THE VENEZIANO n-POINT FUNCTION — SYMMETRIC REPRESENTATIONS AND THEIR IMPLICATIONS

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(Presented at the X^{th} Cracow School of Theoretical Physics, Zakopane, June 12–26, 1970)

I shall limit myself to one aspect of the recent developments in the study of the Veneziano model, *i. e.*, to symmetric representations of the n-point function and their implications. To be self-contained, however, the first half will be an elementary introduction to the conventional way of construction of the Veneziano n-point function [1].

My subject may be deviating from the main lines of this meeting, but, hopefully, not so much. One additional feature of the n-point function is that it satisfies the bootstrap condition, so that external as well as internal particles are equally "composite". In fact some of the characteristic features of this model can be easily understood as these of interaction of composite systems.

1. *Quark diagram and resonances* [2–3]

1.1. For simplicity we shall assume the following:

1) All the external particles are scalar mesons and consist of a pair of (scalar) quark and antiquark.

2) All the resonances are non-exotic and consist of a pair of (scalar) quark and antiquark.

3) Diffractive processes (pomeron exchange) are not included.

4) All the Regge trajectories are linear and have a universal slope[1] $\alpha' > 0$.

5) All the Regge trajectories have a common intercept $\alpha^0 = 1$.

The condition 4) is very essential for the construction of the n-point function. Other conditions can be, to a certain extent, loosened. The condition 5) leads to an unplesant result

$$(m_{\text{scalar}})^2 = -1/\alpha' < 0 \tag{1.1}$$

(including the external particles !), but it very much simplifies the formulation. (Most of the results mentioned below are valid without this condition.)

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[1] We shall often put $\alpha' = 1$.

1.2. Let us consider a reaction of n external mesons, formally all regarded as incoming, in the quark picture. Combine the quark lines at the ends of the external particles and trace them one by one in succession, and rearrange the external particles in this order. Example ($n = 6$). $K^+\pi^-\pi^+\pi^-\bar{K}^0\varnothing$, see Fig. 1.1.

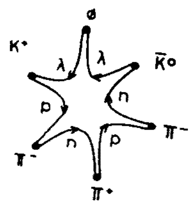

Fig. 1.1

We shall restrict ourselves to such cases, where, as in the above example, we can, in this way, form a closed quark line, which then specifies an ordering of external particles on a circle. (We shall not distinguish clockwise and anticlockwise orders).

Remarks

1) Separate quark line loops correspond to possibility of diffractive process (or forbidden coupling). Example: Fig. 1.2.

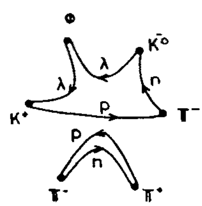

Fig. 1.2

2) No loop diagrams (closed quark line without entering into external particles) will be taken into account.

3) For a given reaction, there are, in general, a number of possible ways of combining quark lines into a single closed one as illustrated above. In the case of pure quark state external mesons, the maximum number is $1/2\,(n-1)\,!$ (all circular ordering without distinction between clockwise and anticlockwise ones).

In the following we examine the Veneziano amplitude ascribed to one such diagram, which is in general only a part of the whole amplitude (even without satellite terms), and, if taken by itself alone, leads to exchange degeneracy.

For convenience of drawing, we shall deform the connected quark diagram (Fig. 1.1) into a circular form (Fig. 1.3). The dots represent the external mesons, and the arcs in between represent quarks.

1.3. In a connected quark diagram as given above, there are n quark lines which can be pair-wise combined into $(1/2)\,n\,(n-1)$ families of resonances. But n out of them are already fixed as external mesons (pairs of neighbouring quark lines), so that the remaining $(1/2)n(n-3)$ resonances can appear in intermediate stages of the reaction. These possible resonances can be indicated by "hooks" which connect a pair of non-neighbouring quark lines. Indeed

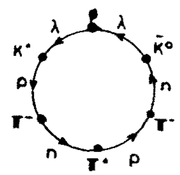

Fig. 1.3

one can imagine, if one likes, that when a hook shrinks and pulls two quark lines together, the diagram is so distorted that it looks like a resonance formation (Fig. 1.4).

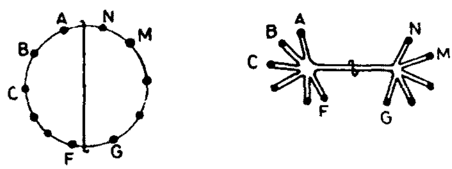

Fig. 1.4

The circular order of external particles specified by the quark diagram has now got the following meaning: Any group of external mesons which are in succession in this order (AB, ABC, MNA, *etc.*) can form a resonance, while those separated in this order (AC, MB, NCF, *etc.*) cannot.

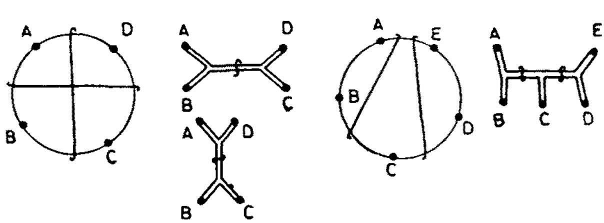

Fig. 1.5

As can be seen from simple examples (Fig. 1.5), some of the resonances can appear simultaneously in a single Feynman diagram, but some of them cannot. The former, compatible resonances, correspond to hooks which do not cross each other, while the latter, non-compatible resonances, are represented by crossing hooks. The maximal number of compatible resonances is $(n-3)$.

2. *Construction of the n-point function* [1]

2.1. Firstly let us recall how this incompatibility of certain resonances (corresponding to crossing hooks) is incorporated in the original Veneziano model ($n = 4$). In the integral representation of the beta function

$$B(-\alpha_s, -\alpha_t) = \int_0^1 dv\, v^{-\alpha_s-1}(1-v)^{-\alpha_t-1},$$

$$(\alpha_s < 0,\ \alpha_t < 0), \tag{2.1}$$

the resonances in the s-channel come from the region $v \sim 0$ and the resonances in the t-channel come from the region $(1-v) \sim 0$. It should be further noticed that, when $v \sim 0$, *i.e.* in the s-channel a resonance appears there is a region where the variable α_t represents essentially the angle $\cos\theta_t$; therefore it should appear only as polynomials and not as an exponential, in order to avoid ancestor resonances. This is in fact guaranteed since $(1-v)$ reduces to 1.

We can formulate the procedure of constructing the four point funtion as follows. Each step will have its counterpart in the construction of the n-point function.

1) For each family of resonances, α_s and α_t, we introduce its conjugate variable, u_s and u_t, respectively; each of them varies from 0 to 1. The integrand is given by

$$\prod_{\lambda=s,t} u_\lambda^{-\alpha_\lambda-1}. \tag{2.2}$$

2) In order to take into account the incompatibility of s-channel and t-channel resonances, we impose a constraint

$$u_s + u_t = 1. \tag{2.3}$$

3) We choose one of the Feynman diagrams which involve a resonance, for instance the one which includes an s-channel resonançe, and correspondingly, choose an independent variable

$$v \equiv u_s \tag{2.4}$$

and solve the constraint equation (2.3), getting

$$u_t = 1 - u_s = 1 - v. \tag{2.5}$$

4) We confirm that a change of independent variable from $v \equiv u_s$ to $v' \equiv u_t$ leaves

$$\int_0^1 dv \ldots = \int_0^1 dv' \ldots \tag{2.6}$$

unchanged.

5) Thus we get

$$B^{(4)} = \int_0^1 du_s^{-\alpha_s-1} u_t^{-\alpha_t-1} \tag{2.7}$$

with

$$u_s = v, \qquad u_t = 1 - v.$$

2.2. Now the generalization to the n-point function is straightforward.

1) For each out of $\frac{1}{2}n(n-3)$ resonance families, $\alpha_{j,k}$ (indices refer to two quark lines), we introduce a conjugate variable $u_{j,k}$ which varies from 0 to 1. The integrand is then given by

$$\prod u_{j,k}^{-\alpha_{j,k}-1}. \tag{2.8}$$

2) In order to take into account the incompatibility of certain resonances, we impose a set of constraints,

$$u_{j,k}+\prod_{\substack{\text{all}\\ \text{crossing}\\ \text{hooks}}} u_{l,m}=1 \quad (\text{for each of } u_{j,k}). \tag{2.9}$$

In the multiple product all the conjugate variables of resonances incompatible with $\alpha_{j,k}$ are included. The variables $\alpha_{l,m}$ of these crossing hooks can be essentially angles $\cos\theta$ when $u_{j,k}\sim 0$ and then each of them should bocome 1.

3) We choose one of the Feynman diagrams which involve $(n-3)$ resonances and take the corresponding conjugate variables as independent ones

$$v_1, v_2, \ldots, v_{n-3}, \tag{2.10}$$

and solve the constraint equations to express the remaining $\frac{1}{2}(n-2)$ $(n-3)$ variables in terms of $v_1, v_2, \ldots, v_{n-3}$.

4) In order that the integral remains invariant when we go over from one set ofindependent variables to another (this is a requirement of multiduality), we need an invariant volume element, which is known to have the following fro [5]

$$dV^{(n)}=\frac{dv_1dv_2\ldots dv_{n-3}}{\prod\limits_{(j,k,l)}^{n-2} F(v_j, v_k, v_l)} \tag{2.11}$$

where the product in the denominator includes one factor for each vertex of the chosen Feynman diagram.[2] (In the case of $n=4$ this factor reduces to 1.)

5) Thus we get

$$B^{(n)}=\int_0^1\ldots\int_0^1 dV^{(n)}u\prod\nolimits_{j,k}^{-\alpha_{j,k}-1} \tag{2.12}$$

$dV^{(n)}$, $u_{j,k}$ are expressed in terms of $v_1, v_2, \ldots, v_{n-3}$.

The n-point function obtained in this way has been examined and shown to have expected resonance structures and Regge asymptotic behaviours.

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[2] We use the convention that external lines have $v=0$. The three arguments in the function F appearing in the denominator of (2.11) are three lines which form the vertex.

3. *Multiperipheral configuration* [1, 6]

3.1. For $n = 4, 5$, the Feynman diagrams involving $(n-3) = 1,2$ resonances have always the same shape (Fig. 3.1). So the change of independent variables amounts to a cyclic change of the names and corresponding 4-momenta of the external particles. For $n \geqslant 6$,

Fig. 3.1

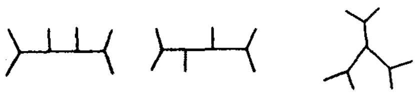

Fig. 3.2

however, there are configurations of different shapes (Fig. 3.2). Any of these can be chosen as the basis of the set of independent variables. These figures correspond to the boundary configuration:

$$v_1 = v_2 = \ldots = v_{n-3} = 0. \tag{3.1}$$

Although all these choices give the identical result, as they should, the multiperipheral configuration is generally preferred for its simplicity. For instance we take

$$v_1 = u_{1,3},\ v_2 = u_{1,4}, \ldots v_{n-3} = u_{1,n-1}, \tag{3.2}$$

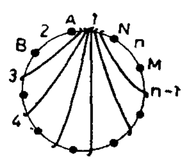

Fig. 3.3

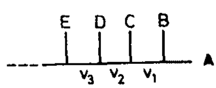

Fig. 3.4

and the rest of the variables as well as the volume element can be expressed in a simple way.

Starting from the well-known 4-point function,

$$\int_0^1 dv_1 v_1^{-\alpha(AB)-1}(1-v_1)^{-2p_B \cdot p_C} \tag{3.3}$$

we can add (Fig. 3.4) one more external line D, and now get the 5-point function

$$\int_0^1 dv_1 v_1^{-\alpha(AB)-1} \int_0^1 dv_2 v_2^{-\alpha(ABC)-1}(1-v_1)^{-2p_B \cdot p_C} \times$$

$$\times (1-v_1 v_2)^{-2p_B \cdot p_D}(1-v_2)^{-2p_C \cdot p_D}. \tag{3.4}$$

For 6-point function, add one more line E, and get

$$\int_0^1 dv_1 v_1^{-\alpha(AB)-1} \int_0^1 dv_2 v_2^{-\alpha(ABC)-1} \int_0^1 dv_3 v_3^{-\alpha(ABCD)-1} (1-v_1)^{-2p_B \cdot p_C}$$

$$(1-v_1 v_2)^{-2p_B \cdot p_D} (1-v_2)^{-2p_C \cdot p_D} \times$$

$$\times (1-v_1 v_2 v_3)^{-2p_B \cdot p_E} (1-v_2 v_3)^{-2p_C \cdot p_E} (1-v_3)^{-2p_D \cdot p_E}. \tag{3.5}$$

The rule of constructions is now obvious.

The added factors have the expected form: the invariant parameter $(p_B \cdot p_E)$, for example, can be regarded as including $(\cos \theta_{BE})$ when any one of the three intermediate stages forms a resonance. Then it should appear only as polynomials and not as exponentials, in order to avoid ancestors. Thus the factor where $(p_B \cdot p_E)$ appears as the exponent should be one, if any of v_1, v_2, v_3 vanishes. The expression $(1-v_1 v_2 v_3)$ is the simplest form that satisfies the requirement.

4. *String model and operator formalism* [7–9]

4.1. Another favourable feature of the multiperipheral configuration is that one can write down a compact factorized expression for the Veneziano n-point function. In order to see this, it will be convenient (though not necessary) to make recourse to an intuitive model proposed[3] by Nambu [7] and Susskind [8].

According to the multiperipheral configuration, we start with the meson A, which is now regarded as a system of a quark and an antiquark combined by an elastic string. The state of this string is described by a set of functions $\chi_\mu(\xi)$, and their canonical conjugates $\pi_\mu(\xi)$, $\mu = 0, 1, 2, 3$, satisfying the periodicity condition.

$$\chi_\mu(\xi+2\pi) = \chi_\mu(\xi),\ \pi_\mu(\xi+2\pi) = \pi_\mu(\xi). \tag{4.1}$$

We quantize the vibration of this string so that

$$\chi_\mu(\xi) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}} [a_{k,\mu} + a_{k,\mu}^{+}] \cos k\xi, \tag{4.2}$$

$$\pi_\mu(\xi) = \sum_{k=1}^{\infty} i \sqrt{\frac{k}{2}} [a_{k,\mu} - a_{k,\mu}^{+}] \cos k\xi, \tag{4.3}$$

where the constant mode $k = 0$ is discarded. Here a and a^+ are the annihilation and creation operators and satisfy

$$[a_{k,\mu}, a_{l,\nu}^{+}] = -\delta_{k,l} g_{\mu,\nu}, \tag{4.4}$$

all the other commutators vanishing. The "Hamiltonian" is given by

$$H = -\sum_{k,\mu} k a_{k,\mu}^{+} a_{k,\mu} = \sum k n_{k,\mu}. \tag{4.5}$$

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[3] H. B. Nielsen had a similar picture from which he developed his fluid model. See Section 7.

The vibrational state of this string, expressed in terms of this infinite set of harmonic oscillators, describes the internal excitation of the meson and the eigenvalue of (4.5) determines the resonance energy.

4.2. At the initial stage, the meson A is in the ground state of the string vibration (spin 0), so we write it $|0\rangle$. Then the external particle B interacts with the quark (or anti-

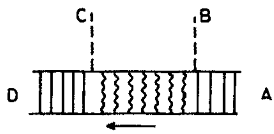

Fig. 4.1

quark) located at $\xi = 0$. See Fig. 4.1. This interaction, we assume, is described by a vertex operator:

$$\Gamma(p_B) = :\exp\left(i2p_B\cdot\chi(0)\right): \tag{4.6}$$

Then the system is excited and the intermediate stage, with the given energy squared, $s_{AB} = (p_A+p_B)^2$, is described by a propagator

$$\Delta(s_{AB}) = \frac{1}{H-\alpha(s_{AB})} = \int_0^1 dv v^{-\alpha(AB)-1+H}. \tag{4.7}$$

Then comes another interaction with the external particle C, which is expressed by $\Gamma(p_C)$ and finally the system is in the state of meson D which is again in the ground state and can be described by $\langle 0|$.

In this way we have got

$$\langle 0|\Gamma(p_C)\Delta(s_{AB})\Gamma(p_B)|0\rangle \tag{4.8}$$

which can be evaluated, using the commutation relations derived from (4.4). We find

$$(4.8) = \int_0^1 dv v^{-\alpha(AB)-1}\times$$

$$\times\left\langle 0\left|\exp\left\{i\sqrt{2}p_C^\mu\sum\frac{a_{k,\mu}}{\sqrt{k}}\right\} v^{-\sum k a^+_{k,\mu} a_{k,\mu}} \exp\left\{i\sqrt{2}p_B^\mu\sum\frac{a^+_{k,\mu}}{\sqrt{k}}\right\}\right|0\right\rangle$$

$$= \int_0^1 dv v^{-\alpha(AB)-1}\exp\left\{(2p_B\cdot p_C)\sum_{k=1}^{\infty}\frac{v^k}{k}\right\}$$

$$= \int_0^1 dv v^{-\alpha(AB)-1}(1-v)^{-2p_B\cdot p_C} \tag{4.9}$$

i.e. the Veneziano 4-point function is reproduced.

It can be verified in a similar way that this prescription works for the case of n-point function, *i.e.*, more external distuıbances coming in, as well. This operator foımalism not only gives a concise expression, but also allows a deeper insight into the nature of the dual models, clearly showing the very high degeneracy of resonance levels in the intermediate stage [7, 10].

4.3. The above results can be in a natural way derived from a field theoretical formulation [7, 8, 11]. Here we quote the proper time equation proposed by Miyamoto [11]. He sets up the Schrödinger equation

$$\left\{ P^2 - H + V(X) - i\frac{\partial}{\partial t} \right\} \Psi(X, u_{k,\mu}; \tau) = 0, \tag{4.10}$$

where X, P are 4-coordinate and 4-momentum of the $q-\bar{q}$-string system and

$$\mathrm{V(X)} = \int \frac{d^3\vec{p}}{\sqrt{p_0}} a_p e^{-ipX} e^{ipX(0)} + \text{herm.conj.} \tag{4.11}$$

a_p being the annihilation operator of the "external" meson with 4-momentum p. By putting

$$\bar{\Psi} = \exp\{-i(P^2 - H)\tau\}\Omega, \tag{4.12}$$

we go over to the "interaction representation" and get

$$\left\{ V(P, \tau) - i\frac{\partial}{\partial \tau} \right\} \Omega(P, n_{k,\mu}; \tau) = 0 \tag{4.13}$$

where $V(P, \tau)$ is the transformed "interaction Hamiltonian" in the momentum representation. Then the n-th term in the expansion of

$$U(\infty, -\infty) = T\exp\{-i \int_{-\infty}^{\infty} V(P, \tau) d\tau\} \tag{4.14}$$

leads to the expression of Veneziano n-point function in the operator formalism.

4.4. In these formulations the external mesons are treated in a very asymmetric way. Two privileged mesons are regarded as the initial and final states of the main system, while others are just external disturbances, although the construction of the n-point function

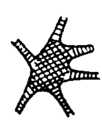

Fig. 4.2

we have followed clearly shows that the content of the expression is in fact symmetric with respect to all the external mesons (under the restriction of the given ordering). What we should have liked would be rather a figure like Fig. 4.2. than the Fig. 4.1. In the following sections we shall try to improve the formulation with respect to this point.

5. *Symmetric representation* [11]

5.1. In Sections 3 and 4 we had to choose a particular set of $(n-3)$ compatible resonances out of $\frac{1}{2}n\,(n-3)$ and, although the n-point amplitude is invariant under a transformation to another set, this was not manifest. So we should like to reformulate it into a more symmetric expression.

Our argument is based on a geometrical interpretation of the conjugate variables u. For this purpose it is convenient to reinterpret the symbolical quark diagram, Fig. 1.3, as a unit circle in the complex plane. The dots are now regarded as a kind of coordinates ascribed to the external mesons, which move on the unit circle maintaining the given ordering.

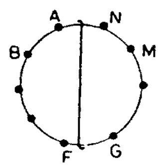

Fig. 5.1

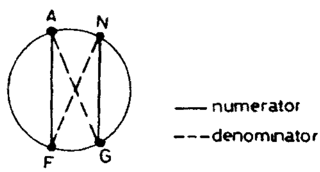

Fig. 5.2

Then we identify the conjugate variable u, shown in Fig. 5.1, with the anharmonic ratio of the 4 points which are adjacent to the relevant hook,

$$u = \frac{(Z_A - Z_F)}{(Z_A - Z_G)} \Bigg/ \frac{(Z_N - Z_F)}{(Z_N - Z_G)} \tag{5.1}$$

where Z_A *etc.* are the coordinates of A *etc.* It is easily seen that u is real and varies between 0 and 1 as the external mesons move on the circle keeping their order (Fig. 5.2.). Utilizing the basic properties of the anharmonic ratio, illustrated in Fig. 5.3, we can verify that the ansatz (5.1) satisfies the constraint equations.

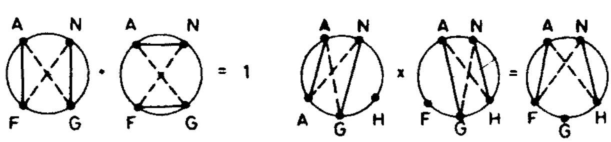

Fig. 5.3

From the form of (5.1) we see that in order that $u = 0$ either A, B, C .... F or G, H, ... N have to come together on the circle. Notice, however, that the inverse is not true, because if A, B ... F are included in a larger resonances, say ABC ... GH, then ABC ... F do not necessarily form a resonance although $Z_A - Z_F \to 0$. This situation is well taken care of in our definition, since in such a case at least one of the denominators also vanishes.

5.2. The integral representation of the n-point function has the form (2.12). The integrand, $\Pi u_{j,k}^{-\alpha_{j,k}-1}$ can be expressed in terms of Z_A, Z_B, ... Z_N by inserting (5.1). The volume element, $dV^{(n)}$, is determined, up to a constant factor, from the bootstrap consistency condition. That is to say, we require that when a pair of neighbouring external mesons form a resonance of spin 0, the residue of the n-point function at this pole should be, apart from a numerical factor, identical with the $(n-1)$-point function of the corresponding circular order. This determines the volume element,

$$dV^{(n)} = \frac{1}{C} \frac{\prod_{j=1}^{n} dZ_j}{\prod_{j=1}^{n} |Z_{j+2} - Z_j|} \tag{5.2}$$

where we have written $Z_1, Z_2, \ldots Z_n$, modulo n, instead of $Z_A, Z_B, \ldots Z_N$.

Inserting (5.1) and (5.2) into (2.12) and after simple algebra we get

$$B^{(n)} = \frac{1}{C} \int \prod_{j=1}^{n} dZ_j \prod_{j>k} (Z_j - Z_k)^{-2p_j \cdot p_k} \tag{5.3}$$

which is completely symmetric with respect to all the mesons.

Since the integrand is even independent of the specified circular order (this is true only under the condition 5) mentioned in *1.1*) we can sum over the terms ascribed to different ordering simply by removing the restriction on the integration region, *i.e.* the order of $Z_A, Z_B, \ldots, Z_N$ on the circle [4].

In this way a symmetric expression for the Veneziano n-point function has been obtained up to a normalization factor. The latter turns out, however, to be by no means trivial.

5.3. When we examine the r.h.s. of (5.3) we find that it is divergent! Thus the normalization factor should be zero, and we need a prescription to extract a sensible result. But the divergence is not a bad one, because, if we choose arbitrarily three out of the external particles, say F, G, H, and perform integration over the $(n-3)$ variables except Z_F, Z_G and Z_H then we get

$$\int \frac{dZ_F dZ_G dZ_H}{(Z_F - Z_G)(Z_G - Z_H)(Z_F - Z_H)} B^{(n)}(\alpha_{j,k}) \tag{5.4}$$

and the divergence appears only when integrating over the remaining 3 variables, and here no dynamical variables $(\alpha_{j,k})$ are involved.

Thus we should, as the "normalization procedure", take arbitrarily three external particles, F, G, H, say, and choose arbitrary (not coinciding) constant values Z_F^0, Z_G^0, Z_H^0 on the unit circle, and insert the factor

$$(Z_F^0 - Z_G^0)(Z_G^0 - Z_H^0)(Z_F^0 - Z_H^0)\,\delta(Z_F - Z_F^0)\,\delta(Z_G - Z_G^0)\,\delta(Z_H - Z_H^0) \tag{5.5}$$

under the integral sign. It is in a way obvious that we have 3 redundant degrees of freedom, because the original integral is $(n-3)$-fold, while we have introduced n variables.

6. *Möbius transformation*

6.1. It is instructive to look more into the origin of the above divergence [11]. The integrand, which is a function of the anharmonic ratios, (5.1), as well as the volume element (5.2) are invariant under the Möbius transformation,

$$Z \to Z' = \frac{\alpha Z+\gamma}{\beta Z+\delta}, \quad \alpha\delta-\beta\gamma = 1, \tag{6.1}$$

and, in particular, those ones which map the unit circle onto itself,

$$Z \to Z' = e^{i\varphi}\,\frac{Z-re^{i\varrho}}{1-re^{-i\varrho}} \tag{6.2}$$

with three real parameters φ, ϱ and $r \neq 1$. Thus, we are counting, so to speak, ∞^3 times too much, and to remove this redundancy three points have to be fixed.

6.2. We can apply the general Möbius transformation to the n-point function (5.4). Then the expression remains the same, but the region of integration is transformed, in general, to another circle. A particularly useful choice is to take the real axis (a circle of infinite radius) as the new integration region. We can then choose one of the fixed points, say Z_N^0, as

$$Z_N = Z_N^0 \to \infty. \tag{6.3}$$

The final result is independent of this choice, as it should be, and all the factors which include Z_N cancel out. Taking, further, the other two points $Z_M = 1$, $Z_A = 0$, in the given ordering A, B, C, ... L, M, N, we get a simple form [11].

$$\int_0^1 \prod'_{j=B,C,\ldots,L} dX_j \prod_{\substack{j>k\\ j,k\neq N}} (X_j-X_k)^{-2p_j p_k}. \tag{6.4}$$

This becomes identical with the conventional expression in the multiperipheral configuration (3, 4, 5) if we make the following change of variables. See Fig. 6.1.

$$\begin{aligned} v_1\, v_2\, v_3 \ldots v_{n-3} &= Z_B \\ v_2\, v_3 \ldots v_{n-3} &= Z_C \\ v_3 \ldots v_{n-3} &= Z_D \\ &\cdots\cdots\cdots \\ v_{n-3} &= Z_L \end{aligned} \tag{6.5}$$

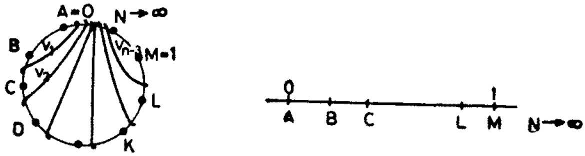

Fig. 6.1

6.3. From the above example we see that a different multiperipheral configuration (for example, Fig. 6.2) can be obtained by a Möbius transformation which maps new three successive points (for instance, B, C, and D in Fig. 6.2) into 1, ∞ and 0, respectively. This

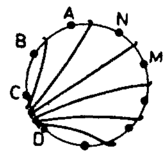

Fig. 6.2

suggests that the dual property of the model is closely related to its invariance under the Möbius transformation.

It is in a way trivial in the above formulation, because this symmetry property is built-in in the expression. But the requirement of this symmetry leads to non-trivial results when applied, for instance, to the operator formalism, where the dual properties of the expression is far from being obvious.

As an example we can take proper time equation (4.13). The Möbius transformation (6.1) consists of three fundamental operations:

$$(i)\ Z \to Z' = cZ \text{ (dilatation and rotation)} \tag{6.6}$$

$$(ii)\ Z \to Z' = Z+b \text{ (translation)} \tag{6.7}$$

$$(iii)\ Z \to Z' = \frac{1}{Z} \text{ (inversion).} \tag{6.8}$$

The proper time variable τ is related to the variable Z by

$$Z = \exp\ (-i\tau) \tag{6.9}$$

and the invariance under the operations (i) and (iii) correspond to the invariance of Eq. (4.13) with respect to the translation and reversal of the proper time, while the invariance under (ii) implies the so-called gauge transformation

$$\Omega'(Z') = S\Omega(Z) \tag{6.10}$$

with

$$S(P) = \exp\ (-bL^+), \tag{6.11}$$

$$L^+ = \sum a_{k+1}^+ a_k \sqrt{k(k+1)} - i\sqrt{2}\, P \cdot a_1^+ \tag{6.12}$$

which plays an essential role in eliminating the unphysical states that appear in the harmonic oscillator formalism [13].

Fubini and Veneziano [14] have made a detailed systematic examination of the transformation properties of the factorized operator formalism and in this way obtained a symmetric (manifestly dual) expression. Similar considerations have been made also by Gervais [15]. Here we cannot enter into details of these interesting works.

6.4. The Möbius transformation is formally closely related to the proper orthochronous Lorentz transformation. As is well known, the latter can be described by

$$\begin{pmatrix} t'+z' & x'-iy' \\ x'+iy' & t'-z' \end{pmatrix} = \begin{pmatrix} \alpha\beta \\ \gamma\delta \end{pmatrix} \begin{pmatrix} t+z & x-iy \\ x+iy t & -z \end{pmatrix} \begin{pmatrix} \alpha^* \gamma^* \\ \beta^* \delta^* \end{pmatrix} \tag{6.13}$$

with

$$\alpha\delta-\beta\gamma = 1. \tag{6.14}$$

So the correspondence is obvious and can easily be visualized. Indeed, the variable Z, which is transformed by (6.1) under the Lorentz transformation (6.13), can be utilized in constructing irreducible representations of the Lorentz group. (See Ref. [16]). So the question arises, whether this relation is just a mathematical isomorphism or has a deeper physical meaning.

Domokos and his co-workers [17] have developed a formalism of interaction of Reggeized hadrons in term of this Z-representation. Their interaction kernel consists of factors of the type

$$\Pi(Z_j-Z_k)^\lambda(Z_j^*-Z_k^*)^\sigma \tag{6.15}$$

and in their recent work [18] they conclude that the Lorentz covariance, the Reggeization (*i. e.* external and virtual particles lying on the Regge trajectories) and the pole approximation lead to the Veneziano formula in the symmetric form (5.3). This would mean that Z-variables introduced in Section 5 are essentially the same as used in the representation of the Lorentz group, *i. e.*, duality and Lorentz invariance are interrelated. A crucial test for this point of view will be to construct the amplitude for particles with spin (in particular) for fermions.

7. Fluid model, (symmetric field theory), conformal invariance

7.1. Finally let me briefly describe[4] a model initiated by Nielsen [19] and elaborated later by himself and also by Fairlie, Susskind, Hsue, Sakita, Virasoro.

The integrand of the symmetric representation of the Veneziano n-point function (5.3) can be put in the form

$$\exp L \tag{7.1}$$

where

$$L = -\sum_{j>k} 2p_j\cdot p_k \ln|Z_j-Z_k|. \tag{7.2}$$

This is formally the same expression, for instance, with the electrostatic energy of a system of n-point charges in this hypothetical two-dimensional space (*i. e.*, infinite line charges in the 3-dimensional space) with the strength $\sqrt{2}\times(p_A, p_B, \ldots, p_N)$, located at the points $Z_A, Z_B, \ldots, Z_N$ (Fig. 7.1 *a*), (The scalar product $(p_j\cdot p_k)$ consists of 4 terms, so that we need

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[4] This presentation slightly deviates from that of the original works.

4 kinds of charges $p_j^\mu(\mu = 0, 1, 2, 3)$. We shall in the following suppress this Lorentz space index μ so far as no confusion arises.)

The "energy" expression (7.2) is the "action at distance" form, and as is well known, we have an alternative description in terms of the electrostatic field (Fig. 7.1 *b*)

$$L = \frac{1}{2}\iint dxdy\{E^2+E_y^2\} = \frac{1}{2}\iint dxdy\left\{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right\}, \tag{7.3}$$

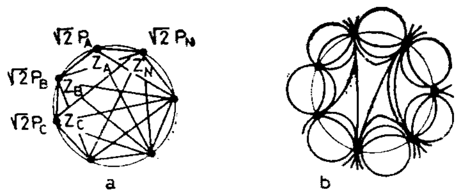

Fig. 7.1

where $\varphi(x, y)$ is the electrostatic potential. The free field equation

$$\text{div}\ \vec{E} = 0,\ \text{or}\ \Delta\varphi = 0,\ (Z \neq Z_j) \tag{7.4}$$

is obtained by the variation principle

$$\delta L = 0. \tag{7.5}$$

An important property of these equations is their invariance under conformal mapping, from which not only the Möbius invariance follows, but also the equivalence of the apparently asymmetric picture like Fig. 4.1 to a more symmetric one like Fig. 4.2 becomes reasonable.

7.2. These simple equations are very familiar to us in the classical physics and we can make recourse to various analogies.

Since the strength of the source is proportional to p_j, it may be more appealing to regard the line of forces in Fig. 7.1 *b* rather as stream lines of hypothetical fluid which carry a component of momentum from one meson to another. (Because of 4-momentum conservation, $\Sigma p_j = 0$.) In fact this was the original picture of Nielsen [19]. In the hydrodynamics of two-dimensional potential flow, one introduces the complex velocity potential

$$\Psi = \Phi + i\psi \tag{7.6}$$

where the real part Φ is the velocity potential and the imaginary part ψ is the stream function, ψ = const representing the stream lines.

Another useful analogy, emphasized by Fairlie and Nielsen [20], is that of stationary electric currents flowing through a metallic disc from and to n electrodes put on the edge. Then (7.2) is proportional to the heat generated by the current and its extremum is realized by the current distribution obeying Ohm's law. This picture may lead to a further interpretation as a fishnet-like Feynman diagram [23]. (The correspondence of the latter to electric circuits is given in Ref. [25].)

7.3. One can further elaborate this formalism. Firstly the sources can be incorporated into the "Lagrangian" by introducing (instead of point singularities) a set of smooth distribution $\varrho_j(x, y)$, which satisfy

$$\iint \varrho_j(x, y)\, dx\, dy = 1 \tag{7.7}$$
$$\varrho_j(x, y) = 0 \quad \text{when} \quad x^2+y^2 = 1.$$

Thus we write

$$\delta \iint \left[\frac{1}{2}\left\{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right\} + \sum_j \sqrt{2}\, p_j \varrho_j(x, y)\varphi(x, y)\right] dxdy = 0 \tag{7.8}$$

and at the end go to the limit where ϱ_j shrinks to the point Z_j from the internal side of the unit circle,

$$\varrho_j \to \lim_{\varepsilon\to 0} \delta\,(x-(1-\varepsilon)\cos\theta_j)\;\delta(y-(1-\varepsilon)\sin\theta_j) \tag{7.9}$$

with

$$Z_j = \exp i\,\theta_j.$$

Secondly, the extremal condition for $\exp L$ can be expressed as an average integration.

In this way Hsue, Sakita and Virasoro [22] have derived the following expression

$$(2\pi)^4\delta^{(4)}(\Sigma p_j)B^{(n)}(p_1, p_2, \dots p_n) = \frac{1}{C}\int_0^{2\pi} d\theta_1 \int_0^{\theta_1} d\theta_2 \dots \int_0^{\theta_{n-1}} d\theta_n \left\langle \exp\left(\sqrt{2\pi}\sum p_j\varphi(Z_j)\right)\right\rangle, \tag{7.10}$$

where C^{-1} represents the vanishing normalization discussed in Section 5.3, and the functional average $\langle\ \rangle$ is defined by

$$\langle \exp \sqrt{2\pi}\sum p_j\varphi(Z_j)\rangle$$
$$= \lim_{\varepsilon\to 0} \frac{\iint \mathscr{D}^{(4)}\varphi(x, y) \exp\left\{-\iint\left[\frac{1}{2}\left\{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right\} + \sum_j p_j p_j \varphi\right] dxdy\right\}}{\exp\left\{\frac{1}{2}\sum_j p_j^2 \iiiint dxdydx'dy'\varrho_j(x, y)\varrho_j(x', y')G(Z-Z')\right\} K} \tag{7.11}$$

where K is a certain constant and G is the Green's function,

$$G(Z-Z') = \frac{1}{2\pi}\ln|Z-Z'|\,|Z-Z'^{*-1}|. \tag{7.12}$$

They have also constructed[5] the generating functional

$$\frac{1}{C}\left\langle \exp g \int d\theta A(x) \sqrt{2\pi}\,\varphi(Z)\right\rangle \tag{7.13}$$

where

$$A(x) = \int d^4p\, e^{ip\cdot x} A_p \tag{7.14}$$

[5] This is only under the condition (5) of Section 1.1.

and the Veneziano n-point function $B^{(n)}(p_1, p_2 \ldots, p_n)$ appears as the coefficient of $(2\pi)^4\delta^4 \times$ $\times (\sum p_j)\, g^n\{\prod_{j=1}^{n} Ap_j\}$ in the expansion of (7.13).

7.4. Let me conclude by reffering to the works which have been and are being performed along these lines.

(*i*) Calculation of the planar and non-planar loop diagrams [20, 22].

(*ii*) Calculation of the slope and intercept of pomeron like object [21].

(*iii*) Attempts to derive the fluid as a limit of very complicated Feynman diagrams [23, 24].

(*iv*) Attempts to combine the fluid with partons [23].

I owe very much to many of the authors quoted in the references who helped me to understand the matter through lectures or personal conversations. In particular I am grateful to H. B. Nielsen for stimulating discussions and for showing me unpublished materials; he is not responsible, however, for any mistakes or incorrect statements that may be included in these talks.

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