words, the considered system exhibits a U(1) gauge symmetry in the configurational space Q for which does not exist a global gauge fixing. Local gauge fixing is possible, i.e. the phase factor can be eliminated locally, in a map on the space $R^8 \backslash \{0\}$. In this framework, the composite variables S^k, I^a, J^{ak} should be regarded as gauge invariant observables of the classical system.

Utilising the variables $\hat{w}$ it is rather easy to find the Lagrangian. It turns out to be singular, partially because of the gauge symmetries. Nevertheless, Dirac's method [6, 7] allows us to construct the corresponding Hamiltonian explicitly.

The plan of this paper is the following. In Section 2 we present the classical equations of motion and the constraints on S^k, I^a, J^{ak}. In Section 3 we write the appropriate Lagrangian and we discuss the gauge invariances of it. In Section 4 we construct the corresponding Hamiltonian. In Appendix we invert the relations (1)–(3), i.e. we express $w^{\alpha\eta}$ by S^k, I^a, J^{ak}. The obtained relation is not smooth, unless we divide the CP^3 space into several patches. This fact reflects the non-triviality of the Hopf fibration $S^7 \to CP^3$.

2. *The classical equations of motion*

We shall consider the following classical equations of motion

$$m\ddot{x}^r = gF^a_{0r}I^a - \frac{g}{c}\,\dot{x}^i F^a_{ri} I^a + \frac{g\hbar}{2mc}\,\Phi_r, \tag{4}$$

where

$$\Phi_i = -\varepsilon_{pks} J^{as} D_i F^a_{pk} + \varepsilon_{pks}\,\frac{\dot{x}^k}{c}\,J^{as}(D_i E^p)^a$$

$$-\varepsilon_{sip} J^{ap}(D_0 E^s)^a - \varepsilon_{pis} J^{as}\,\frac{\dot{x}^r}{c}\,(D_r E^p)^a$$

$$+\frac{g}{4mc^2}\,E^{sa}\varepsilon_{sip}S^q\left(F^a_{pq} + \frac{\dot{x}^p}{2c}\,E^{aq}\right) - \frac{g}{4mc^2}\,\varepsilon_{cad}E^{aq}\left(F^c_{iq} + \frac{\dot{x}^q}{2c}\,E^{ci}\right)I^d, \tag{4'}$$

$$\frac{dI^a}{dt} = \frac{g}{\hbar}\left(\frac{1}{c}\,A^{bi}\dot{x}^i - A^b_0\right)\varepsilon_{bac}I^c + \frac{g}{2mc}\left[-F^c_{it} + \frac{1}{2c}\,(E^{ci}\dot{x}^t - E^{ct}\dot{x}^i)\right]\varepsilon_{its}\varepsilon_{cab}J^{bs}, \tag{5}$$

$$\frac{dS^k}{dt} = \frac{g}{mc}\left[-F^a_{kp} + \frac{1}{2c}\,(E^{ak}\dot{x}^p - E^{ap}\dot{x}^k)\right]J^{ap}, \tag{6}$$

$$\frac{dJ^{ak}}{dt} = \frac{g}{\hbar}\left(\frac{1}{c}\,A^{ci}\dot{x}^i - A^c_0\right)\varepsilon_{cad}J^{dk} + \frac{g}{8mc}\,\varepsilon_{ipk}\varepsilon_{cad}\left(-F^c_{ip} + \frac{\dot{x}^p}{c}\,E^{ci}\right)I^d$$

$$+\frac{g}{4mc}\left[-F^a_{kp} - \frac{1}{2c}\,(E^{ak}\dot{x}^p - E^{ap}\dot{x}^k)\right]S^p, \tag{7}$$

where g is a non-Abelian coupling constant, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \frac{g}{\hbar c}\,\varepsilon_{abc}A^b_\mu A^c_\nu$ is the Yang-Mills field-strength tensor, $(D_\varrho F_{\mu\nu})^a = \partial_\varrho F^a_{\mu\nu} - \frac{g}{\hbar c}\,\varepsilon_{abc}A^b_\varrho F^c_{\mu\nu}$ is its covariant derivative.

The equations (5)–(7) have been obtained in the papers [2] by taking a semiclassical, non-relativistic limit of the Dirac equation with the external Yang-Mills field A^a_μ. The Eq. (4) differs from the corresponding equation presented in [2] by the term $\frac{g\hbar}{2mc}\Phi_r$ on the r.h.s. In our opinion this term has to be negligibly small if the semiclassical approximation is to be a good approximation. Therefore this term was neglected in [2]. However, even such a small term can be important when considering the question whether the system of Eqs (4)–(7) admits a Lagrangian which would exactly reproduce these equations. Actually for the system (4)–(7) we can write the Lagrangian, whereas we are not able to do this for the system consisting of Eqs (5)–(7) and of Eq. (4) with the last term on the r.h.s. excluded.

The classical observables S^k, I^a, J^{ak}, characterising the internal motion, are defined by (1)–(3). From these definitions it follows that they are not independent, e.g. one can prove that ([2])

$$\tfrac{1}{4}I^a - J^{ak}S^k = 0, \tag{8}$$

$$4J^{ak}J^{al} - S^kS^l = (\tfrac{1}{4} - \vec{I}^2)\delta^{kl}. \tag{9}$$

Detailed analysis carried out in the paper [2] (see also Appendix to the present paper) has also showed that knowing (I^a), (S^k), (J^{ak}) one can determine the normalized spinor $\hat{w} = (w^{\alpha\eta})$ up to an overall phase factor. Such a phase factor does not contribute to the expectation values (1)–(3). Therefore, of the eight variables $(w^{\alpha\eta})$ just two and no more are superfluous, namely the phase factor and the normalization factor.

It is easy to check that the Eqs (5)–(7) will be satisfied if the spinor $w^{\alpha\eta}$ obeys the following equation

$$i\hbar\frac{d\tilde{w}}{dt} = g\tilde{w}\hat{A}_0^{\mathrm{T}} - \frac{g}{c}\tilde{w}\hat{A}^{i\mathrm{T}}\dot{x}^i + \frac{g\hbar}{2mc}\varepsilon_{iks}\hat{S}^s\tilde{w}\hat{F}^{\mathrm{T}}_{ik}$$
$$- \frac{g\hbar}{2mc^2}\varepsilon_{iks}\dot{x}^k\hat{S}^s\tilde{w}\hat{E}^{i\mathrm{T}} + \lambda\tilde{w} \equiv H_{in}\tilde{w} + \lambda\tilde{w}, \tag{10}$$

where λ can be any real-number-valued function of time, and $\tilde{w} \overset{\mathrm{df}}{=} \hat{w}/(\mathrm{Tr}\,\hat{w}^\dagger\hat{w})^{1/2}$. It follows from this equation that

$$i\hbar\frac{d\tilde{w}^\dagger}{dt} = -g\hat{A}_0^{\mathrm{T}}\tilde{w}^\dagger + \frac{g}{c}\hat{A}^{i\mathrm{T}}\tilde{w}^\dagger\dot{x}^i - \frac{g\hbar}{2mc}\varepsilon_{iks}\hat{F}^{\mathrm{T}}_{ik}\tilde{w}^\dagger\hat{S}^s$$
$$+ \frac{g\hbar}{2mc^2}\varepsilon_{iks}\dot{x}^k\hat{E}^{i\mathrm{T}}\tilde{w}^\dagger\hat{S}^s - \lambda\tilde{w}^\dagger. \tag{11}$$

The function $\lambda(t)$ will be specified later on. We use here a matrix notation $\hat{A}_\mu = A^a_\mu T^a$, $\hat{F}_{\mu\nu} = F^a_{\mu\nu}T^a$, where $T^a = \sigma_a/2$ are generators of SU(2) group. $\hat{S}^k = \sigma_k/2$ are the spin operators. The terms like $\hat{S}^k\tilde{w}\hat{F}^{\mathrm{T}}_{ik}$, etc., are to be interpreted as ordinary products of 2×2 matrices. Eq. (10) is the basis for the Lagrangian formulation presented in the next Section. Let us notice that from (10), (11) it does not follow that the norm $||\hat{w}|| = (\mathrm{Tr}\,\hat{w}^\dagger\hat{w})^{1/2}$ is constant in time. Actually it can be any function of time.

3. The Lagrangian formulation

It is easy to guess the Lagrangian corresponding to the equations (4), (10),

$$\mathscr{L} = \tfrac{1}{2} m\dot{\vec{x}}^2 + \frac{i\hbar}{2} \operatorname{Tr}(\hat{w}^\dagger \dot{\hat{w}} - \dot{\hat{w}}^\dagger \hat{w})/\operatorname{Tr}(\hat{w}^\dagger \hat{w}) - \operatorname{Tr}(\hat{w}^\dagger H_{\text{in}} \hat{w})/\operatorname{Tr}(\hat{w}^\dagger \hat{w}). \tag{12}$$

Here the dot denotes the time derivative, and $\hat{w}^\dagger H_{\text{in}} \hat{w}$ has the form

$$\operatorname{Tr}(\hat{w}^\dagger H_{\text{in}} \hat{w}) = \operatorname{Tr}\Big(g\hat{w}^\dagger \hat{w}\hat{A}_0^{\mathrm{T}} - \frac{g}{c}\,\hat{w}^\dagger \hat{w}\hat{A}^{i\mathrm{T}}\dot{x}^i$$
$$+ \frac{g\hbar}{2mc}\,\varepsilon_{iks}\hat{w}^\dagger \hat{S}^s \hat{w}\hat{F}_{ik}^{\mathrm{T}} - \frac{g\hbar}{2mc^2}\,\varepsilon_{iks}\dot{x}^k \hat{w}^\dagger \hat{S}^s \hat{w}\hat{E}^{i\mathrm{T}}\Big), \tag{13}$$

which follows from the definition of H_{in} given by (10). The action integral $S = \int \mathscr{L}\,dt$ yields the Eq. (4) under variation with respect to x^i, while the Eqs (10), (11) follow from (independent) variations of $(\bar{w}^{\alpha\eta})$, $(w^{\alpha\eta})$, respectively. In particular, we obtain that

$$\lambda(t) = \left(-\operatorname{Tr}(\hat{w}^\dagger H_{\text{in}} \hat{w}) + \frac{i\hbar}{2} \operatorname{Tr}(\hat{w}^\dagger \dot{\hat{w}} - \dot{\hat{w}}^\dagger \hat{w})\right)\Big/\operatorname{Tr}(\hat{w}^\dagger \hat{w}). \tag{14}$$

The Lagrangian (12) is gauge-invariant with respect to simultaneous SU(2) gauge transformations of the external Yang-Mills field and of the spinor $\hat{w}$. This kind of gauge invariance is very important of course, however in the present context much more interesting is the fact that the Lagrangian (12) exhibits also entirely different types of gauge invariance.

Namely, let us consider the following transformations of the spinors $(w^{\alpha\eta})$, $(\bar{w}^{\alpha\eta})$:

$$\hat{w} \to e^{i\alpha(t)}\hat{w}, \qquad \hat{w}^\dagger \to e^{-i\alpha(t)}\hat{w}^\dagger, \tag{15}$$

where $\alpha(t)$ is a time-dependent, real-number valued phase. The external gauge field does not transform under this new kind of gauge transformations. Thus, the transformations (15) act only in the internal configurational space $R^8 \backslash \{0\}$ of the particle. Under the transformations (15) the Lagrangian changes by a full time-derivative, namely

$$\mathscr{L} \to \mathscr{L}' = \mathscr{L} - \hbar\dot{\alpha}. \tag{16}$$

Thus, $\mathscr{L}$ and $\mathscr{L}'$ are physically equivalent. Because in general $\mathscr{L} \neq \mathscr{L}'$, we shall call this invariance "the weak gauge invariance" (in this we follow Ref. [1]). The equations (4), (10), (11) with λ given by (14) are invariant with respect to the transformations (15).

The Lagrangian (12) is also gauge invariant under a change of the norm of $\hat{w}$, i.e.

$$\hat{w}(t) \to e^{-\beta(t)}\hat{w}(t), \qquad \hat{w}^\dagger(t) \to e^{-\beta(t)}\hat{w}^\dagger, \tag{17}$$

where $\beta(t)$ is a real-valued function of time. In this case we have the gauge invariance in a strong sense, i.e. $\mathscr{L}' = \mathscr{L}$. Eqs (4), (10), (11) with λ given by (14) are invariant with respect to the transformations (17).

It is natural to regard the fact of the presence of the gauge symmetries (15), (17) as an indication that the internal configurational space $R^8\backslash\{0\}$ contains superfluous variables, namely an overall phase of $\hat{w}$ and a normalization factor of w. It turns out that this impression is true as regards the normalization factor, however it is not true in the case of the phase factor. Let us first analyze the case of the norm. Any $(w^{\alpha\eta}) \in R^8\backslash\{0\}$ can be uniquely written in the form

$$w^{\alpha\eta}(t) = e^{\alpha(t)}\tilde{w}^{\alpha\eta}(t), \tag{18}$$

where $\alpha(t)$ is real and

$$\mathrm{Tr}\ \tilde{w}^{\dagger}\tilde{w} = 1, \tag{19}$$

i.e. $(\tilde{w}^{\alpha\eta}) \in S^7$, and

$$\alpha(t) = \tfrac{1}{2}\ln(\mathrm{Tr}\ \hat{w}^{\dagger}\hat{w}). \tag{20}$$

Therefore, we can write

$$R^8\backslash\{0\} = S^7 \times R_+, \tag{21}$$

where $R_+ = \{r : r > 0\}$. It follows from (21) that $R^8\backslash\{0\}$ is a trivial bundle with the base S^7 and the fibre R_+. Therefore, we can find a global, regular section of this bundle, i.e., in other words, a gauge condition which eliminates the arbitrariness of the norm. For example, we can use (18), (19), (20). Substituting (18)–(20) into the Lagrangian (12) we obtain a new, reduced Lagrangian $\tilde{\mathscr{L}}$, which contains only $\tilde{w}^{\alpha\eta}$, $\bar{\tilde{w}}^{\alpha\eta}$ constrained by condition (19). Thus, the resulting configurational space is the sphere S^7. Now the internal dynamical variables are coordinates on this sphere. The new Lagrangian $\tilde{\mathscr{L}}$ leads to the same Eqs of motion for the gauge invariant classical observables S^k, I^a, J^{ak} as those obtained from $\mathscr{L}$, i.e. to Eqs (4)–(7). Therefore, $\tilde{\mathscr{L}}$ is physically equivalent to $\mathscr{L}$.

Now, let us consider the phase factor and the gauge transformations (15). These transformations do not change the norm of $\hat{w}$, i.e. they operate within the sphere S^7. Therefore, in order to eliminate the phase degree of freedom we would have to write the sphere S^7 as a trivial bundle with a fibre $U(1) \equiv S^1$. Then we could use a global, regular section of this trivial bundle in order to eliminate the phase from the Lagrangian. The class of points of the sphere S^7 which differ only by the phase factor can be regarded as a point of CP^3 space. Thus, the question is whether S^7 could be regarded as $CP^3 \times S^1$. It is a well-known fact that it is not the case, [5]. For example, $\pi_1(S^7) = 0$, while $\pi_1(CP^3 \times S^1) = Z \otimes Z$, where Z is the additive group of integers. However, it is true that $S^7 \simeq CP^3 \times S^1$ locally. Therefore, the phase factor can be eliminated only locally, by introducing local patches on S^7. Any effort to eliminate the phase factor globally would necessarily lead to singularities in the corresponding new, reduced Lagrangian. In other words, in the case of the gauge invariance (15) a global gauge fixing is not possible. Let us also recall that the Lagrangian (12) is invariant with respect to (15) in the weak sense because of (16). Therefore, when eliminating the phase factor locally, we can neglect the total time derivative $-\hbar\dot{\alpha}$, where α is the phase factor present in the considered local section of the bundle S^7 (thus, now α is a function of the point of the space CP^3).

The quantities S^k, I^a, J^{ak} are invariant with respect to the gauge transformations (15), (17). For this reason it is natural to call them the classical observables. The formulae (1)–(3) can be looked upon as relations between the classical observables and points of the CP^3 space. In the Appendix we invert this relation. In order to do this, we have to introduce local patches on the CP^3 space. The fact that such inverse relation is possible means that S^k, I^a, J^{ak} form the complete set of classical internal observables — any other observable is a function of them.

Let us stress that from the preceding considerations it follows that it is not possible to express $\mathscr{L}$ in terms of the classical observables S^k, I^a, J^{ak} only, just because it is not possible to reduce $\mathscr{L}$ to a Lagrangian on the CP^3 space. At best we can have the weakly gauge-invariant Lagrangian on the sphere S^7 with the unphysical, gauge degree of freedom (the phase factor) which cannot be globally eliminated for the topological reason.

4. The Hamiltonian formulation

In order to obtain the Hamiltonian formulation we apply the standard procedure presented in many text-books, see, e.g. [6, 7]. We shall restrict our considerations to a concrete local coordinate map on the $R^8\backslash\{0\}$ space. (We could proceed in a coordinate-independent manner. However, this would lead to rather lengthy computations, especially when classifying constraints, with no extra insight gained.) Passage to another map can be done by a standard patching. As the local coordinate map M on the internal $R^8\backslash\{0\}$ space we shall use $(r, \alpha, \xi^k, \zeta^i)$ which are introduced in the following manner. If $(w^{\alpha\eta}) \in M \subset R^8\backslash\{0\}$ then $w^{\alpha\eta}$ can uniquely be written in the form

$$w^{\beta\eta} = re^{i\alpha}v^{\beta\eta}, \tag{22}$$

where

$$\begin{gathered}
r > 0, \quad 0 \leqslant \alpha < 2\pi, \quad \operatorname{Tr} \hat{v}^\dagger \hat{v} = 1, \hat{v} = (v^{\alpha\eta}), \quad \operatorname{Im} v^{11} = 0, \\
\operatorname{Re} v^{11} = \sqrt{1-|v^{12}|^2-|v^{21}|^2-|v^{22}|^2} > 0, \\
\vec{\xi} = (\xi^1, \xi^2, \xi^3) \stackrel{\mathrm{df}}{=} \operatorname{Re}(v^{12}, v^{21}, v^{22}), \\
\vec{\zeta} = (\zeta^1, \zeta^2, \zeta^3) \stackrel{\mathrm{df}}{=} \operatorname{Im}(v^{12}, v^{21}, v^{22}).
\end{gathered} \tag{23}$$

In these coordinates the Lagrangian (12) has the following form

$$\mathscr{L} = \frac{m\dot{\vec{x}}^2}{2} - \hbar\dot{\alpha} + \hbar(\vec{\zeta}\dot{\vec{\xi}} - \vec{\xi}\dot{\vec{\zeta}}) - \operatorname{Tr}(\hat{v}^\dagger H_{\text{in}}\hat{v}). \tag{24}$$

The corresponding canonical momenta are defined as follows

$$\begin{gathered}
p_i = \frac{\partial \mathscr{L}}{\partial \dot{x}^i}, \quad p_\alpha = \frac{\partial \mathscr{L}}{\partial \dot{\alpha}}, \quad p_r = \frac{\partial \mathscr{L}}{\partial \dot{r}}, \\
K_i = \frac{\partial \mathscr{L}}{\partial \dot{\xi}^i}, \quad Z_i = \frac{\partial \mathscr{L}}{\partial \dot{\zeta}^i}, \quad i = 1, 2, 3.
\end{gathered} \tag{25}$$

Poisson bracket has the usual form

$$\{F, G\} = \sum_{I} \left(\frac{\partial F}{\partial q^I} \frac{\partial G}{\partial p_I} - \frac{\partial F}{\partial p_I} \frac{\partial G}{\partial q^I} \right), \tag{26}$$

where we have introduced the abbreviations $(q^I) \equiv (x^i, \alpha, r, \xi^l, \zeta^k), (p_I) \equiv (p_i, p_\alpha, p_r, K_l, Z_k)$, $I = 1, 2, \ldots, 8$. We obtain from (24), (25) the following eight primary constraints

$$p_\alpha + \hbar = 0, \qquad p_r = 0, \tag{27}$$

$$K_i - \hbar\zeta^i = 0, \qquad Z_i + \hbar\xi^i = 0, \tag{28}$$

where $i = 1, 2, 3$. It is easy to check that the constraints (27) are first class constraints, while the constraints (28) are second class constraints. It is also easy to verify that there are no secondary constraints.

The second class constraints can be eliminated by passing to the Dirac bracket $\{F, G\}_{\mathrm{D}}$. We obtain that in our case

$$\{F, G\}_{\mathrm{D}} = \{F, G\} - \frac{1}{2\hbar} \left(\frac{\partial F}{\partial \xi^i} + \hbar \frac{\partial F}{\partial Z_i} \right) \left(- \frac{\partial G}{\partial \zeta^i} + \hbar \frac{\partial G}{\partial K_i} \right)$$

$$+ \frac{1}{2\hbar} \left(- \frac{\partial F}{\partial \zeta^i} + \hbar \frac{\partial F}{\partial K_i} \right) \left(\frac{\partial G}{\partial \xi^i} + \hbar \frac{\partial G}{\partial Z_i} \right). \tag{29}$$

Because it is allowed to use constraint relations inside Dirac bracket, we can altogether eliminate the momenta K_i, Z_i from the theory by using the formulae (28). Then F, G will be functions of $\vec{x}, \vec{p}, r, p_r, \alpha, p_\alpha, \vec{\xi}, \vec{\zeta}$, and the Dirac bracket reduces

$$\{F, G\}_{\mathrm{D}} = \frac{\partial F}{\partial \vec{x}} \frac{\partial G}{\partial \vec{p}} - \frac{\partial F}{\partial \vec{p}} \frac{\partial G}{\partial \vec{x}} + \frac{\partial F}{\partial \alpha} \frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial G}{\partial \alpha}$$

$$+ \frac{\partial F}{\partial r} \frac{\partial G}{\partial p_r} - \frac{\partial F}{\partial p_r} \frac{\partial G}{\partial r} + \frac{1}{2\hbar} \left(\frac{\partial F}{\partial \xi^i} \frac{\partial G}{\partial \zeta^i} - \frac{\partial F}{\partial \zeta^i} \frac{\partial G}{\partial \xi^i} \right). \tag{30}$$

Now, let us find a Hamiltonian H for our system. Following the general recipe, [6, 7] we find that

$$H = \frac{1}{2m} \left(p^i - \frac{g}{c} A^{ai} I^a - \frac{g\hbar}{2mc^2} \varepsilon_{lis} E^{al} J^{as} \right)^2$$

$$+ g A_0^a I^a + \frac{g\hbar}{2mc} \varepsilon_{iks} F_{ik}^a J^{as} + \dot{\alpha}(p_\alpha + \hbar) + \dot{r} p_r. \tag{31}$$

Here we have used the formulae (24), (13), (1–3). I^a and J^{ak} should be regarded upon as functions of the variables $\vec{\xi}, \vec{\zeta}$. The last two terms are due to the first class constraints (27).

$\dot{\alpha}$ and $\dot{r}$ present in these terms should be regarded as new variables independent of the coordinates and momenta, [6]. Analogous terms for the second class constraints (28) have been eliminated in a standard manner, [6, 7]. We may do this because we have already passed to the Dirac bracket. So called physical Hamiltonian H_{F} for our system is given by that part of H which is gauge-invariant with respect to gauge transformations (15), (17). Thus,

$$H_{\mathrm{F}} = \frac{1}{2m}\left(p^i - \frac{g}{c}A^{ai}I^a - \frac{gh}{2mc^2}\varepsilon_{lis}E^{al}J^{as}\right)^2$$

$$+ gA_0^a I^a + \frac{gh}{2mc}\varepsilon_{iks}F_{ik}^a J^{as}. \tag{32}$$

It is easy to check that Hamilton equations of motion

$$\frac{d\eta}{dt} = \{\eta, H\}_{\mathrm{D}},$$

where $\eta = (\vec{x}, \vec{p}, \vec{\xi}, \vec{\zeta})$, coincide with Lagrange equations of motion following from the Lagrangian (24). On the other hand, for $\alpha(t)$ and $r(t)$ we obtain merely identities

$$\frac{d\alpha}{dt} = \dot{\alpha}, \qquad \frac{dr}{dt} = \dot{r},$$

Thus, $\alpha(t)$ and $r(t)$ are not fixed by the equations of motion (plus initial conditions). This reflects the fact that α and r are gauge variables. The only way to fix time dependence of $\alpha(t)$, $r(t)$ is to impose a gauge condition. Of course, such a gauge condition has to be compatible with the equation of motion (10). For r we can take, for example,

$$\dot{r} = 0, \quad \text{i.e.} \quad r = \text{const.}$$

For α any gauge condition has to be local on $\mathrm{R}^8\backslash\{0\}$, otherwise we would have a global, regular section of the S^7 bundle with the base CP^3. This would contradict the non-triviality of the generalized Hopf fibration. In the case of the local map (22), (23), we can take, e.g. $\alpha = \text{const.}$

5. *Remark*

Considerations carried out in this paper could be repeated in the case of a relativistic spinning particle with color. Basic Lorentz covariant equations of motion, which would replace our non-relativistic equations (4), (10), could easily be obtained by considering Heisenberg equations of motion for quantum observables for a particle described by the Dirac equation in Foldy-Wouthuysen representation. Then, in order to obtain Lorentz covariant equations one has to use Lorentz covariant position, spin and colour operators. Such operators have been constructed (in the m^{-2} order of the Foldy-Wouthuysen representation) in the paper [8].

APPENDIX

It is easy to see that formulae (1)–(3) and (19) imply that

$$\tilde{w}^{\dagger}\tilde{w} = \tfrac{1}{2}\sigma_0 + I^a\sigma_a^{\mathrm{T}}, \tag{A1}$$

$$\tilde{w}\tilde{w}^{\dagger} = \tfrac{1}{2}\sigma_0 + S^k\sigma_k, \tag{A2}$$

$$\tilde{w}^{\dagger}\frac{\sigma_k}{2}\tilde{w} = \frac{S^k}{2}\sigma_0 + J^{ak}\sigma_a^{\mathrm{T}}, \tag{A3}$$

$$\tilde{w}\frac{\sigma_a^{\mathrm{T}}}{2}\tilde{w}^{\dagger} = \frac{I^a}{2}\sigma_0 + J^{ak}\sigma_k, \tag{A4}$$

where $\sigma_0 = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, T denotes the matrix transposition, and $\tilde{w} = (\tilde{w}^{\alpha\eta})$ is the normalized spinor introduced by formulae (18)–(20).

Now, let us write

$$\tilde{w} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},$$

$$\alpha = ae^{i\varphi}, \quad \beta = be^{i\psi}, \quad \gamma = ce^{i\chi}, \quad \delta = de^{i\omega},$$

where a, b, c, d are moduli of α, β, γ, δ, correspondingly. It follows from (A1), (A2), (A3) that

$$a^2 = \tfrac{1}{2}(\tfrac{1}{2} + I^3 + S^3 + 2J^{33}), \tag{A5}$$

$$b^2 = \tfrac{1}{2}(\tfrac{1}{2} - I^3 + S^3 - 2J^{33}), \tag{A6}$$

$$c^2 = \tfrac{1}{2}(\tfrac{1}{2} + I^3 - S^3 - 2J^{33}), \tag{A7}$$

$$d^2 = \tfrac{1}{2}(\tfrac{1}{2} - I^3 - S^3 + 2J^{33}). \tag{A8}$$

We see that normalization condition (19), i.e.

$$\mathrm{Tr}\,\tilde{w}^{\dagger}\tilde{w} = a^2 + b^2 + c^2 + d^2 = 1$$

is satisfied.

From (A1)–(A3) it also follows that

$$abe^{-i\varphi}e^{i\psi} = \tfrac{1}{2}(I^1 + iI^2) + J^{13} + iJ^{23},$$

$$ace^{i\varphi}e^{-i\chi} = \tfrac{1}{2}(S^1 - iS^2) + J^{31} - iJ^{32}, \tag{A10}$$

$$ade^{-i\varphi}e^{i\omega} = J^{11} + iJ^{12} + iJ^{21} - J^{22}, \tag{A11}$$

$$cde^{i\omega}e^{-i\chi} = \tfrac{1}{2}(I^1 + iI^2) - J^{13} - iJ^{23}, \tag{A12}$$

$$bce^{i\psi}e^{-i\chi} = J^{11} - iJ^{12} + iJ^{21} + J^{22}, \tag{A13}$$

$$bde^{i\psi}e^{-i\omega} = \tfrac{1}{2}(S^1 - iS^2) - J^{31} + iJ^{32}. \tag{A14}$$

From these relations we can determine the phase factors $e^{i\varphi}$, $e^{i\psi}$, $e^{i\chi}$, $e^{i\omega}$. It is clear that if we multiply these phase factors by a common phase factor $e^{i\vartheta}$, it will cancel itself in Eqs (A9)–(A14). This freedom of the overall phase factor for $\tilde{w}$ can also be seen from (A1)–(A3). Thus, we can actually compute $\tilde{w}$ up to the overall phase factor. Such a class of $\tilde{w}$'s determines a point of the CP^3 space.

The relations (A9)–(A14) can be solved with respect to the phases if the coefficients ab, ac, ... are nonvanishing. Because in general they can vanish, we have to use patches covering the space CP^3. For example, in the open subset A of CP^3 defined by the condition $a \neq 0$ (let us explicitly state that this condition does not exclude vanishing of b, c, d) we can use the formulae (A9)–(A11). Using them we can compute $be^{i\psi}$, $ce^{i\chi}$, $de^{i\omega}$ and, subsequently, we can form the matrix $\tilde{w} \equiv \tilde{w}_A$. The resulting formula for $\tilde{w}_A$ contains $e^{i\varphi}$ as the arbitrary, overall phase factor. In the case of the open subset B of CP^3 defined by $b \neq 0$ we can use the formulae (A9), (A13), (A14) from which we can compute $ae^{i\varphi}$, $ce^{i\chi}$, $de^{i\omega}$ — in this case $e^{i\psi}$ is the arbitrary phase factor. The resulting matrix $\tilde{w}$ we shall denote by $\tilde{w}_B$. Similarly one can consider the other cases, i.e. $c = 0$ and $d = 0$.

In the overlap region $A \cap B$, i.e. when $ab \neq 0$, the spinors $\tilde{w}_A$, $\tilde{w}_B$ are related by the following non-trivial phase factor (in the gauges $\varphi = 0$ in A and $\psi = 0$ in B)

$$e^{i\lambda} = \frac{I^1 - iI^2 + 2(J^{13} - iJ^{23})}{\sqrt{(\frac{1}{2}+S^3)^2 - (I^3 + 2J^{33})^2}}.$$

In order to check this formula it is very convenient to use identities (8), (9) and the following two identities

$$\varepsilon_{bac} I^a J^{cs} = \varepsilon_{skr} S^k J^{br}, \tag{A15}$$

$$\varepsilon_{abc} J^{bk} J^{cs} = \tfrac{1}{4}\,\varepsilon_{ksq} I^a S^q + \tfrac{1}{4}\,\varepsilon_{kqs} J^{aq}. \tag{A16}$$

The identity (A15) follows from

$$\begin{aligned}
\mathrm{Tr}\left(\tilde{w}^\dagger \frac{\sigma_k}{2}\tilde{w}\frac{\sigma_c^{\mathrm{T}}}{2}\tilde{w}^\dagger\tilde{w}\right) &= \mathrm{Tr}\left[\left(\frac{S^k\sigma_0}{2} + J^{bk}\sigma_b^{\mathrm{T}}\right)\frac{\sigma_c^{\mathrm{T}}}{2}\left(\tfrac{1}{2}\sigma_0 + I^a\sigma_a^{\mathrm{T}}\right)\right] \\
&= \mathrm{Tr}\left(\tilde{w}\tilde{w}^\dagger\frac{\sigma_k}{2}\tilde{w}\frac{\sigma_c^{\mathrm{T}}}{2}\tilde{w}^\dagger\right) \\
&= \mathrm{Tr}\left[\tfrac{1}{2}(\sigma_0 + S^p\sigma_p)\frac{\sigma_k}{2}\left(\frac{I^c}{2}\sigma_0 + J^{cr}\sigma_r\right)\right],
\end{aligned}$$

where we have used formulae (A1)–(A3). In order to prove identity (A16) consider

$$\begin{aligned}
&\mathrm{Tr}\left(\tilde{w}^\dagger\frac{\sigma_k}{2}\tilde{w}\frac{\sigma_a^{\mathrm{T}}}{2}\tilde{w}^\dagger\frac{\sigma_s}{2}\tilde{w}\right) \\
&= \mathrm{Tr}\left[\left(\frac{S^k}{2}\sigma_0 + J^{bk}\sigma_b^{\mathrm{T}}\right)\frac{\sigma_a^{\mathrm{T}}}{2}\left(\frac{S^s}{2}\sigma_0 + J^{cs}\sigma_c^{\mathrm{T}}\right)\right]
\end{aligned}$$

$$= \mathrm{Tr}\left(\frac{\sigma_k}{2}\,\tilde{w}\,\frac{\sigma_a^{\mathrm{T}}}{2}\,\tilde{w}^{\dagger}\,\frac{\sigma_s}{2}\,\tilde{w}\tilde{w}^{\dagger}\right)$$

$$= \mathrm{Tr}\left[\frac{\sigma_k}{2}\left(\frac{I^a}{2}\,\sigma_0+J^{ap}\sigma_p\right)\frac{\sigma_s}{2}\,(\tfrac{1}{2}\,\sigma_0+S^r\sigma_r)\right].$$

The fact that we have to use the local patches reflects the non-triviality of the generalized Hopf fibration $S^7 \to CP^3$. Namely, let us assume for a while that there exists a single smooth function $\tilde{w}(S^k, I^a, J^{ak})$ of the variables S^k, I^a, J^{ak} such that its values lie on the sphere S^7. According to the definitions (1)–(3) S^k, I^a, J^{ak} can be regarded as a smooth function on the CP^3 space. Therefore, $\tilde{w}(S^k, I^a, J^{ak})$ can be regarded as a smooth function on the CP^3 space with values in the sphere S^7. Thus, we obtain a smooth section of the bundle S^7 with the base CP^3. This contradicts the non-triviality of the generalized Hopf fibration. Therefore, such a single smooth function $\hat{w}(S^k, I^a, J^{ak})$ cannot exist.

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