EFFECTIVE FORCE FOR THE NILSSON MODEL CALCULATIONS OF THE COLLECTIVE NUCLEAR PHENOMENA

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A theoretical estimate of the strength and deformation dependence of the two-body interaction producing the nuclear field is discussed and tested by a RPA calculation. The deformation dependent two-body interaction is used to describe the nonaxial quadrupole vibrational state for the large values of the quadrupole deformation of nuclei.

1. *Introduction*

In 1955 Bohr [1] proposed the channel theory of fission. According to this theory the fissioning nucleus in the saddle point of its deformation may exhibit excited states which are similar to those at equilibrium deformation. The different excited states correspond to the different channels. Some experiments have been undertaken in order to find the position of this transition-state spectrum. The group of Britt [2] measured the probability of fission as the function of the excitation energy. The experimental results indicated the presence of a low-lying vibrational spectrum. They also measured the angular correlations for (α, pf) and (α, tf) reactions for ^{233}U, ^{235}U and ^{239}Pu and found the position of some collective vibrational states at the saddle point (the octupole and γ vibrational states). It is therefore interesting to look at the theoretical estimate of the energy of the γ-vibrational 2^+ state in the saddle point. If one makes, for example, the microscopic calculation described in detail in the next section, one would find that at the saddle point the nucleus seems to be extremely stiff with respect to the non-axial quadrupole deformation, *i.e.*, the calculated energy is about 2 MeV while the experimental one is about 0.6 MeV. In this kind of calculation there is only one free parameter — the strength of the quadrupole force which is responsible for the existence of this vibrational level and the value of this parameter is usually fitted so as to account for the energies of the equlibrium deformation.

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The simplest way to make the nucleus softer with respect to the nonaxial γ-vibrations at the saddle point is to increase the value of the strength of the quadrupole coupling constant. This means that one needs this value to be deformation-dependent. We are thus faced with one of the central problems in every microscopic calculation of the collective nuclear phenomena: the choice of the effective two-body interaction.

This problem becomes especially important when an effective interaction is used in the study of collective processes taking place far from the equilibrium, *e.g.* at deformations much larger than the equilibrium deformation. Fine effects, such as the density dependence of the effective interaction, which are often accounted for at equilibrium by the choice of the phenomenological parameters of the two-body force, lead to non-negligible changes of the interaction strength. In many cases these changes play an important role in understanding the nuclear properties far from equilibrium.

Thus it is important to test different approaches to the problem of the dependence of the effective interaction strength on parameters defining the density distribution in the nucleus.

The approach to the effective interaction problem, which is tested here, is that of Bohr and Mottelson [3] generalized to include the deformation dependence of the interaction strength [4], [5].

The strength of the two-body force can be obtained from the experimental data as a function of the parameters which define the density distribution by fitting calculated (*e.g.* by the RPA method) energies of the collective excitations to the experimental values for a number of nuclei and a number of excitations. The calculation must include all transitions which give significant contribution (*e.g.* $\Delta N = 0, 2$ transitions for sufficiently large number of shells in the case of quadrupole excitations) in order to make "renormalization" of the force strength negligibly small. This is essential not only for testing the strength of the force but also for its dependence on deformation which can be strongly affected by large renormalization.

First, we present this kind of calculations performed for quadrupole excitations in deformed nuclei with the use of the Nilsson model potential. Afterwards we calculate the effective two-body force directly from a given single-particle potential; we discuss the deformation-dependence of the multipole interaction and compare the results obtained by both methods.

In the last section we use the calculated values of the strength of the non-axial quadrupole force in estimating the position of the γ-vibrational state for large deformations (for the saddle point and in the second potential-energy well).

2. *Determination of the quadrupole force strength from the position of* 0^+ *and* 2^+ *vibrational states of deformed nuclei*

In order to connect the strength of the quadrupole force with the energies of 0^+ and 2^+ vibrational states, the method described in detail by Bès [6] and used in Ref. [7] has been applied.

Nucleons are assumed to move in a single-particle field approximated by the "new"

Nilsson potential [8] and interact through the pairing and quadrupole two-body forces. Thus the Hamiltonian has the form

$$H = H_0 + H_{\text{pair}} - \tfrac{1}{2}\varkappa \hat{Q}^{\dagger}\hat{Q} \tag{1}$$

where H_0 is the Nilsson model Hamiltonian, H_{pair} represents the pairing interaction in its standard form [9], $\varkappa$ is the strength of the quadrupole force and $\hat{Q}$ is defined as $\hat{Q} = \sum_{i=1}^{A} \hat{q}_i$ where

$$\hat{q} = \begin{cases} \sqrt{\dfrac{16\pi}{5}}\, r^2 Y_{20} & \text{in the case of } \beta \text{ vibration} \\ \dfrac{1}{\sqrt{2}}\sqrt{\dfrac{16\pi}{5}}\, r^2 (Y_{22} + Y_{2-2}) & \text{in the case of } \gamma \text{ vibration} \end{cases} \tag{2}$$

The energies of the vibrational states are obtained in the adiabatic approximation by calculating the stiffness and mass parameters C and B.

$$B = \hbar^2 \frac{\Sigma_3}{2\Sigma_1^2} \tag{3}$$

$$C = \frac{1}{2\Sigma_1} - \varkappa \tag{3a}$$

where

$$\Sigma_1 = \sum_{\nu\nu'} \frac{|q_{\nu\nu'}|^2 (u_\nu v_{\nu'} + u_{\nu'} v_\nu)^2}{E_\nu + E_{\nu'}} \tag{4}$$

$$\Sigma_3 = \sum_{\nu\nu'} \frac{|q_{\nu\nu'}|^2 (u_\nu v_{\nu'} + u_{\nu'} v_\nu)^2}{(E_\nu + E_{\nu'})^3}. \tag{5}$$

In the present calculation the sums in (4) and (5) were extended over all the single-particle states with positive magnetic quantum number values of $N = 0, 1, \ldots, 7$ and $N = 0, 1, \ldots, 8$ harmonic oscillator shells for protons and neutrons respectively, in the rare earth region and of $N = 0, 1, \ldots, 8$ and $N = 0, 1, \ldots, 9$ oscillator shells, respectively, in the actinide region.

In the case of γ vibrations the matrix elements of $\hat{q}$ between states with the opposite signs of the magnetic quantum numbers were also included. The E_ν are the quasiparticle energies and u, v are pairing amplitudes of the state ν. The parameters of the Nilsson potential are those of Ref. [8]. The pairing force was diagonalized by the BCS method within 24 double-degenerated single-particle states with pairing force strength $G_p = 32.2/A$ MeV for protons and $G_n = 26.5/A$ MeV or $G_n = 26.04/A$ MeV for neutrons in the rare earth or actinide regions, respectively.

TABLE I

Strength of the $K = 0$ component of the quadrupole force.

The first four columns give the nucleus, quadrupole deformation parameter ε, hexadecapole deformation parameter ε_4 and the experimental position of the β-vibrational state, respectively. In column five we present the $\varkappa^0_{20}$ value calculated from the experimental value of the energy of the β vibration while the last column gives the value of $\varkappa^0_{20}$ obtained from formula (22)

Nucleus	ε	ε_4	E_β [MeV]	$\varkappa^{0\,\mathrm{exp}}_{20}$	$\varkappa^{0\,\mathrm{theor}}_{20}$
$_{62}Sm^{152}$	0.25	− 0.02	0.685	0.173	0.191
$_{62}Sm^{154}$	0.29	− 0.03	1.100	0.166	0.188
$_{64}Gd^{154}$	0.24	− 0.04	0.680	0.173	0.199
$_{64}Gd^{156}$	0.28	− 0.02	1.048	0.172	0.194
$_{64}Gd^{158}$	0.30	− 0.02	1.449	0.162	0.189
$_{66}Dy^{156}$	0.25	− 0.02	0.674	0.171	0.202
$_{66}Dy^{158}$	0.27	− 0.01	0.993	0.174	0.199
$_{68}Er^{162}$	0.27	0.00	1.081	0.179	0.202
$_{69}Er^{164}$	0.27	0.02	1.245	0.187	0.208
$_{68}Er^{166}$	0.28	0.02	1.460	0.193	0.205
$_{72}Hf^{176}$	0.25	0.07	1.250	0.238	0.227
$_{72}Hf^{178}$	0.24	0.07	1.199	0.253	0.230
$_{76}Os^{188}$	0.17	0.09	1.086	0.276	0.258
$_{78}Pt^{188}$	0.17	0.05	0.800	0.256	0.246
$_{78}Pt^{190}$	0.15	0.05	0.922	0.253	0.252
$_{78}Pt^{192}$	0.15	0.05	1.195	0.247	0.252
$_{78}Pt^{194}$	0.15	0.05	1.267	0.250	0.252
$_{78}P^{196}$	0.12	0.05	0.135	0.259	0.262

TABLE II

Strength of $K = 2$ component of the quadrupole force.

The first four columns give the nucleus, quadrupole deformation parameter ε, hexadecapole deformation parameter ε_4 and the experimental position of the γ-vibrational state, respectively. In column five we give the $\varkappa^0_{22}$ value calculated from the experimental position of the γ-vibrational state while the last column presents the value of $\varkappa^0_{22}$ obtained from formula (23)

Nucleus	ε	ε_4	E_γ [MeV]	$\varkappa^{0\,\mathrm{exp}}_{22}$	$\varkappa^{0\,\mathrm{theor}}_{22}$
$_{62}Sm^{152}$	0.25	− 0.02	1.090	0.465	0.405
$_{62}Sm^{154}$	0.29	− 0.03	1.450	0.486	0.428
$_{66}Dy^{160}$	0.26	− 0.02	0.970	0.486	0.410
$_{68}Er^{164}$	0.27	0.02	0.840	0.485	0.419
$_{68}Er^{170}$	0.27	0.04	0.950	0.488	0.421
$_{70}Yb^{170}$	0.27	0.06	1.230	0.481	0.422
$_{70}Yb^{172}$	0.27	0.06	1.470	0.475	0.422
$_{70}Yb^{176}$	0.27	0.07	1.260	0.457	0.423
$_{76}Os^{188}$	0.17	0.09	0.633	0.424	0.371
$_{90}Th^{232}$	0.19	− 0.02	0.790	0.459	0.374
$_{92}U^{232}$	0.20	− 0.02	0.870	0.456	0.378
$_{92}U^{234}$	0.21	− 0.02	0.920	0.459	0.383
$_{94}Pu^{238}$	0.22	− 0.02	1.030	0.467	0.387
$_{94}Pu^{240}$	0.23	− 0.02	0.940	0.486	0.394

The relation

$$E=\hbar\sqrt{\frac{C}{B}}=\sqrt{\frac{\frac{1}{2\Sigma_1}-\varkappa}{\frac{\Sigma_3}{2\Sigma_1^2}}} \tag{6}$$

relates $\varkappa$, which is the only free parameter of this calculation, to the vibration energy values.

By taking the experimental values of E we obtain

$$\varkappa=\frac{1}{2\Sigma_1}\left(1-\frac{\Sigma_3}{\Sigma_1}E_{\mathrm{exp}}^2\right). \tag{7}$$

The calculations were performed for experimental values of quadrupole deformation ε. The experimental values of the hexadecapole deformation ε_4 were used in the rare earth region only [10]. The values of ε_4 in the actinide region were taken from calculations of Ref. [6].

The results of our calculations are presented in Tables I and II where the values of

$$\varkappa^0=\varkappa A^{4/3}\left(\frac{\hbar}{M\omega_0}\right)^2(\hbar\omega_0)^{-1} \tag{8}$$

are given. Column five of Table I gives the strength $\varkappa_{20}^0$ of the $K=0$ component of the quadrupole force which is responsible for β vibrations, while column five of Table II shows the strength $\varkappa_{22}^0$ of the $K=2$ component responsible for γ-vibrations.

3. Deformation dependence of the multipole-multipole interaction

According to Ref. [4], the field-producing two-body force used in the RPA calculation of the collective excitations with a self-consistent single-particle potential U is

$$F_{12}=F(r_1r_2)=\frac{\delta U(r_1)}{\delta\varrho(r_2)} \tag{9}$$

where ϱ is the density distribution of the nucleus.

The components of F_{12} corresponding to collective modes of given symmetry $(\lambda\mu)$ may be obtained by restricting the variation of the density distribution to the variation having this symmetry. For a self-consistent potential U a deformation parameter determines both the density variation and the corresponding distortion of the potential. Thus the relation (9) may be rewritten in the form [11]

$$F_{12}^{(\lambda\mu)}=\frac{\partial U_1}{\partial\alpha_{\lambda\mu}}\frac{\delta\alpha_{\lambda\mu}}{\delta\varrho_2}. \tag{10}$$

It is convenient to define the deformation parameter $\alpha_{\lambda\mu}$ by the relation

$$u_1^{\lambda\mu}=f(\alpha_{\lambda\mu})\hat{q}_{\lambda\mu}(1) \tag{11}$$

where

$$\hat{q}_{\lambda\mu}=g(r)\,Y_{\lambda\mu}(\vartheta,\varphi) \tag{12}$$

$U_1^{\lambda\mu}$ is the part of the single particle potential of the $(\lambda\mu)$ symmetry type, and $f(\alpha_{\lambda\mu})$ is such a function of $\alpha_{\lambda\mu}$ that the equipotential surfaces of the distorted potential are given by

$$r = r_0(1 + \sum_{\lambda\mu} \alpha_{\gamma\mu} Y_{\lambda\mu}) \tag{13}$$

We have then

$$F_{12} = \frac{\partial f(\alpha_{\lambda\mu})}{\partial \alpha_{\lambda\mu}} \frac{\partial \alpha_{\lambda\mu}}{\partial Q_{\lambda\mu}} \frac{\delta Q_{\lambda\mu}}{\delta \varrho_2} \hat{q}_{\lambda\mu}(1) = \frac{\partial f(\alpha_{\lambda\mu})}{\partial \alpha_{\lambda\mu}} \frac{\partial \alpha_{\lambda\mu}}{\partial Q_{\lambda\mu}} \hat{q}_{\lambda\mu}(1)\hat{q}_{\lambda\mu}(2) \tag{14}$$

where

$$Q_{\lambda\mu} = \int \hat{q}_{\lambda\mu}(2)\varrho_2 dV_2 \tag{15}$$

is the average value of the operator $\hat{q}_{\lambda\mu}$.

Thus the multipole force strength is given by

$$\varkappa_{\lambda\mu} = \frac{\partial f(\alpha_{\lambda\mu})}{\partial \alpha_{\lambda\mu}} \frac{\partial \alpha_{\lambda\mu}}{\partial Q_{\lambda\mu}} \tag{16}$$

where the derivatives should be taken at the value of the deformation parameters α which correspond to the nuclear state just considered.

In general, both the $f(\alpha_{\lambda\mu})$ and $Q_{\lambda\mu}$ are not linear in $\alpha_{\lambda\mu}$ so the strength parameter $\varkappa_{\lambda\mu}$ changes with the deformation of the nucleus. It should be stressed that, as a rule, the parameters $\varkappa_{\lambda\mu}$ change differently with deformation not only for different λ components but also for different μ components of the same λ. For example, in the quadrupole case, $\varkappa_{22}$ increases while $\varkappa_{20}$ decreases with the increasing quadrupole deformation [5].

In order to avoid calculating the derivatives of the expresion (16) for given values of deformation one can define the operators

$$\hat{q}_{\lambda\mu}(\tilde{r}) = g(\tilde{r}) Y_{\lambda\mu}(\tilde{\vartheta}, \tilde{\varphi}) \tag{17}$$

in such a streched coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ in which the potential and the density distribution become spherical again. In this coordinate system we have

$$F_{\tilde{1}\tilde{2}} = \frac{\partial f(\tilde{\alpha}_{\lambda\mu})}{\partial \alpha_{\lambda\mu}} \frac{\partial \tilde{\alpha}_{\lambda\mu}}{\partial \tilde{Q}_{\lambda\mu}} \hat{q}_{\lambda\mu}(\tilde{r}_1)\hat{q}_{\lambda\mu}(\tilde{r}_2) \tag{18}$$

where the derivatives should be taken at $\tilde{\alpha}_{\lambda\mu} = 0$.

For $g(\tilde{r}) = \tilde{r}^\lambda$ the value of

$$\varkappa_{\lambda\mu} = \left(\frac{\delta f(\tilde{\alpha}_{\lambda\mu})}{\partial \tilde{\alpha}_{\lambda\mu}} \frac{\partial \tilde{\alpha}_{\lambda\mu}}{\partial \tilde{Q}_{\lambda\mu}} \right)_{\tilde{\alpha}_{\lambda\mu}=0} \tag{19}$$

coincides with the estimate of Bohr and Mottelson [1] for the multipole force strength of spherical nuclei.

The deformation dependence of the force given by (14) is determined by the transformation of the expression (18) from the space $(\tilde{x}, \tilde{y}, \tilde{z})$ to the space (x, y, z).

In the case of an ellipsoidal nucleus with the semi-axis

$$a_1 = R_0 e^{\sigma/2}$$
$$a_2 = R_0 e^{\sigma/2} \tag{20}$$
$$a_3 = R_0 e^{\sigma}$$

described by the harmonic oscillator potential, this transformation has the form

$$\tilde{x} = x e^{\sigma/2}$$
$$\tilde{y} = y e^{\sigma/2} \tag{21}$$
$$\tilde{z} = z e^{-\sigma}$$

where

$$\sigma = 0.631\beta(1+0.045\beta+\ldots)$$

and

$$\varepsilon \approx 0.95\beta.$$

The quadrupole force dependence on σ in the (x, y, z) system implied by this transformation is given in Ref. [5].

A static hexadecapole deformation introduces an additional distortion of the density. In the space (x, y, z) this introduces the dependence of the multipole force strength on the hexadecapole deformation parameter ε_4. In the formula (14) this dependence is determined by the dependence of the multipole moment $Q_{\lambda\mu}$ on ε_4. In the case of quadrupole force with $g(r) = r^2$ the complete formulas for the strength with ε_4 dependence included have the form:

$$\varkappa_{20} = \frac{4\pi}{3} \frac{M\omega_0^2}{AR_0^2} \frac{2e^{-\sigma}+e^{\sigma}}{2e^{2\sigma}+e^{-\sigma}} \left[1+\varepsilon_4 \frac{1.12+2.7\sigma}{1+\sigma}\right] \tag{22}$$

$$\varkappa_{22} = \frac{4\pi}{3} \frac{M\omega_0^2}{AR_0^2} e^{2\sigma}[1+0.19\,\varepsilon_4] \tag{23}$$

where R_0 is the radius of the nucleus.

The values of $\varkappa_{20}^0$ and $\varkappa_{22}^0$ obtained from Eqs (22) and (23) are listed in column six of Tables I and II, respectively.

The quadrupole force strength values calculated from the experimental energies of β and γ vibrational states are in good agreement with the values predicted on the basis of the simple oscillator model according to formulas (22) and (23). Both the absolute values of the strength and their deformation dependence agree within an average accuracy of a few per cent in the case of β-vibration, and 10–20 per cent in the case of γ-vibration with the predicted number. In particular, the calculation shows the opposite tendencies in the deformation dependence of $\varkappa_{20}^0$ and $\varkappa_{22}^0$ in accordance with formulas (22) and (23). Both methods of calculation give nearly the same difference between these two components of the quadrupole force.

It is interesting to notice that even the slight dependence on ε_4 predicted by formula (22) appears in the value of the strength calculated from the experimental data. It appears quite clearly in the splitting of $\varkappa_{20}^0$ value for nuclei with the same quadrupole deformation ε but different hexadecapole deformation ε_4.

The results obtained above support the conclusion that the force extracted from the phenomenological potentials may be used in the calculations of the collective nuclear properties.

4. *Vibrational states for large deformation*

The method described in Section 2 was used to study the behaviour of the γ-vibrational state for large deformation. The quadrupole coupling constant $\varkappa_{22}$ is taken from formula (23). In this case we took into account nine harmonic oscillator shells for protons and ten for neutrons. The pairing force was diagonalized within Z and N double degenerate levels for protons and neutrons, respectively, and the pairing force strength was assumed to be proportional to the surface of the nucleus [12]. The results of these calculations are shown in Fig. 1. We see that the energy of the 2^+ vibrational state has two maxima which correspond very nicely to the minima of the total energy of the nucleus calculated with the use of the

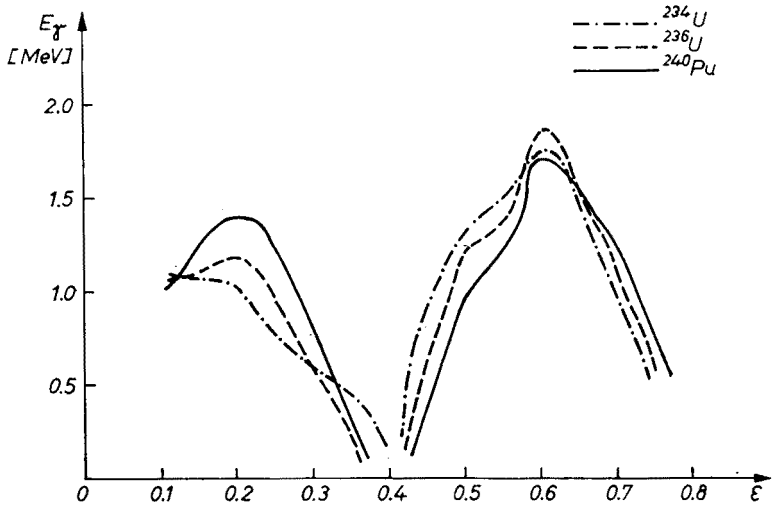

Fig. 1. The energy of the γ-vibrational state for ^{234}U, ^{236}U and ^{240}Pu is drawn as the function of the deformation parameter ε for the $\varkappa_{22}^0$ value calculated from formula (23). The haxadecapole deformation parameter $\varepsilon_4 = 0.06$

Nilsson potential by the method of Strutinsky [13]. The position of the 2^+ vibrational state in the second minimum is of the same order of magnitude as at the equilibrium. It seems very interesting to notice that we have no real solution at the deformation corresponding to the saddle points. This seems to be in agreement with the result obtained by Pashkevich [14] who calculated the total energy of the nucleus with the inclusion of the γ deformation parameter. He concluded that while in both minima the nucleus seems to be axially symmetric

the saddle point occurs for $\gamma \neq 0$, so if there is any experimental evidence about the γ-vibrational state at the saddle point [2] this kind of vibration may exist only if $\gamma \neq 0$.

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