Vol. B2 (1971) ACTA PHYSICA POLONICA Fasc. 6

ON A NONLOCAL THEORY OF SCALAR FIELDS

By Trân hũ'u Phát*

Department of Physics, Hanoi University

(*Received February 1, 1971*)

A nonlocal theory of scalar fields is considered. In this theory, by using generalized analytic functions the utraviolet divergences are eliminated without any regularization. S-matrix, satisfying the unitary and macrocausality conditions is constructed.

1

The possibility of removing the divergence difficulties by constructing a nonlocal, relativistic field theory was considered for the first time by Peierls [1], Rayski [2] and Rzewuski [3]. More recently a nonlocal theory of scalar fields, in which the ultraviolet infinities are absent, was constructed by Efimov [4].

However Efimov's theory encounters difficulty related to introducing a regularization procedure: the results of the theory depend on the form of the regularization function. The cause of this defect is as follows. For all entire functions, there is always at least a sector, in which they increase to infinity. Consequently, there is no analytic function, playing the role of formfactor, to be regular in the whole plane of the complex variable except for a constant.

As will be seen below, such a function exists only in the set of the generalized analytical functions, and this fact suggests the use of these functions as formfactors instead of the ordinary analytical functions.

As it is well known, to formulate the macrocausality principles of S-matrix is one of the fundamental difficulties of the nonlocal theory. Following [4] it is possible to find a S-matrix, satisfying the macrocausality principle. Let us illustrate this by studying a scalar field.

Let $\varphi(x)$ be the operator of a scalar field. Then the macrocausality principle can be formulated as follows:

$$\frac{\delta}{\delta\varphi(x)}\left(\frac{\delta S}{\delta\varphi(y)}\, S^{-1}\right) = 0 \tag{1.1}$$

* Present address: Instytut Fizyki; Uniwersytet Łódzki, Kopcińskiego 16/18, Poland.

besides the regions G and G_l, where

$$G : x^0 \geqslant y^0, \quad (x-y)^2 > 0$$

$$G_l : -l^2 \leqslant (x-y)^2 \leqslant l$$

and l means a certain "elementary" length.

Moreover, it is necessary to state the complementary condition: The expression (1.1) in the region G_l is proportional to a certain relativistic invariant generalized function $A_l(x-y)$ to have the following property. Arbitrary functions $f(x)$, nonvanishing in a certain limited region G_f of space-time transform to functions

$$F(x) = \int d^4y A_l(x-y) f(y)$$

nonvanishing only in a limited region of space-time $G_F = G_f + \delta G_f$, where δG_f is limited and belonging entirely to the interior of region G_{fl}, so that $x \in G_{fl}$ if and only if

$$-l^2 \leqslant (x-y)^2 \leqslant l^2, \text{ here } y \in G_f.$$

Such a definition of macrocausality principle by Efimov is very satisfactory.

We shall prove below, that such generalized functions exist and the theory is free from the ultraviolet divergent difficulties without any regularization.

2

It follows that in the region G_l of space-time, microcausality is violated. In the local theory of quantized fields, basing on the microcausality principle [5, 6], one proved that the scattering amplitudes are analytical functions of complex energy variable E and they increase slower than a certain polynomial for $E \to \infty$.

Following [7, 8, 9] it is possible that there are two ways of introducing the violation of causality.

a) Either the scattering amplitudes increase faster than a certain polynomial for $E \to \infty$, *i.e.* they verify the following unequality

$$|f(E)| > A e^{\alpha |E|^\beta} \text{ as } E \to \infty.$$

b) Or the scattering amplitudes still increase slower than a certain polynomial at infinity, but they are generalized analytical functions of Vekua type [10] and therefore their real and imaginary parts verify the following equation:

$$\partial_{\bar{E}} f(E) + A(E) f(E) + B(E) \bar{f}(E) = 0.$$

For simplify, we shall further restrict ourselves to studying the above equation to have vanishing B, *i.e.*

$$\partial_{\bar{E}} f(E) + A(E) f(E) = 0 \tag{2.1}$$

Then $f(E)$ can be represented in the following form [10]:

$$f(E) = e^{\omega(E)} \psi(E), \tag{2.2}$$

where $\psi(E)$ is a certain analytical function and $\omega(E)$ is given by

$$\omega(E) = \frac{1}{\pi} \iint\limits_{G} \frac{A(E')}{E'-E}\, dx'dy', \qquad E' = x' + iy'.$$

As an example, let us study the relation between the generalized analyticity of scattering amplitudes and the violation of causality in the one-dimension case.

Let $M(t)$ be the scattering amplitude in coordinate space. Its Fourier transform is given by

$$M(t) = \int_{-\infty}^{+\infty} e^{-iEt} f(E) dE. \tag{2.3}$$

The causality principle means that

$$M(t) = 0 \quad \text{for} \quad t < 0. \tag{2.4}$$

From (2.3) and (2.4) one deduces that $f(E)$ is the analytical function of E and it increases slower than a certain polynomial for $E \to \infty$, that is

$$|f(E)| < A|E|^n \quad \text{as} \quad E \to \infty.$$

Assume now the scattering amplitude $f(E)$ to be a certain generalized analytical function of the Vekua type. It is then of the form (2.3); here we suppose that $\psi(E)$ increases slower than a certain polynomial at infinity and $\omega(E)$ has the form

$$\omega(E) = \ln\left(1 + \frac{\alpha e^{-\lambda|E|}}{\psi(E)}\right)$$

from which we obtain

$$f(E) = \psi(E) + \alpha e^{-\lambda|E|}$$

which increases also slower than a polynomial for $E \to \infty$. Then

$$M(t) = \int_{-\infty}^{+\infty} e^{-iEt}(\psi(E) + \alpha e^{-\lambda|E|}) dE$$

and the causality principle is violated

$$M(t) = \frac{2\alpha}{\lambda^2 + t^2}\, \lambda \quad \text{for} \quad t < 0.$$

By this example we can conclude that the violation of causality principle can be described entirely by the generalized analyticity of scattering amplitudes.

Therefore, it is possible that two ways of introducing nonlocality exist:

1) Propagator of scalar field is deplaced as follows

$$\frac{1}{k^2 + m^2} \to \frac{V(k^2)}{k^2 + m^2}$$

in which the formfactor $V(k^2)$ is an entire function decreasing for $k^2 \to \infty\,\alpha$ and in order to converge the Feynman integrals it is necessary to introduce a procedure of regularization by using a regularization function R^ε.

2) The formfactor $V(z)$ is a certain generalized analytical function, decreasing faster than $1/|z|^n$ for $z \to \infty$.

For example, we can choose such a function as follows:

$$V(z) = e^{-l^2|z|^2}\psi(z)$$

here $\psi(z)$ is a certain analytical function of finite order. It is easily seen that $V(z)$ satisfies the equation of form (2.1):

$$\partial_{\bar{z}} V(z) + l^2 z V(z) = 0.$$

It is clear that $V(z) \to 0$ along any direction of the z plane as $z \to \infty$.

3

The Lagrangian of the scalar field $\varphi(x)$ is written in the following form

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I$$

where $\mathscr{L}_0$ is the Lagrangian of the free field and $\mathscr{L}_I$ describes the autointeraction of field $\varphi(x)$ and is a certain polynomial of $\varphi(x)$, such as $\lambda\varphi^4$. Following [4], we assume that there is no participation of field $\varphi(x)$ to the interaction Lagrangian, except the field $\Phi(x)$ defined by

$$\Phi(x) = \int d^4y\, V(x-y)\varphi(y) = V(\Box_x)\varphi(x)$$
$$V(x-y) = V(\Box_x)\delta^4(x-y). \tag{3.1}$$

Then, we have

$$D^c(x-y) = \underbrace{\Phi(x)\Phi(y)} = V(\Box_x)V(\Box_y)\underbrace{\varphi(x)\varphi(y)}$$
$$= V(\Box_x)V(\Box_y)\Delta^c(x-y) = \int e^{ip(x-y)}\, \frac{[V(p^2)]^2}{p^2+m^2}\, d^4p$$

where $\Delta^c(x)$ is the causal function of the scalar field $\varphi(x)$. The commutator is given by

$$[\Phi(x), \Phi(y)]_- = \int d^4x' d^4y'\, V(x-x')V(y-y')\Delta(x'-y')$$
$$= [V(m^2)]^2\Delta(x-y) = D(x-y).$$

The commutator function $D(x)$ coincides with that of the local field if we put $V(m^2) = 1$. It is possible to obtain the same results for $D^{\pm}(x)$.

Next, let us study the generalized functions

$$V(x-y) = V(\Box_x)\delta^4(x-y)$$

defined in the space of fundamental functions D.

This space is defined as follows. It is such space that the functionals

$$\langle V, f\rangle = \int d^4y\, V(x-y)f(y) = V(\Box_x)f(x)$$

are defined univalently for each $f(x) \in D$. In order to satisfy this demand, $f(x)$ must be infinitely differentiable functions of x in the real axis and their derivatives of n-order must increase slower than $|x|^p$ at infinity for a certain definite natural number p. Otherwise, if one is to demand that the functionals $\langle V, f\rangle$ are defined univalently in every system of reference, then $f(x)$ must represent the values on the real axis of certain entire functions $f(z_i)$ with respect to each argument z_i.

To summarize, the space of fundamental functions is defined as follows:

1. Each $f(x) \in D$ is the value on the real axis of a certain entire function $f(z)$ increasing slower than $|x|^p$ for $z \to \infty$.

2. An arbitrary suite of functions of $D\{f_n | f_n \in D\}$ converged to zero in a certain region G, if and only if all the functions of this suite converge uniformly to zero in G.

Let us now study the local properties of generalized functions $V(x-y)$. To do this, let us choose a formfactor to behave like the following function

$$V(z) = e^{-l^2}|z|^2\psi(z)$$

where $\psi(z)$ is the formfactor given by Efimov [1]:

a) $\psi(\Box_x) = \int_{\varrho^2<l^2} d^4\varrho a(\varrho^2) \exp(i\varrho_0\delta_0 + \vec{\varrho}\vec{\delta})$

b) $\psi(\Box_x) = \int_{\varrho^2<l^2} d^4\varrho a(\varrho^2) \exp(\varrho_0\delta_0 + i\vec{\varrho}\vec{\delta})$

and let us examine the suite of functions belonging to $D\{f_\nu(x, y)\}$ so that the limit function

$$f(x, y) = \lim_{\nu\to 0} f_\nu(x, y)$$

does not belong to D and equals zero for all $x \neq y$. Otherwise $f_\nu(x, y)$ is normalized by

$$\int d^4x f_\nu(x, y) = 1.$$

Then that suite $\{f_\nu(x, y)\}$ gives a δ-function representation in the space D. Such a function was studied by [4]. Thus for every $f_\nu(x-y)$ so that $\lim_{\nu\to 0} f_\nu(x-y) = 0$ for all $x \neq y$, the function

$$g_\nu(x) = V(\Box_x) f_\nu(x)$$

has the demanded property as follows:

$$g(x) = \lim_{\nu\to 0} g_\nu(x) = 0$$

beside the region $x_0 = 0$, $\vec{x}^2 < l^2$. Therefore, these are the generalized functions stated in the first paragraph.

Finally, an other important property of the generalized functions is discovered. This is that the product of two arbitrary generalized functions can be defined univalently without regularization:

$$C(x-y) = V^{(1)}(x-y)V^{(2)}(x-y).$$

The proof is similar to that of [4]. However, there is a feature of this theory, which is absent in the Efimov's theory: if $V^{(1)}$ and $V^{(2)}$ belong to two different types (a) and (b), then their product is still defined entirely.

4

Let us now study the members of the perturbation theory series. Formally, S-matrix can be represented as follows:

$$S = T\exp\{-i\int \mathscr{L}_I d^4x\}. \tag{4.1}$$

Then the matrix element of a diagram in n-th-order approximation is of the form:

$$F(x_1, \dots, x_n) = \prod_{i,j} D^c(x_i - x_j). \tag{4.2}$$

The above amplitude $F(x_1, \dots, x_n)$ is a generalized function of D' space dual to D since

$$D^c(x-y) = V(\Box_x)V(\Box_y)\Delta^c(x-y)$$

and therefore it is integrable in the space of fundamental functions.

The Fourier transform of $F(x_1, \dots, x_n)$ is given by

$$F(p_1, \dots, p_n) = \int \dots \int \prod_i d^4 l_i \prod_j \frac{[V(k_j^2)]^2}{k_j^2 + m^2 + i\varepsilon} \tag{4.3}$$

here k_j — four-momenta corresponding to the interior lines of the diagram and l_i — four-momentums of integration.

With the formfactor $V(k^2)$ given at the third paragraph, the above integral converges without any regularization.

Thus, all the members of the series of the perturbation theory converge and thereby the ultraviolet infinity difficulties appearing in the local field theory is overcome.

5

We prove next the unitarity of S-matrix in each approximation of perturbation theory on the mass shell, that is

$$\langle a|SS^+|b\rangle = \langle a|b\rangle \tag{5.1}$$

for all two arbitrary physical states $|a\rangle$ and $|b\rangle$.

It is easily seen that in our theory the unitary equality (5.1) is still true. In effect, equality (5.1) was proved by [4] in the case where $V(k^2)$ has the form

$$V(k^2) = \psi(k^2)$$

here $\psi(k^2)$ is given at *3*.

In our theory $V(k^2)$ is only different from $\psi(k^2)$ by a real factor. Then the unitary equality (5.1) is not violated.

Therefore, in the general case, formfactor $V(k^2)$ is chosen as follows. It is of the form (2.2), in which $\psi(k^2)$ is Efimov's formfactor and $\exp\,\omega(k^2)$ is a real function of complex variable k^2 and decreasing at infinity to zero along any direction of the k-plane.

In particular, if we choose $V(k^2)$ to have the simple form

$$V(k^2) = e^{\omega(k^2)}$$

then the truth of the above obtained results is not violated and S-matrix becomes a unitary operator, that is it verifies the equality

$$SS^{+} = 1$$

which is deduced easily by using the form of S-matrix (3.1).

6

To end, the causality of S-matrix is considered. To do this, let us study the following operator

$$C(x, y) = \frac{\delta}{\delta\varphi(x)} \left(\frac{\delta S}{\delta\varphi(y)} S^{-1} \right). \tag{6.1}$$

In the local field theory $C(x, y)$ equals zero for $(x-y)^2 < 0$. Now we shall prove that $C(x, y)$ in a non-local theory is equal to zero apart from a region of the form given at first paragraph.

In effect, by decomposing the S-matrix (3.1) with respect to the degress of interaction constant λ, we obtain

$$S = \sum_{n=0}^{\infty} \frac{1}{n!} \int \dots \int \lambda^n S(x_1, \dots, x_n) : \Phi(x_1) \dots \Phi(x_n) : dx_1 \dots dx_n \tag{6.2}$$

where $S(x_1, \dots, x_n)$ contains the Green function $D^c(x_i - y_j)$. When writing in the N-products of field operators, $\Phi(x_1) \dots \Phi(x_2) \dots \Phi(x_n)$: give the $D^{\pm}$-functions which are not different from $\Delta^{\pm}$. The substitution of (6.2) into (6.1) gives an expression of $C(x, y)$ being different from that in local field theory by deplacing $D^c(x)$ by $\Delta^c(x)$.

On the other hand, basing on the localization of the formfactor $V(x-y)$, which was mentioned in 3, we can conclude that the function $D^c(x-y)$, whose form is given as follows

$$D^c(x-y) = \int d^4x' d^4y' V(x-x') V(y-y') \Delta^c(x'-y')$$

behaves asymptotically like the causal function $\Delta^c(x-y)$ for

$$|x^0 - y^0| \gg l \quad \text{and} \quad |\vec{x} - \vec{y}| \gg l$$

Following [11], that is the proof of the macrocausality of S-matrix:

$$C(x, y) = 0$$

apart from the regions G and G_l:

$$G : x^0 > y^0, \ (x-y)^2 > 0$$

$$G_l : -l^2 \leqslant (x-y)^2 \leqslant l^2.$$

It is necessary to note that in the approximation of high order, the acausal region is enlarged, for instance, in the n-order approximation the acausal region G_l is deplaced by G_{nl}. This situation is similar to that of Efimov's theory [4]. It is possible that the violation of causality in those region G_{nl} can be neglected since there are no remarkable contributions of high order members to the final results.

In resuming, by using the generalized analytical functions, playing the role of form-factors, the nonlocal theory of scalar field was constructed satisfactorily:

1. The ultraviolet infinities without regularization is overcome.
2. The unitarity of S-matrix is conserved.
3. The causality of S-matix at large distances is guaranteed.

REFERENCES

[1] R. Peierls, H. McManus, *Proc. Roy. Soc.*, **A195**, 323 (1948).
[2] J. Rayski, *Phil. Mag.*, Ser. 7, **42**, 1289 (1951); J. Rayski, *Acta Phys. Polon.*, **13**, 95 (1954).
[3] J. Rzewuski, *Nuovo C mento*, **10**, 182, 789 (1953).
[4] G. V. Efimov, *Commun. Math. Phys.*, **5**, 42 1967).
[5] N. N. Bogolyubov, D. V. Shirkov, *Introduction to the theory of quantized field*, Gostekhizdat, Moscow 1957.
[6] N. N. Bogolyubov, B. V. Medvedev, M. K. Polivanov, *Problems in the theory of dispersion relations*, Fizmatgiz, Moscow 1958.
[7] Trân hữ'u Phát, *Scientific communications of Hanoi University*, Ser. Phys., 3, 1970.
[8] Trân hữ'u Phát, *Acta Phys. Polon.*, **B1**, 323 (1970).
[9] Trân hữ'u Phát, *Ann. Phys.* (Germany), **27**, 33 (1971).
[10] I. N. Vekua, *Generalized analytic functions*, Fizmatgiz, Moscow 1959.
[11] D. I. Blokhinsev, *Space and time in microworld*, Moscow 1970.