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ON COUPLING CONSTANT DEPENDENCE OF GAUGE FIELDS*

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Classical gauge fields (pure, coupled to the Dirac, scalar and gravitational fields) are investigated in the weak-coupling and strong-coupling limits. Several results concerning coupling constant dependence of fields in these regions are given. In particular, validity of the weak-coupling perturbative techniques is questioned for dynamical and non-singular solutions to the field equations.

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1. Introduction

Since the 't Hooft-Polyakov monopole solution and instanton have been found there is still interest in classical Yang-Mills (YM) theory [1]. There are two main directions of investigations. The first one is to study YM theory with (arbitrary) external sources [2, 3]. The second line of attack is to investigate the selfcontained dynamical theory. This is much more ambitious program, but a rather modest progress has been done. In fact, the only solutions whose existence was proved are from the monopole family (especially, those in the Prasad-Sommerfeld limit) and a few rather special recently found YM-Dirac and YM--scalar field solutions [4].

It seems to us that there is a little hope to find new and interesting (i.e. physically relevant) solutions to the dynamical YM models explicitly. Nevertheless, it is possible to obtain some general results (e.g. existence and non-existence theorems).

In our view, the most interesting results are those connected with the coupling constant (g) dependence of solutions. This is because of the role played by the coupling constant

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in any physical theory and because of specific behaviour of the effective (running) coupling constant in the standard gauge theory. This behaviour is rather well-established (asymptotic freedom, confinement). Hence, investing this knowledge to the classical theory, new quantitative semi-classical approximations could result for different energy regions.

It is a well-known fact, that in gauge theories the coupling constant g can be rescaled using the transformation: $A \to A' = gA$ and g is absent in the new field equations. In the presence of matter fields an additional matter-fields and/or masses and coupling constants rescaling may also be necessary. However, the above rescaling property does not mean that all solutions to the field equations are of the form: $A(x, g) \sim 1/g$. On contrary, explicit solutions are known of different g-dependence. E.g. Coleman's plane wave ($A \sim g^0$) or 't Hooft-Polyakov monopole ($A \sim g^{-1}\, e^{-g}$). We would like to stress, that complicated coupling constant dependence of fields discussed in this paper is not an artifact of boundary conditions.

To study the classical YM theory in the vicinity of the coupling constant $g = g_0$ one can expand the relevant equations in a formal power series in $(g-g_0)$. This method was applied to non-dynamical (i.e. with external sources) theory in the vicinity of $g_0 = 0$ [2] and then for $g_0 = \infty$, which is the non-perturbative region and one meets the singular perturbation problem [3]. In this way, an infinite chain of equations for the (functional) expansion coefficients is obtained. Although this infinite set of partial differential equations is extremely complicated, some conclusions can be drown from it.

The formal-expansion approach can be complemented by more refined mathematical methods based on some functional inequalities, namely the Sobolev inequality [5].

This paper is organized as follows. In the following Section basic notions and some general results are given. Section 3 is entirely devoted to the strong-coupling region of the theory. Section 4 deals with the weak-coupling region. In the last Section summary and conclusions are given.

2. *General results*

In this Section we analyze some general properties of non-singular, smooth classical gauge fields, preferably with finite total energy. We expect them to be the most important from the physical point of view.

We are interested in pure (i.e. source free) YM, YM-Dirac and Maxwell-Dirac and YM-Higgs theories. The YM-Dirac field equations read:

$$\partial^\mu F^a_{\mu\nu} + g f_{abc} A^b_\mu F^{\mu c}_{\ \ \nu} = g \bar{\psi}^s_A \gamma^\nu_{AB} T^a_{st} \psi^t_B, \tag{2.1}$$

$$\gamma^\mu_{AB} \partial_\mu \psi^s_B + m\psi^s_A - ig\gamma^\mu_{AB} A^a_\mu T^a_{st} \psi^t_B = 0. \tag{2.2}$$

Here, the following notation is used: f_{abc} are structure constants of the gauge group G, the field-strength tensor $F^a_{\mu\nu}$ is defined below:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu, \tag{2.3}$$

T's are the gauge group generators. The indices $a, b, c = 1, \ldots, n$ are the gauge group indices in the adjoint representation, the indices $s, t, u = 1, \ldots, N$ are the gauge group indices for the fundamental representation, $A, B, C = 1, 2, 3, 4$ are the Dirac bispinor indices and $\mu, \nu, \varrho = 0, 1, 2, 3$ are the Minkowski indices. The indices i, j, k will be used as space indices. Bold-faced letters denote vectors: in the (adjoint) group space when used for gauge fields and in the physical space when used for the space variable x. The Dirac matrices are in the spinor representation:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}. \tag{2.4}$$

The covariant derivative for fields in the adjoint representation reads:

$$(D^\mu \phi)^a \equiv \partial^\mu \phi^a + g f_{abc} A^{\mu b} \phi^c, \tag{2.5a}$$

and for fields in the fundamental representation:

$$(D^\mu \psi)_s \equiv \partial^\mu \psi_s - i g T^a_{st} A^{a\mu} \psi_t, \tag{2.5b}$$

where T's are generators in the fundamental representation here.

The Maxwell-Dirac equations are obtained by putting $f^{abc} = 0$ and neglecting all gauge group indices in (2.1) and (2.2).

Putting $\psi \equiv 0$ in (2.1) one obtains the pure (i.e. source-free) YM equations. In terms of potentials they are of the following form:

$$\Box A^a_\nu - \partial^\mu \partial_\nu A^a_\mu + g f_{abc}[\partial^\mu (A^b_\mu A^c_\nu) + A^b_\mu \partial^\mu A^c_\nu - A^{b\mu} \partial_\nu A^c_\mu] + g^2 f_{abc} f_{cde} A^{b\mu} A^d_\mu A^e_\nu = 0. \tag{2.6}$$

Finally, the YM-Higgs equations are:

$$(D^\mu F_{\mu\nu})^a = g f_{abc} \phi^b (D_\nu \phi)^c, \tag{2.7}$$

$$\Box \phi^a + 2 g f_{abc} A^{b\mu} \partial_\mu \phi^c + g f_{abc} \partial^\mu A^b_\mu \phi^c + g^2 f_{abc} f_{cde} A^b_\mu A^{d\mu} \phi^e - \frac{\partial V[\phi]}{\partial \phi^a} = 0, \tag{2.8}$$

where the Higgs potential V is:

$$V[\phi] = \tfrac{1}{4} \lambda g^2 (\phi^a \phi^a - \mu^2)^2, \tag{2.9}$$

and μ, λ are positive constants.

To have unique solutions to the equations (2.1)–(2.8) the initial or boundary value conditions should be added. In the standard gauge theory the following conditions must be satisfied:

$$A^a_\mu(x, g) \to 0, \quad \psi^s_A(x, g) \to 0, \quad \|\phi^a(x, g)\| \to \mu \text{ when } |x| \to \infty; \tag{2.10}$$

the falloff at the spatial infinity should be sufficiently fast, to ensure finiteness of the field energy.

At first, let us notice that for free, linear field theories, where interaction terms are absent (as free Maxwell, free Dirac), the solutions are coupling constant independent.

Due to the linearity the coupling constant dependence can be at most of the trivial, factorial form:

$$A(x, g) = f(g)A(x), \tag{2.11}$$

where f is an arbitrary function and A stands for an arbitrary field. On the other hand, the coupling constant dependence in more realistic, interacting (i.e. non-linear) field theories can be highly complicated. E.g. for the 't Hooft-Polyakov monopole there is a simple pole at $g = 0$ and essential singularity at $g = \infty \left(A(x, g) \sim \frac{1}{g} e^{-g} \right)$.

Another property of fields, which seems to us very important, besides the coupling constant dependence, is their symmetry. There are several results which deal with symmetries of gauge fields. It should be remembered, that there are two main types of symmetries: usual, which reflects invariance of fields under some group of transformations and generalized symmetry — when invariance is maintained thanks to compensating action of an appropriate gauge transformation [6, 7].

According to the Deser-Coleman theorem [8] there is no finite-energy and smooth solution to the static (i.e. invariant under time translations) pure YM equations. For pure Maxwell equations still stronger result holds: there is neither smooth localized and finite energy nor smooth and static solution. Here, the symmetry group is the group of time translations. Also, spherically symmetric solutions to static YM-Dirac equations do not exist [9].

A special class of gauge fields consists of Abelian potentials, i.e. with constant direction in the gauge group space:

$$A_\mu(x) = A\alpha_\mu(x), \tag{2.12}$$

Abelian static and sourceless YM fields have to have infinite energy or have to be singular. Also, this is because of the high symmetry (in the group space) of these potentials. Hence, to have physically relevant, interesting solutions we should avoid too strong symmetry conditions.

3. *Strong-coupling region*

In this Section we consider solutions to field equations with well-defined strong-coupling limit:

$$\lim_{g \to \infty} A_\mu(x, g) = A_\mu(x), \tag{3.1}$$

where $A_\mu(x)$ is a smooth function fulfilling the field equations after taking the limit $g \to \infty$. The strong-coupling limit is associated with low energies (infrared limit) in the standard YM theory, but with high energies (ultraviolet limit) in electrodynamics.

Here, the problem is much more serious than for weak couplings (see Section 4). This is because of the form of the field equations (see e.g. (2.1)–(2.11)). The coupling constant multiplies the interaction terms, whereas the kinetic terms are those with the highest order derivatives of the fields. Hence, in the strong-coupling region the kinetic (derivative) terms are "perturbations" and we meet the difficult problem of singular perturbations (see also [3] and references therein). Especially, for the existing field theory models this problem is far from being solved.

With the above remarks in mind, we have to limit ourselves to the formal methods with the boundary layer problem being left. Hence, we cannot claim to solve the strong-coupling region problems, in general. However, there is quite a number of problems which can be solved rigorously using our formal methods. These are, for instance, the 'no-go' theorems or 'constructive' problems (i.e. where the solution is constructed explicitly and the boundary value problem can be checked a posteriori).

There is a special class of fields satisfying the condition (3.1) — which are analytic in g at $g = \infty$ $(1/g = 0)$. We define for them the expansion coefficients by:

$$A_\mu(x, g) = \overset{(0)}{A}_\mu(x) + \frac{1}{g}\overset{(1)}{A}_\mu(x) + \frac{1}{g^2}\overset{(2)}{A}_\mu(x) + \dots, \tag{3.2}$$

and the same for the matter fields ψ and ϕ.

Using the expansion (3.2) the following results can be easily proved, for the free YM theory:

Theorem 1: For free YM theory any solution analytic in g at $g = \infty$ should satisfy:

$$\overset{(0)}{A}_\mu(x) = A(x)\alpha_\mu(x), \tag{3.3}$$

where $A(x)$ and $\alpha_\mu(x)$ are arbitrary functions obeying the field equations.

Theorem 2: The monomial potentials of the form $A_\mu(x, g) = 1/g^n \overset{(n)}{A}_\mu(x)$ have to be Abelian, unless $n = 1$. In the case $n = 1$ non-Abelian solutions are also possible.

Theorem 3: For polynomial or infinite-series potentials of the form:

$$A_\mu(x, g) = \sum_{i=0} \frac{1}{g^{n_i}} \overset{(n_i)}{A}_\mu(x), \qquad n_i \in \mathbf{N}, \tag{3.4}$$

if for all i: $n_i \neq 1$ and $n_{i+1} \geqslant 3n_i - 1$, then they are Abelian.

These theorems can be proved using the formal expansion (3.2). We obtain an infinite series of equations for coefficients $\overset{(i)}{A}{}^a_\mu(x)$, instead of the single field equation (2.1). They are of the form:

$$\{\overset{(n)}{A}\} + \sum_{i=0}^{n+2} \{\overset{(i)}{A}, \overset{(n-i+1)}{A}\} + \sum_{\substack{i=0 \\ i+k \leqslant n+2}}^{n+2} \sum_{k=0}^{n+2} \{\overset{(i)}{A}, \overset{(j)}{A}, \overset{(n-i-j+2)}{A}\} = 0 \tag{3.5}$$

(n-th order in $1/g$). Here, $\{\dots\}$ symbolize the linear, bilinear and trilinear (in gauge fields) terms of the equation (2.1), respectively [3].

If the desired solution is to be of the monomial form:

$$A^a_\mu(x, g) = \frac{1}{g^N} \overset{N}{A^a_\mu}(x) = \sum_{i=0}^{\infty} \frac{1}{g^i} \overset{i}{A^a_\mu}(x)\delta_{iN}, \tag{3.6}$$

then (3.5) reduces to the three equations:

$$\{\overset{(N)}{A}\} = 0, \tag{3.7a}$$

$$\{\overset{(N)}{A}, \overset{(N)}{A}\} = 0, \tag{3.7b}$$

$$\{\overset{(N)}{A}, \overset{(N)}{A}, \overset{(N)}{A}\} = 0. \tag{3.7c}$$

Hence, because of (3.7a) the monomial potentials should satisfy the Maxwell equations (as linear part of (2.1) is exactly the free Maxwell equation). If the same initial and boundary conditions are imposed for all group components of the potential it must be Abelian. In this case Eqs. (3.7b) and (3.7c) are obeyed automatically. This proves
Lemma 1: The lowest order equations are of the form:

0-th order:
$$\{\overset{(0)}{A}, \overset{(0)}{A}, \overset{(0)}{A}\} = 0, \tag{3.8}$$

which is solved by (3.3) and proves *Theorem 1*. In the first order we have:

1-st order:
$$\{\overset{(0)}{A}, \overset{(0)}{A}\} + \sum_{P} \{\overset{(0)}{A}, \overset{(0)}{A}, \overset{(1)}{A}\} = 0, \tag{3.9}$$

where the symbol $\sum_P$ means the sum over all permutations of arguments of the $\{\ldots\}$-symbol.

Analogously, results concerning polynomial solutions were obtained.

Therefore, according to the above theorems, to have non-Abelian solutions of the form (3.2) either the first-order term should be present or at least two non-zero terms in the expansion which are sufficiently 'close' to each other (i.e. condition $n_{i+1} < 3n_i - 1$ should be satisfied). For example, any solution of the form (3.2) with $n_i = 3^{i-1}(n_1 - 1/2) + 1/2$ and $n_1 > 1$ is Abelian.

The YM theory with matter fields is more complicated. But even here, some results can be established. Using the formal expansion (3.2) for YM fields and for matter fields we can prove the following theorems:
Theorem 4: For the Maxwell-Dirac system the only analytic at $g = \infty$, non-zero and with non-zero current solutions possible have to obey the following necessary conditions:

(*i*) $\overset{(0)}{\psi}(x) \equiv 0,$

(*ii*) $\overset{(0)}{A^\mu}(x) \equiv 0,$

(*iii*) $\overset{(1)}{\psi}(x)$ and $\overset{(1)}{A^\mu}(x)$ satisfy the full Maxwell-Dirac equations with $g = 1$.

Theorem 5: For the YM-Dirac system the only analytic at $g = \infty$ non-zero solutions possible have to obey:

(*i*) $\overset{(0)}{\psi}(x) \equiv 0,$

(*ii*) $\overset{(0)}{A^\mu}(x) \equiv 0,$

(*iii*) $\overset{(1)}{\psi}(x)$ and $\overset{(1)}{A^\mu}(x)$ satisfy the full YM-Dirac equations with $g = 1$.

Here, again using the formal expansion (3.2) the above results can be easily proved. The essential point is to notice that the first-order Dirac equation, which is of the form:

$$i\gamma^\mu_{AB} \overset{(0)}{A^a} \tfrac{1}{2} \sigma^a_{st} \overset{(0)}{\psi^t_B} = 0, \tag{3.10}$$

has the only solutions: either $\overset{(0)}{A} \equiv 0$ or $\overset{(0)}{\psi} \equiv 0$. On the other hand, the first-order YM equation implies that $\overset{(0)}{A} \equiv 0 \Rightarrow \overset{(0)}{\psi} \equiv 0$. Moreover, if $\overset{(0)}{\psi} \equiv 0$ and $\overset{(0)}{A} \neq 0$ then it can be shown by induction, that all coefficients $\overset{(i)}{\psi} \equiv 0$. Hence, such solutions would be trivial (pure YM and $\psi(x, g) \equiv 0$).

4. *Weak-coupling region*

Here, we focus our attention on solutions with the well-defined weak-coupling limit:

$$\lim_{g \to 0} A_\mu(x, g) = A_\mu(x), \tag{4.1}$$

where $A_\mu(x)$ is an arbitrary smooth function. In the standard YM theory the weak-coupling limit is associated with high energies (ultraviolet region), due to the asymptotic freedom. This is in contrast to electrodynamics, where the weak-coupling limit describes the low energy (infrared) region.

The special class of fields with the property (4.1) are those analytic in g at $g = 0$, i.e.

$$A_\mu(x, g) = \overset{(0)}{A_\mu}(x) + g\overset{(1)}{A_\mu}(x) + g^2\overset{(2)}{A_\mu}(x) + \dots . \tag{4.2}$$

For the pure YM equations (2.6) in the $g \to 0$ limit we have result, which is true independently of the coupling constant values for the free Maxwell theory (see Section 2): *Theorem 1*: For pure YM theory there is no smooth and static solution (with energy not necessarily finite) satisfying (2.12) and analytic in g at $g = 0$.

This can be proved, using the expansion (4.2) in the way analogous as in Section 3. With the aid of this formal methods further results can be obtained. We have theorems analogous with Theorem 2 and 3 of Section 3:

Theorem 2: The monomial potentials of the form: $A_\mu(x, g) = g^n \overset{(n)}{A_\mu}(x)$ are Abelian (if exist).

Theorem 3: The polynomial or infinite-series potentials of the form:

$$A_\mu(x, g) = \sum_{i=0}^{\infty} g^{n_i} \overset{(n_i)}{A}_\mu(x), \qquad n_i \in \mathbf{N}, \tag{4.3}$$

are Abelian when non-zero expansion terms are sufficiently dilute: $n_{i+1} \geqslant 3n_i - 1$ (for all i).

Up to now, we have used the formal expansion method. However, for the weak--coupling (or weak-field) region we meet non-singular perturbation problems and "constructive" methods can be applied. This will be done in the following.

At first, let us rewrite the formulae (2.2) in a more convenient form for the gauge group SU(2). Defining

$$\psi^{\pm} = \varphi_\mu^{\pm} \sigma^\mu \sigma^2, \qquad \psi = \begin{bmatrix} \psi^+ \\ \psi^- \end{bmatrix}, \tag{4.4}$$

we obtain the following equations for the Dirac field components $\varphi_\mu^{\pm}$:

$$\mp \partial_0 \varphi_0^{\mp} + \partial_i \varphi_i^{\mp} \pm \frac{1}{2i} gA_0^a \varphi_a^{\mp} - \frac{1}{2i} gA_a^a \varphi_0^{\mp} - \tfrac{1}{2} g\varepsilon_{ika} A_i^a \varphi_k^{\mp} \mp im\varphi_0^{\pm} = 0, \tag{4.5a}$$

$$\mp \partial_0 \varphi_k^{\mp} + \partial_k \varphi_0^{\mp} - i\varepsilon_{kil} \partial_i \varphi_l^{\mp} - \tfrac{1}{2} g\varepsilon_{iak} A_i^a \varphi_0^{\mp} - \frac{1}{2i} gA_a^k \varphi_a^{\mp}$$

$$+ \frac{1}{2i} gA_a^a \varphi_k^{\mp} - \frac{1}{2i} gA_k^a \varphi_a^{\mp} \mp im\varphi_k^{\pm} = 0. \tag{4.5b}$$

From now on we shall restrict our discussion to stationary Dirac fields, i.e. satisfying

$$\partial_0 \varphi_\mu^{\mp} = i\varepsilon \varphi_\mu^{\mp}, \tag{4.6}$$

and to static YM potentials:

$$\partial_0 A_\mu^a = 0 \tag{4.7}$$

in the Coulomb gauge $\partial_i A_i^a = 0$. Multiplying Eqs. (4.5a) and (4.5b) by $(\partial_k \bar{\varphi}_k^{\mp} \pm im\bar{\varphi}^{\pm} - i\varepsilon\bar{\varphi}_0^{\mp})$ and $(\partial_k \bar{\varphi}_0^{\mp} - i\varepsilon_{kil}\partial_i \bar{\varphi}_l^{\mp} \mp i\varepsilon\bar{\varphi}_k^{\mp} \pm im\bar{\varphi}_k^{\pm})$ respectively and integrating over space, one arrives at

$$\int [|\nabla \varphi_\mu^M|^2 + (m^2 - \varepsilon^2)\, |\varphi_\mu^M|^2 - \varepsilon A_\mu^a j^{a\mu}] d^3x = g^2 \int (\bar{C}^M C^M + \bar{D}_k^M D_k^M) d^3x. \tag{4.8}$$

Here j_μ^a is the YM current,

$$j_0^a = \sum_M g(\bar{\varphi}_0^M \varphi_a^M + \varphi_0^M \bar{\varphi}_a^M + i\bar{\varphi}_i^M \varphi_j^M \varepsilon_{ija}), \tag{4.9a}$$

$$j_j^a = \sum_M g[M|\varphi_0^M|^2 \delta^{ja} + i\varepsilon_{aji} M(\bar{\varphi}_0^M \varphi_i^M - \bar{\varphi}_i^M \varphi_0^M)$$

$$+ M(\bar{\varphi}_a^M \varphi_j^M - \delta^{ja} |\varphi_j^M|^2 + \varphi_a^M \bar{\varphi}_j^M)]; \tag{4.9b}$$

$M = \pm$ and sum over M is assumed; C's and D's are defined below:

$$C^M = -A_0^a \frac{M}{2i} \varphi_a^M - \frac{1}{2i} A_a^a \varphi_0^M - \tfrac{1}{2} \varepsilon_{ika} A_i^a \varphi_k^M, \tag{4.10a}$$

$$D_k^M = -\frac{M}{2i} A_0^k \varphi_0^M - \tfrac{1}{2} \varphi_k^M A_0^l \varepsilon_{ilk} - \tfrac{1}{2} \varepsilon_{iak} A_i^a \varphi_0^M - \frac{1}{2i} A_a^k \varphi_a^M$$

$$+ \frac{1}{2i} A_a^a \varphi_k^M - \frac{1}{2i} A_k^a \varphi_a^M. \tag{4.10b}$$

Now, we are ready to formulate our basic results.

Lemma 1. Suppose that Dirac and YM fields are static and $\varphi_\mu^\mp \in W_{1,2} \cap L_6$ [1], $A_\mu^a \in L_3$. Define $\eta = \lim_{g\to 0} |g| \, ||A||_{L_3}$. Then, the nonzero solutions of the Dirac equation (4.5) are absent for sufficiently small η.

For potentials analytic in g at $g = 0$ the above limit is equal to zero. Hence, in such a case the solutions of the Dirac equation (4.5) are trivial, $\psi = 0$. Using the Deser no-go theorem [8] one arrives at the following.

Theorem 4. Suppose that Dirac and YM fields are static and $\varphi_\mu^\mp \in W_{1,2} \cap L_6$. Let the YM potential be analytic in g at $g = 0$. Then the SU(2) YMD equations have no nontrivial solutions.

Similar results hold for stationary Dirac fields but under stronger conditions imposed on gauge potentials:

Lemma 2: Suppose that Dirac field is stationary, $\varphi_\mu^\mp \in W_{1,2} \cap L_6$, gauge potentials are static and $A_\mu^a \in L_{3,2}$. If $m^2 - \varepsilon^2 \geqslant 0$ then nonzero solutions of the Dirac equation (3.5) are absent for sufficiently small η.

Theorem 5. Under the conditions stated in Lemma 2 the SU(2) YMD equations have no nonzero solutions.

To prove Ths 4 and 5, let us notice that Lemmas 1 and 2 imply $\psi = 0$ for η sufficiently small; thus, the YM current $j_\mu^a = 0$. As it was remarked, in this case Deser showed the absence of finite energy solutions [8].

Now we will prove Lemma 1. At first, we use Hölder and Minkowski inequalities to estimate the right hand side of (4.8) by

$$g^2 C ||A^2||_{L_{3/2}} ||\varphi||_{L_3} = g^2 C ||A||_{L_3}^2 ||\varphi||_{L_6}^2. \tag{4.11}$$

Here, C is a real, positive constant.

The crucial point of the proof is the "energy" Sobolev estimation [5, 10]

$$||\varphi||_{L_6}^2 \leqslant \frac{1}{5.478} ||\nabla\varphi||_{L_2}^2. \tag{4.12}$$

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[1] Here, $W_{1,2}$ is the Sobolev space, i.e. the completion of the set $\{f \in C_0^\infty(R^3)\colon ||f||_{1,2} = [\int (|f|^2 + |\nabla f|^2) d^3x]^{1/2}\}$ in the $||\cdot||_{1,2}$ norm. L_p is the set of functions completed in the norm $||f||_p = [\int |f|^p d^3x]^{1/p}$.

Using (4.12), we arrive at

$$\int d^3x(|\nabla\varphi_\mu^M|^2+m^2|\varphi_\mu^M|^2) \leqslant Cg^2 \; \|A\|_{L_3}^2 \int d^3x(|\nabla\varphi_\mu^M|^2 +m^2|\varphi_\mu^M|^2). \tag{4.13}$$

Assuming that g is small so that

$$Cg^2\|A\|_{L_3}^2 < 1$$

and

$$\eta < \frac{1}{(C)^{1/2}}, \tag{4.14}$$

the only possibility to satisfy both (4.13) and (4.14) is to put $\varphi_\mu^M = 0$. That concludes our proof.

The proof of Lemma 2 is similar. The difference is due to the occurrence of the term proportional to "ε" in the Eq. (4.8); this results in a stronger assumption $A_\mu^a \in L_{3/2}$, instead of $A_\mu^a \in L_3$.

The conditions imposed on ψ are well justified for the massive Dirac theory, but the conditions on gauge potentials A_μ^a need an explanation. We have assumed $A_\mu^a \in L_3(R^3)$ in the static case (and even stronger condition for stationary Dirac fields); this seems to be not too restrictive, since, as it is well known, the L_p integrability of φ does not imply $\varphi(\infty) = 0$ in fact, one can find an oscillatory $\varphi \in L_p$ which is not bounded at spatial infinity [11]. But on the other hand, the important configurations with $A_\mu^a = O(1/r)$, $r \to \infty$ are not cubic integrable. We give arguments that this type of configurations should be excluded because its presence can spoil the Lorentz invariance. Let us consider the global charge of colour fields, $Q = \int d^3x(j^0+g[A_i, E_i]+g[A_i, A^0])$. Notice that Q depends on gauge potentials, in contrast to the classical electrodynamics. The global charge is not invariant under Lorentz transformations if $\partial_i A_\mu^a = 0(1/r^2)$ [12]. To restore the Lorentz invariance the required falloff at spatial infinity should be $\partial_i A_\mu^a = o(1/r^2)$, that is (in a suitable gauge) $A_\mu^a = o(1/r)$, unless special symmetry assumptions are made which assure the desired falloff of charge densities even under weaker conditions on the falloff of potentials. The Gauss--like equations $D_i F^{i0} = j^0$ imply now $Q = 0$. For a few known solutions of YM-Higgs equations, e.g., the Bogomolny-Prasad-Sommerfield monopoles, which are highly symmetric, the global charge vanishes although the asymptotic behaviour is not excluded. Let us add also that one is interested in short-ranged solutions, to reveal the so-called "total colour confinement" [13].

Remark. Above we have assumed the SU(2) gauge group. It was done only to simplify the notation. In fact, our results hold for any semi-simple compact gauge group.

Similar results are obtained for the Maxwell-Dirac equations. We have the following.

Lemma 3. Let $\psi \in W_{1,2} \cap L_6$ and be stationary, $\psi = \exp(iEt)\psi_0$, while the electromagnetic potential is static and

(*i*) $\quad A_\mu \in L_3 \quad$ for $\quad E = 0$

or

(*ii*) $\quad A_\mu \in L_3 \cap L_{3/2} \quad$ for $\quad E \neq 0$.

If in the case (i) the quantity $\frac{2}{5.478}\,||A||^2_{L_3} < 1$ or in the case (ii) $\frac{2}{5.478}\,(||A||^2_{L_3} + |eE|\,||A||_{L_{3/2}}) < 1$, where $A = A_0$ for $eE > 0$ and $A = A_k$ for $eE < 0$, nonzero solutions of the Dirac equation with the external potential A_μ do not exist.

Hence, according to Lemma 3, there do not exist fermion bound states for the Dirac equation with an external electromagnetic potential analytic in "e" at $e = 0$. Before going to the study of the full Maxwell-Dirac system a remark is in order. For a single fermion coupled to a Maxwell field the conditions $A_\mu \in L_3$, $F_{\mu\nu} \in L_2$ imply $\psi = 0$[2]. (If one integrates the Gauss equation $F_{i0} = -e\bar{\psi}\psi$, then the left hand side vanishes, due to the quick asymptotic falloff and therefore $\psi = 0$.) Therefore, we admit more general possibility – a collection of fermion fields $(\psi) = (\psi_1, \psi_2, \dots \psi_l)$. As in the YM case the conditions imposed on potentials result in vanishing of the first few multipoles (the total charge and dipole moments).

Applying Lemma 3, we obtain

Theorem 6. Under assumptions as in Lemma 3, there do not exist finite energy nonzero solutions of the Maxwell-Dirac system.

Proofs of Lemma 3 and Theorem 3 are analogous to those presented in Lemmas 1, 2 and Theorems 4, 5; therefore we omit them. In the case of dynamical Yang-Mills-Higgs (YMH) model the absence of perturbative solutions can be proved without employing the field equations. To ensure finiteness of the energy it is necessary to impose (apart from oscillatory solutions which can give finite energy even if (4.15) is not satisfied; we suspect that such solutions are not realistic)

$$D_\mu \varphi = o(1/r^{3/2}), \tag{4.15}$$

where φ is a Higgs field in the adjoint representation of a gauge group G, G being arbitrary compact and semisimple Lie group. At spatial infinity φ tends to an angle-dependent vector function a^a, such that $(a^a a^a)^{1/2} = 1$; thus $\partial_i \varphi = O(1/r)$. Therefore, $D_i\varphi^a = \partial_i \varphi^a + gf^{abc}A^b_i\varphi^c$ cannot recede as $1/r^{3/2}$ for $r \to \infty$ in the limit $g \to 0$, assuming that the potential A^a_i is nonsingular at $g = 0$. Hence we arrive at the following.

Theorem 7. Assume that the energy of YMH system is finite and the gauge group is compact and semisimple. Then, topologically nontrivial smooth solutions are absent if $\lim_{g \to 0} gA^a(x, g) = 0$.

The case of angle-dependent function $a^a = \lim_{r\to\infty} \varphi^a$ corresponds to topologically nontrivial sectors of YMH theory, where many monopole solutions have been found. All the monopoles are singular at $g = 0$.

In the case of constant a^a's the above conclusion does not hold. To ensure that the energy is finite we need now a stronger condition on potentials:

$$fa^{abc}A^b_i a^c = o(1/r^{3/2}).$$

Those components of A^a_i that are not parallel to (a^a) should decrease as $1/r^{3/2+\varepsilon}$. None

[2] We ignore the possibility that A is oscillatory at spatial infinity; we assume that $A_\mu \in L_3$ and $A_\mu \partial_k A_\mu = o(1/r^2)$.

solution is explicitly known in that sector (which is topologically trivial) although Taubes has proved the existence of an unstable solution [14].

To complete our list of results, we give the following theorem for Einstein-YM system which is a direct consequence of [15].

Theorem 8. Let the falloff of A_i^a, A_0^a, $\partial_l A_k^a$ at spatial infinity be in an adapted system of co-ordinates [15]:

$$A_i^a = O(1/r^{1+\varepsilon}), \quad \partial_i A_k^a = O(1/r^2), \quad A_0^a \partial_i A_0^a = O(1/r^{2+\varepsilon}).$$

Suppose that the contracted Christoffel's system $\Gamma^\mu_{i\mu}$ is cubic integrable. Then the EYM system does not possess static perturbative solutions for sufficiently small $\Gamma^\mu_{i\mu}$.

We would like to emphasize that Lemmas 1–3 and Theorems 4–6, 8 are valid not only for solutions which are analytic in the coupling constant "g" or "e" but for more general configurations. In fact, the crucial role is played by a parameter $\eta = \lim\limits_{g \to 0} |g|\ ||\varphi||$, where $||\cdot||$ denotes certain functional norm and φ is a solution of YMD/MD or EYM equations. We have proven that for sufficiently small η the only possible solution is the trivial one, $\varphi = 0$. Thus a nonzero solution (if any does exist) should be singular in g, so that $||\varphi|| = \dfrac{A}{g} + o(1/g)$ and the constant A cannot be arbitrarily small.

5. *Summary*

In this paper the coupling-constant dependence of gauge fields has been investigated. In particular, several results concerning the coupling-constant dependence of solutions of gauge-matter equations have been obtained for the strong-coupling ($g \to \infty$) and weak-coupling ($g \to 0$) regions.

From the strong-coupling analysis (Section 3) the general suggestion can be drawn that either fields are nonanalytic at $g = \infty$ or the $1/g$-dependence should be quite complicated (Theorems 2 and 3 of Sect. 3), to have non-trivial solutions. It is interesting to notice the exceptional role of the first order term $(\overset{(1)}{A}(x)/g)$ which is of the same form as the instanton solution. This term contributes mostly to the gauge and matter fields in the strong-coupling limit (Theorems 4 and 5 of Sect. 3) as well as it has strongly non-Abelian nature.

On the other hand, we have proven that nonzero solutions of YM-Dirac (YM-Higgs, YM-Einstein) equations must be sufficiently singular in g at $g = 0$ (at least as $1/g$). This is in full agreement with all explicitly known solutions. This result puts into question attempts to use perturbative techniques to the study of weak-coupling region of gauge theories. At least, these methods cannot be applied for systems with dynamical sources. The same conclusion is true for any selfconsistent field theory provided that the equations of motion are elliptic and the selfinteraction is "sufficiently" nonlinear [16].

Further implication of our results concerns the total colour screening solutions. All proposed solutions with this "confining" property are analytic at $g = 0$ [13]. These fields are excluded, supposing that the external currents are of dynamical origin, because of Theorems of Sect. 4. However, the screening effect for fields analytic at $g = \infty$ [3] is not excluded.

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