ON THE LOWER BOUND FOR THE σ_{el}/σ_{tot} RATIO AT HIGH ENERGIES

By L. Łukaszuk

Institute for Nuclear Research, Warsaw*

(Received July 13, 1970; Revised paper received November 9, 1970)

A new method for the derivation of the lower bound for the elastic-to-total cross-section ratio at high energies is presented. It is shown that the constant factor can be improved down to the latest value of the constant in the Froissart bound: *e.g.* for processes like elastic scattering of πN, $\pi\pi$ *etc.* one has:

$$\sigma_{el}/\sigma_{tot} \geqslant \frac{m_\pi^2\,\sigma_{tot}}{\pi}\cdot\frac{1}{(\ln s/c\,\sigma_{tot})^2}$$

1. *Introduction*

It was shown by Martin [1] that, in the high energy region, the following inequality holds:

$$\frac{\sigma_{el}}{\sigma_{tot}^2} > \frac{1}{C_1(\ln s/s_0)^2}.$$

The assumptions leading to this inequality are:
1. unitarity, $1 \geqslant \mathrm{Im}\, f_l \geqslant |f_l|^2$
2. Analyticity of the absorption part in the neighbourhood of $0 \leqslant t < t_0$ in the t plane and the polynomial bound:

$$A_s(s_1\, t) < C_0 s^N \quad \text{for} \quad 0 < t < t_0, \quad s > s_1$$

where s_1 is sufficiently large.

The assumption 2 is known to result from the Axiomatic Field Theory (AFT) for some processes (*e.g.* $\pi\pi$, πN elastic scattering) [2].

The minimal value of constant C_1 was estimated [1], within the Mandelstam relations, as:

$$C_1 = \frac{16\pi}{t_0}\left(P + \frac{M}{2} + \eta\right)^2$$

P is connected with the polynomial bound on the spectral functions; asymptotically

* Instytut Badań Jądrowych, Warszawa, Hoża 69, Poland.

$st > |\varrho(s,t)|^{1/P}$, M is such that

$$\sigma_{tot} > s^{-M} \quad \text{for} \quad S \to \infty$$

The inequality for $\sigma_{el}/\sigma_{tot}^2$ can be reduced to the bound on the total cross-section alone:

$$\sigma_{tot} < \frac{16\pi P^2}{t_0} (\ln s/s_0)^2.$$

However, this result does not saturate the Froissart bound [3]

$$\sigma_{tot} < \frac{4\pi(N-1)^2}{t_0} (\ln s/s_0)^2$$

with N taken from 2.

An improvement has been done in this direction recently [5], [6] and, given the AFT assumptions plus the value of the ratio $\left|\frac{\text{Re}\, F(s,0)}{\text{Im}\, F(s,0)}\right|$ the relation between σ_{tot} and σ_{el} can be written as [6]

$$\frac{\sigma_{tot}^2}{\sigma_{el}} \leqslant \frac{4\pi}{t_0} \ln^2\left(\frac{s}{C_1\sigma_{el}}\right)\left(1+\left|\frac{\text{Re}\, F(s,0)}{\text{Im}\, F(s,0)}\right|^2\right)^{-1}.$$

Another method [7] was also used to derive the lower bound for σ_{el} and we shall present here the extended version of Ref. [7]. The input information consists of the value of t_0, σ_{tot} and the maximal number of subtractions $N = 2$. Therefore the result

$$\frac{\sigma_{tot}^2}{\sigma_{el}} \leqslant \frac{4\pi}{t_0} (\ln (s/s_1 \ln s/s_1))^2$$

should be compared with the case of [6] when $\left[1+\left|\frac{\text{Re}\, F}{\text{Im}\, F}\right|^2\right]$ is replaced by 1 — we do not use the information about $\frac{\text{Re}\, F}{\text{Im}\, F}$. Taking our expression with the same degree of accuracy as in [6], one gets

$$\frac{\sigma_{tot}^2}{\sigma_{el}} \leqslant \frac{4\pi}{t_0} (\ln s/\sigma_{tot})^2.$$

Due to the fact that $\sigma_{tot} \geqslant \sigma_{el}$ one obtains an improvement and if, moreover $\frac{\sigma_{el}}{\sigma_{tot}} \to 0$, the scaling factors are considerably changed. We can now write the bound for σ_{tot}

$$\sigma_{tot} \leqslant \sqrt{\frac{4\pi\sigma_{el}}{t_0}} \ln\left(\frac{s}{C\sigma_{tot}} \ln \frac{s}{C\sigma_{tot}}\right)$$

which leads to

$$\sigma_{tot} \leqslant \frac{4\pi}{t_0} \ln^2 s/s_0, \qquad \text{with } s_0 = c_0 \ln s.$$

This is similiar to the bound obtained by Common [8] for the averaged cross-section $\bar{\sigma}_{tot}$. The scaling factor can be given in [8] in amore detailed form due to the fact that for the averaged quantity $\bar{\sigma}_{\text{tot}}$ one can express constant c_0 in terms of d-wave scattering length in the third channel.

In this paper we make advantage of the usual notation: s is (energy tot. in c.m.s.)^2, t is (Momentum transfer)^2, t_0 corresponds to the lowest value of (mass)^2 in the t channel.

2

We shall use the assumptions 1 and 2 in our derivation. One has to answer the following question:

Given σ_{tot} and $A_s(s, t)$ satisfying 1, 2, what is the minimal σ_{el} for fixed, sufficiently large s?

The absorptive part can be written as

$$A_s(s,t) = \frac{\sqrt{s}}{k} \sum_{l=0}^{\infty} (2l+1) P_l(x) a_l \tag{1}$$

$$x = 1 + \frac{t_0}{2k^2} > 1, \qquad k \text{ is c.m.s. momentum} \tag{1a}$$

$$1 \geqslant a_l = \operatorname{Im} f_l \geqslant 0. \tag{1b}$$

Because

$$P_l(x) = \frac{1}{\pi} \int_0^{\pi} (x + \sqrt{x^2-1}\cos\Phi)^l d\Phi$$

and $(x+\sqrt{x^2-1}\cos\Phi)$ is a positive decreasing function of Φ for $0 \leqslant \Phi \leqslant \pi$, $(x > 1)$, the following inequality is fulfilled:

$$P_l(x) \geqslant \varepsilon(x+\sqrt{x^2-1}\cos\pi\varepsilon)^l \tag{1c}$$

for arbitraty small, fixed ε.

Let us introduce

$$A \underset{\text{df}}{=} \sum_{l=0}^{\infty} (2l+1) a_l y \tag{2}$$

with

$$y = x + \sqrt{x^2-1}\cos\varepsilon\pi \tag{2a}$$

then

$$\bar{c}s^N \geqslant \frac{1}{\varepsilon} A_s(s,t) \frac{k}{\sqrt{s}} \geqslant A \tag{2b}$$

It is convenient to replace σ_{tot}, σ_{el} by B, D defined below:

$$B \underset{\text{df}}{=} \frac{\sigma_{\text{tot}} k^2}{4\pi} = \sum_{l=0}^{\infty} (2l+1) a_l \tag{3}$$

$$\frac{\sigma_{\text{el}} k^2}{4\pi} \geqslant D \underset{\text{df}}{=} \frac{\sigma_{\text{el.im.}} k^2}{4\pi} = \sum_{l=0}^{\infty} (2l+1) a_l^2. \tag{4}$$

The equations (2), (3), (4) together with the condition (1b) allow us to find the distribution $\{a_l\}$ minimizing D at high energies.

3

Let us notice that the minimal D will be obtained for the largest possible A *i.e.* $A = \bar{c}s^N$ (see Appendix A).

The minimal solution — obtained with the usual Lagrange multiplier technique — $a_l = \lambda_1(s) - \lambda_2(s)y^l$ (Further on we shall write λ_1, λ_2 instead of $\lambda_1(s)$, $\lambda_2(s)$) leads, for large l to negative a_l.

Putting there $a_l = 0$ one arrives at:

$$a_l = \lambda_s - \lambda_2 y^l \quad \text{for} \quad l \leqslant L \tag{5}$$
$$a_l = 0 \quad \text{for} \quad l > L$$

with

$$y^{L+1} > \frac{\lambda_1}{\lambda_2} \geqslant y^L \tag{5a}$$

and

$$a_l \leqslant 1. \tag{5b}$$

In the Appendix B we show that — among $\{a_l\}$ satisfying condition $a_l \geqslant 0$ — the distribution (5) gives, for fixed B, A the minimal value of D. We shall also show below (see Eqs (15), (16) that $a_l \leqslant 1$ is satisfied in our case).

Now, using Eqs (2), (3), (4) one gets:

$$A = \lambda_1 \sum_{l=0}^{L} (2l+1)y^l - \lambda_2 \sum_{l=0}^{L} (2l+1)y^{2l} \tag{6}$$

$$B = \lambda_1 \sum_{l=0}^{L} (2l+1) - \lambda_2 \sum_{l=0}^{L} (2l+1)y^l \tag{7}$$

$$D = \lambda_1^2 \sum_{l=0}^{L} (2l+1) - 2\lambda_1\lambda_2 \sum_{l=0}^{L} (2l+1)y^l + \lambda_2^2 \sum_{l=0}^{L} (2l+1)y^{2l} \tag{8}$$

It is convenient to write Eqs (5a) as

$$\lambda_1 = \lambda_2 y^{L+\delta} \quad 0 < \delta < 1 \tag{9}$$

For $k^2 \to \infty$

$$y = 1 + \eta = 1 + \cos \varepsilon\pi \sqrt{\frac{t}{k^2}} + 0\left(\frac{t}{k^2}\right). \tag{10}$$

Let us notice that

$$g(y) \underset{\text{df}}{=} \sum_{l=0}^{L} (2l+1)y^l = \frac{(2L+1)y^{L+1} - 1}{y-1} - \frac{2y(y^L - 1)}{(y-1)^2} \tag{10a}$$

$$\sum_{l=0}^{L} (2l+1) = (L+1)^2 \tag{10b}$$

Now Eqs (6), (7), (8) can be written as (we are replacing $(1+\eta)^{\delta}$ by 1):

$$A \approx \lambda_2[y^L g(y) - g(y^2)] \tag{11}$$

$$B \approx \lambda_2[y^L (L+1)^2 - g(y)] \tag{12}$$

$$D \approx \lambda^2{}_2[y^{2L}(L+1)^2 - 2y^L g(y) + g(y^2)] \tag{13}$$

Because we are considering the maximal A consistent with AFT, therefore, because of the Froissart bound, the ratio

$$\frac{A}{B} \geqslant s^{N-1}(\ln s)^{-2} \text{ const.}$$

Hence, from (11), (12) and (10a) one has

$$L\eta \to \infty \quad \text{for} \quad s \to \infty$$

In this limit, Eqs (10a) gives:

$$g(y) = \frac{2L\eta \exp(\eta L)}{\eta^2}\left(1 + o\left(\frac{1}{L\eta}\right)\right).$$

Now, the Eqs (11) and (12) read:

$$A = \lambda_2 \frac{L\eta \exp(2\eta L)}{\eta^2}\left(1 + o\left(\frac{1}{L\eta}\right)\right) \tag{11a}$$

$$B = \lambda_2 \frac{(L\eta)^2}{\eta^2} \exp(\eta L)\left(1 + o\left(\frac{1}{L\eta}\right)\right). \tag{12a}$$

Hence

$$\frac{A}{B} = \frac{\exp(\eta L)}{\eta L}\left(1 + o\left(\frac{1}{L\eta}\right)\right) \tag{14}$$

and

$$\eta L = \left(\ln \frac{A}{B}\right)\left(1 + o\left(\frac{\ln \ln \frac{A}{B}}{\ln \frac{A}{B}}\right)\right) \tag{14a}$$

where

$$\lim_{\frac{A}{B} \to \infty} \frac{o\left(\dfrac{\ln \ln \frac{A}{B}}{\ln \frac{A}{B}}\right)}{\dfrac{\ln \ln \frac{A}{B}}{\ln \frac{A}{B}}} = 1. \tag{14b}$$

The unitarity condition $a_l \leqslant 1$ is satisfied for our distribution

$$a_l \leqslant \lambda_1 - \lambda_2 \approx \lambda_2 \exp(\eta L) \approx \frac{B\eta^2}{\left(\ln \frac{A}{B}\right)^2} \leqslant \frac{\sigma_{\rm tot} t_0}{4\pi \left[\ln \frac{c A_s(s, t_0)}{s\sigma_{\rm tot}}\right]^2}. \tag{15}$$

The r.h.s. of (15) does not exceed unity (see Eqs (16) below) which means that

$$a_l \leqslant 1.$$

From Eqs (13), (14a) one gets:

$$D = \left[\frac{\lambda_2 L\eta \exp(\eta L)}{\eta}\right]^2 \left(1 + o\left(\frac{1}{\eta L}\right)\right).$$

Therefore the minimal value of D is equal to:

$$D = \left(\frac{B\eta}{L\eta}\right)^2 \left(1 + o\left(\frac{1}{\eta L}\right)\right).$$

Hence, taking advantage of Eqs (3), (4), (14*a*):

$$\sigma_{\rm el} \geqslant \sigma_{\rm el.im.} = \frac{4\pi}{k^2} D \geqslant \frac{\sigma_{\rm tot}^2 t_0 (1-\varepsilon^2)}{4\pi \left(\ln \frac{c A_s(s, t_0)}{s\sigma_{\rm tot}}\right)^2} \left(1 - o\left(\frac{\ln \ln \frac{A_s(s, t)}{s\sigma_{\rm tot}}}{\ln \frac{A_s(s, t)}{s\sigma_{\rm tot}}}\right)\right)^2 \tag{16}$$

where c is a constant proportional to $1/\varepsilon$, very large for small ε but independent of energy. We shall therefore replace $(1-\varepsilon^2)$ by 1, in Eqs (16).

Now, for the maximal A:

$$A_s = c_0 s^N \quad \text{for} \quad t \leqslant t_0$$

one gets therefore

$$\sigma_{\rm el} \geqslant \frac{\sigma_{\rm tot}^2 t_0}{4\pi} \frac{1}{\left(\ln \frac{c s^{N-1}}{\sigma_{\rm tot}}\right)^2} \left(1 - o\left(\frac{\ln \ln \frac{s^{N-1}}{\sigma_{\rm tot}}}{\ln \frac{s^{N-1}}{\sigma_{\rm tot}}}\right)\right)^2. \tag{16a}$$

For the $\sigma_{\rm el} = \sigma_{\rm tot}$ this inequality saturates the Froissart limit [3], [8]:

$$\sigma_{\rm tot} \leqslant \frac{4\pi}{t_0} (N-1)^2 \left(\ln \frac{s}{s_0}\right)^2 \quad \text{with } s_0 = c_0 \ln s \tag{16b}$$

In the case of πN, $\pi\pi$, πK, KK, $\Lambda\pi$ elastic scattering, it follows from the AFT [2] that $t_0 = 4m_\pi^2$ and [4] $N \leqslant 2$.

We get for this case:

$$\frac{\sigma_{el}}{\sigma_{tot}} \geqslant \frac{\sigma_{tot} m_\pi^2}{\pi} \frac{1}{\left(\ln \frac{s}{c\sigma_{tot}}\right)^2} \left(1 - o\left(\frac{\ln \ln \frac{s}{\sigma_{tot}}}{\ln \frac{s}{\sigma_{tot}}}\right)\right)^2$$

$$\approx \frac{\sigma_{tot}}{60 \text{ mb}} \frac{1}{\left(\ln \frac{s}{c\sigma_{tot}} + \ln \ln \frac{s}{c\sigma_{tot}}\right)^2} = \frac{\sigma_{tot}}{60 \text{ mb}} \frac{1}{\ln^2 \left[\frac{s}{c\sigma_{tot}} \left(\ln \frac{s}{c\sigma_{tot}}\right)\right]}. \quad (17)$$

We should emphasize that our extremal distribution $\{a_l\}$ satisfies conditions $0 \leqslant a_l \leqslant 1$; therefore one cannot improve the inequality (17) without introducing new subsidiary conditions.

APPENDIX A

We shall prove the following lemma:
Let $D^{\min}$ be the minimal value of D for fixed B and A (compare Eqs (1b), (2)–(4)). Let $\bar{D}^{\min}$ be such a value for the same B but $\bar{A} > A$.
Then

$$\bar{D}^{\min} < D^{\min}.$$

Proof:

Let us assume that the distribution $\{a_l\}$ minimizes D for given A, B:

$$A = \sum_{l=0}^{\infty} (2l+1) a_l y^l, \quad B = \sum_{l=0}^{\infty} (2l+1) a_l,$$

$$D^{\min} = \sum_{l=0}^{\infty} (2l+1) a_l^2 \quad (A.1)$$

At first, let us notice that for any $a_p \neq 0$ one can find such $L > p$ that for $l \geqslant L$

$$a_p - a_l > d > 0 \quad (A.2)$$

where d is some constant. This is so because of the convergence of series for A, B, D together with the condition $a_l \geqslant 0$.

Next let us notice that:

Any $\bar{A} > A$ can be obtained from $\{a_l\}$ (without changing B) by change of two $a_l's$: $\bar{a}_l \neq a_l$ for $l = p, L$ only and $0 \leqslant \bar{a}_l \leqslant 1$. Because both distributions give the same B, one has:

$$\delta a_l = \frac{h}{2L+1}, \quad \delta a_p = \frac{h}{2p+1} \quad (A.3)$$

where

$$\bar{a}_p = a_p - \delta a_p, \quad \bar{a}_l = a_l + \delta a_l.$$

In order to have

$$0 \leqslant \bar{a}_l \leqslant 1$$

it is enough that $a_p > \bar{a}_p > \bar{a}_L > a$. These inequalities will be fulfilled if (compare Eqs (A.2), (A.3)):

$$0 < \left(\frac{1}{2L+1} + \frac{1}{2p+1}\right) < d/h. \tag{A.4}$$

Next, $\bar{A}-A = h(y^L - y^p)$ *i.e.*

$$h = \frac{\bar{A}-A}{y^L - y^p}. \tag{A.5}$$

Choosing L large enough ($y > 1$) one can, for any $(\bar{A}-A)$, make h sufficiently small to fulfill condition (A.4).

In the end let us notice that

$$\bar{D} - D^{\min} = h\left[2(a_L - a_p) + h\left(\frac{1}{2L+1} + \frac{1}{2p+1}\right)\right]$$

and using (A.4):

$$\bar{D} - D^{\min} \leqslant h(a_L - a_p) < 0. \tag{A.6}$$

Hence

$$\bar{D}^{\min} \leqslant \bar{D} < D^{\min} \tag{A.7}$$

which proves our lemma.

APPENDIX B

We shall show that the distribution (5):

(5) $$a_l = \lambda_1 - \lambda_2 y^l \quad \text{for} \quad l \leqslant L$$

$$a_l = 0 \quad \text{for} \quad l > L$$

(5a) $$y^{L+1} > \frac{\lambda_1}{\lambda_2} \geqslant y^L \quad (y > 1)$$

with $0 \leqslant a_l$ minimizes D for given A, B (cp. Eqs (2), (3), (4)).

Of course, among all distributions with $a_l = 0$ for $l > L$ the distribution (5) is the extremal one and it is not difficult to see that one obtains the minimum.

Let us consider another distribution $\{a_l\}$ leading to some value $\bar{D}$ instead of D.

This new distribution,

$$\bar{a}_l = a_l + \delta a_l \quad (\text{for } l > L \ \bar{a}_l = \delta a_l \geqslant 0) \tag{B.1}$$

must give unchanged values of A, B. Therefore:

$$\sum_{r=L+1}^{\infty}(2r+1)\delta a_r y^r = -\sum_{l=0}^{L}(2l+1)\delta a_l y^l \tag{B.2}$$

$$\sum_{r=L+1}^{\infty}(2r+1)\delta a_r = -\sum_{l=0}^{L}(2l+1)\delta a_l \tag{B.3}$$

Now,

$$\bar{D}-D=\sum_{l=0}^{\infty}(2l+1)(\bar{a}_l^2-a_l^2) > 2\sum_{l=0}^{\infty}(2l+1)a_l\delta a_l = 2\sum_{l=0}^{L}(2l+1)a_l\delta a_l$$

Next, due to the Eqs (5) and (B.1)–(B.3):

$$\bar{D}-D > 2\sum_{l=0}^{L}(2l+1)\delta a_l(\lambda_1-\lambda_2 y^l) > 2\lambda_2\sum_{r=L+1}^{\infty}(2l+1)\delta a_r(y^r-y^{L+1})$$

Therefore, the conditions: $y>1$, $\delta a_r \geqslant 0$ for $r>L$ lead to:

$$\bar{D}>D$$

which proves our assertion.

REFERENCES

[1] A. Martin, *Nuovo Cimento*, **29**, 993 (1963).
[2] A. Martin, *Nuovo Cimento*, **42**, 30 (1966). For further references, see *e.g.* Epstein, *Proceedings of Conference on High Energy Collisions of Hadrons*, CERN,, Vol. I, p. 290 (1968).
[3] M. Froissart, *Phys. Rev.*, **123**, 1053 (1961); the value of constant factor is taken from L. Łukaszuk, A. Martin, *Nuovo Cimento*, **52A**, 122 (1967).
[4] Y. Jin, A. Martin, *Phys. Rev.*, **135 B**, 1375 (1964).
[5] A. A. Logunov, N. van Hieu, *Proc. of Topical Conf. on High Energy Collisions*, vol. II, CERN 1968.
[6] V. Singh, S. M. Roy, *Ann. Phys.*, **57**, 461 (1970).
[7] L. Łukaszuk, *Report* "*P*" No. 1118/VII/PH, 1969.
[8] A. K. Common, *Nuovo Cimento*, **69A**, 115 (1970).