AN INVARIANCE PROPERTY OF FIELD THEORIES

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A new invariance property of tensorial functions of the field variables ψ_A and their derivatives is presented. This invariance arises from the possibility of replacing the partial derivatives of ψ_A by covariant derivatives with respect to a symmetric second order tensor $\gamma_{\mu\nu}$ posessing properties similar to that of a flat space-time metric tensor. The resulting identities and conservation laws are then discussed. As an example the theory is applied to the Ricci scalar curvature invariant in General Relativity and the resulting conservation law turns out to be the Rosen-Papapetrou identity. In the final section the new method is compared with the more usual Noether method and the differences are given interpretation.

1. Introduction

A great deal of work has been done on the invariance properties of physical theories, most of which stemmed from Noether's theorem [1]. In particular, Bergmann's work [2] gave much insight into the nature of the conservation laws and identities found in such theories. Several years ago, new methods of obtaining conservation laws were developed [3] inspired mainly by Lipkin's discovery of the Zilch conservation law [4] in Maxwell's theory. We shall here discuss another method of producing conservation laws which does not rely on Noether's theorem but on another very natural invariance property of field theories. Before we can proceed we need a few definitions and consequences thereof. Section two will be devoted to this end. The general conservation laws will be derived in Section three and an application of the method to General Relativity will be given in Section four. Section five will be set aside for discussion.

2. Preliminaries

Consider a general Riemannian space-time R and let K be a co-ordinate system of R. Now let $\lambda^1_{\mu\nu}$ be a two-index symmetric quantity defined at all points of R in the frame K. Suppose x^μ are the co-ordinates of a point $P \in R$ in the frame K, and let x'^μ be the co-ordin-

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[1] A partial derivative is denoted by a comma, or the usual $\frac{\partial}{\partial x^\mu}$. Greek indices take the values 0, 1, 2 3.

ates of P in another (arbitrary) frame K'. Then we define an analogous quantity $\gamma'_{\mu\nu}(x'^{\lambda})$ in K' by:

$$\gamma'_{\mu\nu}(x'^{\lambda}) = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \gamma_{\alpha\beta}(x^{\lambda}). \tag{2.1}$$

Thus we have extended the definition of $\gamma_{\mu\nu}$ by allowing it to transform as a symmetric covariant second order tensor. We will then call $\gamma_{\mu\nu}(x^{\lambda})$ a f l a t m e t r i c[2] in K, if there exists a frame $\bar{K}$ in which $\gamma_{\mu\nu}(x^{\lambda})$, on transformation from K to $\bar{K}$ assumes the form $\eta_{\mu\nu} = \text{diag}\,(-1-1-1+1)$. We will then denote by Λ_K the set of all "flat metrics" in the frame K. The following results are easily derived, being consequences of the above definitions:

A. Let $\gamma_{\mu\nu}(x^{\lambda})$, $\bar{\gamma}_{\mu\nu}(x^{\lambda}) \in \Lambda_K$. Then there exists a frame $\tilde{K}$ such that:

$$\frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}} \gamma_{\alpha\beta}(x^{\lambda}) = \bar{\gamma}_{\mu\nu}(\tilde{x}^{\lambda})$$

where x^{λ}, $\tilde{x}^{\lambda}$ denote the co-ordinates of K and $\tilde{K}$ respectively. In words, $\gamma_{\mu\nu}(x^{\lambda})$ can be transformed to a frame $\tilde{K}$ where it assumes the same functional form of $\tilde{x}^{\lambda}$ as $\bar{\gamma}_{\mu\nu}(x^{\lambda})$ is of x^{λ}.

B. Let $\gamma_{\mu\nu}(x^{\lambda}) \in \Lambda_K$ and suppose that under a co-ordinate transformation to a new frame $\tilde{K}$ it becomes $\tilde{\gamma}_{\mu\nu}(\tilde{x}^{\lambda})$. Then $\tilde{\gamma}_{\mu\nu}(x^{\lambda}) \in \Lambda_K$.

Let us now consider a field theory with variables ψ_A defined on R (here the index A denotes any number of tensor indices of any type). Let $P^B(\psi_A\, \psi_{A,\mu}\, \psi_{A,\mu\nu} \ldots)$ be a set of functions of the ψ_A and their derivatives[3]. Suppose now that in a definite co-ordinate system K we select a member $\gamma_{\mu\nu}$ of Λ_K and denoting the covariant derivative with respect to $\gamma_{\mu\nu}$ by a stroke, we can construct the quantity $P^B(\psi_A\, \psi_{A/\mu}\, \psi_{A/\mu\nu} \ldots)$ which is formed by directly replacing $\psi_{A,\mu}\, \psi_{A,\mu\nu} \ldots$ by $\psi_{A/\mu}\, \psi_{A/\mu\nu} \ldots$

Now if under a co-ordinate transformation we allow $\gamma_{\mu\nu}(x^{\lambda})$ to transform as a second order symmetric covariant tensor then we will assume that $P^B(\psi_A\, \psi_{A/\mu}\, \psi_{A/\mu\nu} \ldots)$ transforms as a tensor or weighted tensor. In general, in a given frame K, $P^B(\psi_A\, \psi_{A/\mu}\, \psi_{A/\mu\nu} \ldots)$ will depend on the $\gamma_{\mu\nu}(x^{\lambda})$ $(\in \Lambda_K)$ chosen to construct the "stroke" derivative. If however it is independent of such a choice, we will call $P^B(\psi_A\, \psi_{A/\mu}\, \psi_{A/\mu\nu} \ldots)$ "γ-i n v a r i a n t". The following results are easily proved:

C. If $P^B(\psi_A\, \psi_{A,\mu} \ldots)$ is a tensor or weighted tensor then in any frame K, $P^B(\psi_A\, \psi_{A/\mu} \ldots)$ is γ-invariant. In particular for any given frame K:

$$P^B(\psi_A \psi_{A,\mu} \ldots) = P^B(\psi_A \psi_{A/\mu} \ldots)$$

independently of the $\gamma_{\mu\nu} (\in \Lambda_K)$ used to construct the "stroke" derivative. Conversely if for any given frame K the tensor (weighted tensor) $P^B(\psi_A \psi_{A/\mu} \ldots)$ is γ-invariant then $P^B(\psi_A \psi_{A,\mu} \ldots)$ is a tensor (weighted tensor)[4].

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[2] No geometrical interpretation is to be given here to the name "flat metric".

[3] Again B denotes any number of any type of index, but no assumption is yet made on the transformation properties of $P^B(\psi_A, \psi_{A,\mu}\, \psi_{A,\mu\nu} \ldots)$.

[4] One can also display connections between quantities $P^B(\psi_A \psi_{A/\mu} \ldots)$ which are only partially γ-invariant and quantities $P^B(\psi_A, \psi_{A,\mu} \ldots)$ which transform covariantly only under certain transformations.

It should be noted that the results A, B and C rest heavily on the fact that the quantity $\gamma_{\mu\nu}$ can be transformed to a frame where it takes the form $\eta_{\mu\nu}$. Also C depends on the fact that $P^B(\psi_A \psi_{A,\mu} \dots)$ satisfies a homogeneous transformation law and does not depend on the particular form of such a law (that is it does not depend on the nature of the block index B).

3. *The conservation laws*

Let $P^B(\psi_A \psi_{A,\mu} \dots)$ be a tensor of arbitrary weight. In any given frame K, with co-ordinates x^λ let $\gamma_{\mu\nu}(x^\lambda)$ and $\bar{\gamma}_{\mu\nu}(x^\lambda)$ be in Λ_K. Then by A there exists a frame $\bar{K}$ with co-ordinates $\bar{x}^\lambda$ such that

$$\frac{\partial x^\mu}{\partial \bar{x}^\alpha} \frac{\partial x^\nu}{\partial \bar{x}^\beta} \gamma_{\mu\nu}(x^\lambda) = \bar{\gamma}_{\alpha\beta}(\bar{x}^\lambda). \tag{3.1}$$

Suppose also that

$$\bar{\gamma}_{\mu\nu}(x^\lambda) - \gamma_{\mu\nu}(x^\lambda) = h_{\mu\nu}(x^\lambda) \tag{3.2}$$

where $h_{\mu\nu}$ are first order small quantities. Then we may consider the transformation from x^μ to $\bar{x}^\mu$ to be of the form

$$\bar{x}^\mu = x^\mu + \mathscr{E}^\mu(x^\lambda) \tag{3.3}$$

where $\mathscr{E}^\mu$ are a set of first order small quantities.

Then, (3.1) (3.2) and (3.3) lead, to first order small quantities, to:

$$\bar{\delta}\gamma_{\mu\nu} \overset{\text{def}^n}{=} \bar{\gamma}_{\mu\nu}(x^\lambda) - \gamma_{\mu\nu}(x^\lambda) = -\gamma_{\mu\alpha}\mathscr{E}^\alpha_{,\nu} - \gamma_{\nu\alpha}\mathscr{E}^\alpha_{,\mu} - \gamma_{\mu\nu,\alpha}\mathscr{E}^\alpha. \tag{3.4}$$

Further, by C we have in K,

$$P^B(\psi_A \psi_{A/\mu} \dots) = P^B(\psi_A \psi_{A//\mu} \dots) \tag{3.5}$$

where in (3.5), the single and double strokes refer to covariant derivatives with respect to $\gamma_{\mu\nu}(x^\lambda)$ and $\bar{\gamma}_{\mu\nu}(x_\lambda)$ respectively. Then we can write (3.5) in the form:

$$P^B(\psi_A(x^\lambda)\psi_{A,\mu}(x^\lambda)\dots\gamma_{\mu\nu}(x^\lambda)\gamma_{\mu\nu,\omega}(x^\lambda)\dots) = P^B(\psi_A(x^\lambda)\psi_{A,\mu}(x^\lambda)\dots\bar{\gamma}_{\mu\nu}(x^\lambda)\bar{\gamma}_{\mu\nu,\omega}(x^\lambda)\dots) \tag{3.6}$$

and hence we have:

$$\frac{\partial P^B}{\partial \gamma_{\mu\nu}} \bar{\delta}\gamma_{\mu\nu} + \frac{\partial P^B}{\partial \gamma_{\mu\nu,\lambda}} \bar{\delta}(\gamma_{\mu\nu,\lambda}) + \dots = 0 \tag{3.7}$$

$$\frac{\delta P^B}{\delta \gamma_{\mu\nu}} \bar{\delta}\gamma_{\mu\nu} + A^{B\mu}_{,\mu} = 0 \tag{3.8}$$

where $\dfrac{\delta}{\delta\gamma_{\mu\nu}}$ denotes the Hamiltonian derivative with respect to $\gamma_{\mu\nu}$ and where $A^{B\mu}_{,\mu}$ denotes an ordinary divergence term. On substituting (3.4) in (3.8) we obtain an expression of the form:

$$E_\alpha \mathscr{E}^\alpha + E^\nu_\alpha \mathscr{E}^\alpha_{,\nu} + E^{\mu\nu}_\alpha \mathscr{E}^\alpha_{,\mu\nu} + \dots = 0. \tag{3.9}$$

The arbitrariness of $\mathscr{E}^\alpha$ and its derivatives then gives:

$$E = 0,\ E^\nu_\alpha = 0,\ E^{\mu\nu}_\alpha + E^{\nu\mu}_\alpha = 0 \dots \tag{3.10}$$

The particular conservation law we seek can be derived from (3.10) with much labour. However, a quicker way to obtain it is to integrate (3.8) over an arbitrary four dimensional region Ω of R, upon the "surface" S of which, $\mathscr{E}^\alpha$ is supposed to vanish. Then the use of (3.4), and two simple applications of Gauss' law yields:

$$F^{B\mu\nu}\gamma_{\mu\nu,\lambda}-2\,(F^{B\mu\nu}\gamma_{\mu\lambda})_{,\nu}=0 \tag{3.11}$$

where we have use the abbreviation

$$F^{B\mu\nu}=\frac{\delta P^B}{\delta\gamma_{\mu\nu}}. \tag{3.12}$$

In particular, if P^B is an invariant density, then (3.11) yields

$$F^{B\mu\nu}_{\ /\nu}=0 \tag{3.13}$$

where the stroke denotes a $\gamma_{\mu\nu}$ convariant derivative.

We can now eliminate $\gamma_{\mu\nu}$ in (3.11) by setting $\gamma_{\mu\nu}=\eta_{\mu\nu}$ thus obtaining[5]

$$\overset{\prime}{F}{}^{B\mu\nu}_{\ ,\nu}=0,\quad \overset{\prime}{F}{}^{B\mu\nu}=[F^{B\mu\nu}]_{\gamma_{\mu\nu}=\eta_{\mu\nu}}. \tag{3.14}$$

A further class of conservation laws can be constructed by writing (3.6) in the form:

$$P^B(\psi_A(x^\lambda)\psi_{A,\mu}(x^\lambda)\ldots\bar{\Gamma}^\alpha_{\beta\sigma}(x^\lambda)\,\bar{\Gamma}^\alpha_{\beta\sigma,\mu}(x^\lambda)\ldots)=P^B(\psi_A(x^\lambda)\psi_{A,\mu}(x^\lambda)\ldots\Gamma^\alpha_{\beta\sigma}(x^\lambda)\,\Gamma^\alpha_{\beta\sigma,\mu}(x^\lambda)\ldots)$$

where

$$\Gamma^\alpha_{\beta\sigma}=\frac{\gamma^{\alpha\omega}}{2}\,(\gamma_{\omega\beta,\sigma}+\gamma_{\omega\sigma,\beta}-\gamma_{\beta\sigma,\omega})\quad \bar{\Gamma}^\alpha_{\beta\sigma}=\frac{\bar{\gamma}^{\alpha\omega}}{2}\,(\bar{\gamma}_{\omega\beta,\sigma}+\bar{\gamma}_{\omega\sigma,\beta}-\bar{\gamma}_{\beta\sigma,\omega}).$$

Now it is not difficult to show

$$\bar{\delta}\Gamma^\alpha_{\beta\sigma}=\bar{\Gamma}^\alpha_{\beta\sigma}(x^\lambda)-\Gamma^\alpha_{\beta\sigma}(x^\lambda)=A^{\alpha\nu\lambda}_{\beta\sigma\mu}\mathscr{E}^\mu_{,\nu\lambda}+B^{\alpha\nu}_{\beta\sigma\mu}\mathscr{E}^\mu_{,\nu}+C^\alpha_{\beta\sigma\mu}\mathscr{E}^\mu$$

where

$$A^{\alpha\nu\lambda}_{\beta\sigma\mu}=-\tfrac{1}{2}\,[\delta^\alpha_\mu\delta^\nu_\beta\delta^\lambda_\sigma+\delta^\alpha_\mu\delta^\lambda_\beta\delta^\nu_\sigma]\quad C^\alpha_{\beta\sigma\mu}=-\Gamma^\alpha_{\beta\sigma,\mu}$$

$$B^{\alpha\nu}_{\beta\sigma\mu}=[-\delta^\alpha_\omega\delta^\varrho_\beta\delta^\tau_\mu\delta^\nu_\sigma-\delta^\alpha_\omega\delta^\tau_\sigma\delta^\varrho_\mu\delta^\nu_\beta+\delta^\varrho_\beta\delta^\tau_\sigma\delta^\nu_\omega\delta^\alpha_\mu]\,\Gamma^\omega_{\varrho\tau}.$$

Then (3.15) can be written in the form:

$$F^{B\beta\sigma}_\alpha\bar{\delta}\Gamma^\alpha_{\beta\sigma}+\bar{A}^{B\mu}_{,\mu}=0,\quad F^{B\beta\sigma}_\alpha=\frac{\delta P^B}{\delta\Gamma^\alpha_{\beta\sigma}}.$$

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[5] There is an important difference between the conservation laws (3.13) and (3.14) namely that in (3.13) $\gamma_{\mu\nu}$ transforms as a tensor whereas in (3.14) it takes the fixed value $\gamma_{\mu\nu}=\eta_{\mu\nu}$ in all reference frames. Conservation laws such as (3.13) are therefore analogous to those found in bimetric relativity [5], [6], [8] whereas those such as (3.14) take the non-covariant form typical of orthodox General Relativity. We note that we could obtain other (non-covariant) conservation laws from (3.13), besides (3.14), by selecting a $\gamma_{\mu\nu}$ other than $\gamma_{\mu\nu}=\eta_{\mu\nu}$, which has the same functional form in all frames and satisfies the flatness condition mentioned earlier. These in general however would lead to a covariant divergence differential law rather than an ordinary partial differential law as in (3.14).

If the divergence term $\bar{A}^{\beta\mu}_{,\mu}$ is written out in full, and use made of (3.17), one can obtain identities similar to those in (3.10), again with much labour. However, we may obtain our conservation law by integrating (3.20) through an arbitrary four volume Ω upon the "surface" S of which $\mathscr{E}^\alpha$ vanishes. The usual applications of Gauss' law yield:

$$F^{B\beta\sigma}_{\alpha} C^{\alpha}_{\beta\sigma\mu} - (F^{B\beta\sigma}_{\alpha} B^{\alpha\nu}_{\beta\sigma\mu})_{,\nu} + (F^{B\beta\sigma}_{\alpha} A^{\alpha\nu\lambda}_{\beta\sigma\mu})_{,\lambda\nu} = 0. \tag{3.15}$$

Then putting $\gamma_{\mu\nu} = \eta_{\mu\nu}$ and defining $F'^{B\beta\sigma}_{\alpha} = [F^{B\beta\sigma}_{\beta}]_{\gamma_{\mu\nu}=\eta_{\mu\nu}}$ we find

$$\theta^{B\nu}_{\mu,\nu} = 0, \quad \theta^{B\nu}_{\mu} = (F'^{B\beta\sigma}_{\alpha} A^{\alpha\nu\lambda}_{\beta\sigma\mu})_{,\lambda} = -F'^{B\nu\lambda}_{\mu,\lambda}. \tag{3.16}$$

4. *Application to General Relativity*

For an example of the above ideas, we will consider the application of (3.13) to General Relativity theory. In this case, the field variables ψ_A become the metric tensor of the Riemannian space-time $g_{\mu\nu}$. If we now put $P^B = \mathscr{R}$, where $\mathscr{R}$ is the Ricci invariant curvature density, we can as before construct the γ-invariant quantity $P^B(\psi_A \psi_{A/\mu} \ldots) = \mathscr{R}(g_{\alpha\beta}\, g_{\alpha\beta/\mu}\, g_{\alpha\beta/\mu\nu})$ and then (3.13) yields:

$$F^{\mu\nu}_{/\nu} = 0 \quad F^{\mu\nu} = \frac{\delta\mathscr{R}}{\delta\gamma_{\mu\nu}}. \tag{4.1}$$

In fact we have:

$$F^{\mu\nu} = \frac{\sqrt{-\gamma}}{2} [\tilde{g}^{\mu\nu}\gamma^{\varrho\sigma} + \tilde{g}^{\varrho\sigma}\gamma^{\mu\nu} - \tilde{g}^{\mu\varrho}\gamma^{\nu\sigma} - \tilde{g}^{\nu\sigma}\gamma^{\mu\varrho}]_{/\sigma\varrho} \tag{4.2}$$

where $\gamma = \det\ (\gamma_{\mu\nu})$, $\quad g = \det\ (g_{\mu\nu})$, $\quad \tilde{g}^{\mu\nu} = \dfrac{\sqrt{-g}}{\sqrt{-\gamma}}\, g^{\mu\nu}$.

The conservation law (4.1) has been discussed by Rosen [5] in his bimetric theory of Relativity. Clearly the analogous conservation law to (3.13) can be obtained from (4.2) by setting $\gamma_{\mu\nu} = \eta_{\mu\nu}$.

Also from (4.2) we can construct the conservation law:

$$P^{\mu\nu}_{,\nu} = 0, \quad P^{\mu\nu} = [\sqrt{-g}\,(g^{\mu\nu}\eta^{\varrho\sigma} + g^{\varrho\sigma}\eta^{\mu\nu} - g^{\mu\varrho}\eta^{\nu\sigma} - g^{\nu\sigma}\eta^{\mu\varrho})]_{,\varrho\sigma} \tag{4.3}$$

which is the conservation law given by Papapetrou [6].

5. *Discussion*

We have seen that from a very natural invariance property of tensorial quantities we can construct conservation laws of arbitrary order. Those arising from (3.13) have the property that the quantities $F^{B\mu\nu}$, $F'^{B\mu\nu}$ are symmetric in μ and ν. Indeed Papapetrou used the expression (4.3) to construct a conservation law for angular momentum. The identities in (3.16) have the feature that the quantity $\theta^{B\nu}_{\mu}$ is derivable from a "superpotential" $-F'^{B\nu\lambda}_{\mu}$, which possesses symmetry in the indices ν, λ, (instead of the usual antisymmetry).

It is worthwhile to compare our invariance laws with the more usual ones based on Noether's theorem. The latter arise in a form dependent on the specific nature of the transformation law satisfied by P^B and ψ_A under (3.3). If P^B is a form invariant function of ψ_A and its derivatives and if P^B and ψ_A satisfy homogeneous transformation laws for co-ordinate changes then under (3.3) we have:

$$P^B(\bar{\psi}_A(\bar{x}^\lambda), \bar{\psi}_{A,\mu}(\bar{x}^\lambda) \ldots) - P^B(\psi_A(x^\lambda)\, \psi_{A,\mu}(x^\lambda) \ldots) = M^{B\nu}_{A\mu} P^A \mathscr{E}^\mu_{,\nu} = \bar{\delta} P^B + P^B_{,\mu} \mathscr{E}^\mu \tag{5.1}$$

$$\bar{\delta} P^B = P^B(\bar{\psi}_A(x^\lambda) \bar{\psi}_{A,\mu}(x^\lambda) \ldots) - P^B(\psi_A(x^\lambda) \psi_{A,\mu}(x^\lambda) \ldots) \tag{5.2}$$

$$\bar{\psi}_A(\bar{x}^\lambda) - \psi_A(x^\lambda) = Y^{B\nu}_{A\mu} \psi_B \mathscr{E}^\mu_{,\nu} \Rightarrow \bar{\delta}\psi_A = \bar{\psi}_A(x^\lambda) - \psi_A(x^\lambda) = Y^{B\nu}_{A\mu} \psi_B \mathscr{E}^\mu_{,\nu} - \psi_{A,\mu} \mathscr{E}^\mu \tag{5.3}$$

where the $Y^{B\nu}_{A\mu}$, $M^{B\nu}_{A\mu}$ are constants dependent on the transformation laws satisfied by ψ_A and P^B (that is dependent on the nature of the block indices A nad B). Then (5.1), (5.2) and (5.3) give:

$$\frac{\partial P^B}{\partial \psi_A} \bar{\delta}\psi_A + \frac{\partial P^B}{\partial \psi_{A,\mu}} \bar{\delta}(\psi_{A,\mu}) + \ldots + P^B_{,\mu} \mathscr{E}^\mu - M^{B\nu}_{A\mu} P^A \mathscr{E}^\mu_{,\nu} = 0. \tag{5.4}$$

If we substitute for $\bar{\delta}\psi_A$ in (5.4) using (5.3) we obtain an expression analogous to (3.9) from which the identities are obtained as in (3.10). In practice one normally takes P^B to be the Lagrangian invariant density of the theory[6]. Now it is clearly seen that the identities (5.4) contain, amongst others, terms dependent on $\frac{\partial P^B}{\partial \psi_A}$, $Y^{B\nu}_{A\mu}$ and $M^{B\nu}_{A\mu}$. However our identities in section three were derived purely on the basis that $P^B(\psi_A \psi_{A,\mu} \ldots)$ satisfied a homogeneous transformation law and did not depend on the explicit nature of that law (that is on the nature of the block index B). Also since ψ_A itself satisfied a homogeneous transformation law (whereas $\psi_{A,\mu}\psi_{A,\mu\nu} \ldots$ do not) one would suspect that only the functional form of P^B with respect to $\psi_{A,\mu}\psi_{A,\mu\nu} \ldots$ would be important in assuring that P^B satisfied such a law. Thus we will investigate the possibility that our identities (3.14) and (3.16) are expressible in terms of only $\psi_A \psi_{A,\mu} \ldots \frac{\partial P^B}{\partial \psi_{A,\mu}} \frac{\partial P^B}{\partial \psi_{A,\mu\nu}} \ldots$ and not $\frac{\partial P^B}{\partial \psi_A}$, $Y^{B\nu}_{A\mu}$ and $M^{B\nu}_{A\mu}$. Before doing so we will need to distinguish between the different forms we have used for P^B more precisely. We will call the original tensor $P^B(\psi_A \psi_{A,\mu} \ldots)$ and denote the corresponding tensor function of the $\psi_A \psi_{A/\mu} \ldots$ by $\bar{P}^B(\psi_A \psi_{A/\mu} \ldots)$. For the tensor function obtained from $\bar{P}^B$ by expanding the stroke derivative we will use the symbol $\tilde{P}^B(\psi_A \psi_{A,\mu} \ldots \gamma_{\mu\nu} \gamma_{\mu\nu,\lambda} \ldots)$. We note that since $\psi_{A/\mu} \cdot \psi_{A/\mu\nu} \ldots$ all contain terms involving ψ_A then in general $\frac{\partial \bar{P}^B}{\partial \psi_A} \neq \frac{\partial \tilde{P}^B}{\partial \psi_A}$. Returning to the identities (3.11) and (3.15) we see that they are expressible in terms of $\gamma_{\mu\nu}\ \frac{\partial \tilde{P}^B}{\partial \gamma_{\mu\nu}}\ \frac{\partial \tilde{P}^B}{\partial \gamma_{\mu\nu,\lambda}} \ldots$

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[6] The Noether formulation can be applied in some cases when P^B is not tensorial. In particular it has been applied to Lagrangians which are not invariant densities [7].

and their derivatives with respect to the x^{α}. However since

$$\left.\begin{aligned}\frac{\partial \tilde{P}^B}{\partial \gamma_{\mu\nu}} &= \frac{\partial \bar{P}^B}{\partial \psi_{A/\mu}} \frac{\partial(\psi_{A/\mu})}{\partial \gamma_{\mu\nu}} + \frac{\partial \bar{P}^B}{\partial \psi_{A/\mu\nu}} \frac{\partial(\psi_{A/\mu\nu})}{\partial \gamma_{\mu\nu}} + \ldots \\ \frac{\partial \tilde{P}^B}{\partial \gamma_{\mu\nu,\lambda}} &= \frac{\partial \bar{P}^B}{\partial \psi_{A/\mu}} \frac{\partial(\psi_{A/\mu})}{\partial \gamma_{\mu\nu,\lambda}} + \frac{\partial \bar{P}^B}{\partial \psi_{A/\mu\nu}} \frac{\partial(\psi_{A/\mu\nu})}{\partial \gamma_{\mu\nu,\lambda}} + \ldots\end{aligned}\right\} \tag{5.5}$$

and similar expressions for the other derivatives of $\tilde{P}^B$ and since the quantities $\frac{\partial(\psi_{A/\mu})}{\partial \gamma_{\mu\nu}}$ $\frac{\partial(\psi_{A/\mu\nu})}{\partial \gamma_{\mu\nu}} \ldots \frac{\partial(\psi_{A/\mu})}{\partial \gamma_{\mu\nu,\lambda}} \frac{\partial(\psi_{A/\mu\nu})}{\partial \gamma_{\mu\nu,\lambda}} \ldots$ *etc.* can easily be calculated when ψ_A is given, the identities (3.11) are identities between $\gamma_{\mu\nu}\, \gamma_{\mu\nu,\lambda} \cdots \psi_A\, \psi_{A,\mu} \cdots \frac{\partial \bar{P}^B}{\partial \gamma_{A/\mu}} \frac{\partial \bar{P}^B}{\partial \gamma_{A/\mu\nu}} \ldots$ and their derivatives with respect to the x^{λ}. Thus on putting $\gamma_{\mu\nu} = \eta_{\mu\nu}$ we see that the identities (3.14) are identities between $\psi_A\, \psi_{A,\mu} \ldots$ and $\frac{\partial P^B}{\partial \psi_{A,\mu}} \frac{\partial P^B}{\partial \psi_{A,\mu\nu}} \ldots$ only and do not involve the quantities $\frac{\partial P^B}{\partial \psi_A}$ $Y^{B\nu}_{A\mu}$ and $M^{B\nu}_{A\mu}$. Thus the identities (3.14) are to a large extent independent of how P^B depends on ψ_A. For example if the quantities $\frac{\partial P^B}{\partial \psi_{A,\mu}} \frac{\partial P^B}{\partial \psi_{A,\mu\nu}} \ldots$ are given then one can construct the identities (3.14) yet P^B is determined only to within an arbitrary additive function of the ψ_A. Similar arguments can be given concerning the identities (3.16).

We can now reconsider the situation. Noether's theorem is derived by considering an actual infinitesimal transformation (3.3) and values of P^B are considered in two different coordinate systems. In the present method, the transformation (3.3) is a mental operation introduced in order to give a convenient expression for $\bar{\delta}\gamma_{\mu\nu}$. That Noether's identities contain the extra quantities $\frac{\partial P^B}{\partial \psi_A}$, $Y^{B\nu}_{A\mu}$ and $M^{B\nu}_{A\mu}$ is hardly surprising.

We finally point out that the identities (3.11) and (3.15) have a special significance in Rosen's bimetric Relativity theory [8] and it is hoped that this will be dicussed further in a future paper.

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