A COSMOLOGICAL MODEL IN THE PROJECTIVE THEORY OF RELATIVITY

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A simple system of field-equations of the projective theory of relativity is solved for a homogeneous, isotropic cosmological model in the case of incoherent matter. The consequences of this projective cosmology are discussed and compared with those of the classical Einsteinian and projective Jordanian theories.

The five-dimensional projective theory of relativity is a unified theory of gravitation, electromagnetism, and a scalar field. By projection of the projective field-equations or the projective Lagrangian one obtains a four-dimensional formulation. The field-equations proposed by Schmutzer [1] can be derived in the case of an absent or negligible electromagnetic field from the four-dimensional variational principle

$$\delta \int (SR+S_0L)\sqrt{g}\, d\tau = 0. \tag{1}$$

S is a scalar field, S_0 a constant, R the scalar curvature and L the Lagrangian of the matter which is independent of S. A more rigorous motivation of the variational integral can be given within the framework of the projective theory [1].

By variation of S and g_{ik} one obtains the field-equations:

$$R = 0 \tag{2}$$

$$R_{ik}+S^{-1}S_{,i;k} = \varkappa_0 S_0 S^{-1}(T_{ik}-1/3T^l_l g_{ik}). \tag{3}$$

The condition of integrability

$$T^{ik}{}_{;k} = 0 \tag{4}$$

coincides with that of the Einsteinian theory.

It is important that there appears the coupling factor

$$\varkappa = \varkappa_0 S_0 S^{-1}, \tag{5}$$

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which is interpreted as a variable gravitational coefficient. In the projective theory [1] this connection between the scalar field S and the gravitational coefficient $\varkappa$ follows from an analysis of the equations of the electromagnetic field and its coupling to the gravitational field, *i. e.* it depends on the feature of the projective theory to be a unified field theory. The system of field-equations (2, 3, 4) shall be solved for a cosmological model. We shall start these investigations with the cosmological principle of homogeneity, which states that for constant cosmic time the three-dimensional local space is equally filled with matter.

Mathematically, we express this principle by the Robertson-Walker-metric

$$ds^2 = \varrho^2(T)\, dl^2 - dT^2. \tag{6}$$

The line-element dl defines the metric of a three-dimensional space, which shall not depend on the universal cosmic time $T = ct$.

By substituting (6) in (2, 3, 4) and taking as energy-momentum-tensor that of incoherent matter

$$T_{ik} = \mu u_i u_k, \tag{7}$$

where the density of matter μ depends only on T and the relativistic velocity u_i has in the considered co-moving coordinates only the non-vanishing component $u_4 = ic$, we find that the local space described by dl is a space of constant curvature

$$dl^2 = \varrho_0^2 \left(\frac{dr^2}{1 - \varepsilon r^2} + r^2 d\Omega^2 \right) \tag{8}$$

where ϱ_0 — constant curvature, r — radial coordinate (measured in units of ϱ_0), $d\Omega$ — solid angle, ε — sign of curvature ($\varepsilon = 0, \pm 1$).

It is a homogeneous and isotropic space.

Furthermore, from $R = 0$ follows the cosmological equation

$$\ddot{\varrho}\varrho + \dot{\varrho}^2 + \varepsilon \equiv \tfrac{1}{2}(\varrho^2)^{\cdot\cdot} + \varepsilon = 0 \tag{9}$$

and after some combination we obtain from the other field-equations the analogon to Friedman's equation

$$\left(\frac{\dot{\varrho}}{\varrho}\right)^2 + \frac{\varepsilon}{\varrho^2} + \frac{\dot{S}\dot{\varrho}}{S\varrho} = \frac{\varkappa \mu c^2}{3}. \tag{10}$$

The dot means differentiation with respect to T.

According to (4) the conservation law has the same form as in the Einsteinian theory

$$\frac{\dot{\mu}}{\mu} + \frac{3\dot{\varrho}}{\varrho} = 0. \tag{11}$$

The variant of a projective cosmology represented by the system of differential equations (9), (10), (11) differs essentially from the Jordanian, even if one disregards the different relation between scalar field and gravitational coefficient. Compared with Jordan's extraordinarily complicated differential equation [2] the cosmological equation (9) is quite simple. Moreover it does not depend on the state of the matter, which is supposed to be an ideally fluid medium. This supposition must be given up, if one wants to remove the singularities of $\varrho(T)$. For, contrary of the Einsteinian and Jordanian theories, even the consideration of the pressure terms, which is certainly indispensable on the extreme conditions at the start of the world, does not modify the cosmological equation.

In the Einsteinian cosmology the Friedman-equation is sufficient to calculate the world-radius $\varrho(T)$, if one regards also the equation of integrability (11). Here the Friedman-equation is enlarged by the scalar field ($\dot{S} \neq 0$).

For incoherent matter the integration (A the constant of integration) of the conservation law (11) gives us the result

$$\mu c^2 \varrho^3 = 3 A S_0^{-1}, \tag{12}$$

which for the closed model ($\varepsilon = +1$) means finite, constant total mass of the universe

$$M = \mu V = 2\pi^2 \mu \varrho^3 = \frac{6\pi^2 A}{S_0 c^2}. \tag{13}$$

The cosmological differential equation of second order determines ϱ as a function of T with two constants of integration (T_0, ϱ_0).

It is invariant under translation (and reflection) of time, so that by a time-scale-transformation T_0 can be made to vanish:

$$\varrho^2 = \begin{cases} 2\varrho_0(T-T_0) \text{ resp. } \varrho_0^2, & \varepsilon = 0 \\ \Delta\varrho_0^2 - \varepsilon(T-T_0)^2, & \varepsilon = \pm 1, \ \Delta = 0, \pm 1 \end{cases} \tag{14}$$

This solution corresponds essentially to that of the Einsteinian radiation cosmos (parabola, semicircle, hyperbola).

The temporal behaviour of the world-radius induces us to define the world-age T_w (start of the expansion or vanishing world-volume at $T_w = 0$):

$$T_w = \begin{cases} T & \varepsilon = 0 \\ T + \varrho_0 & \varepsilon = +1 \\ T & \varepsilon = -1, \ \Delta = 0, +1 \\ T - \varrho_0 & \varepsilon = -1, \ \Delta = -1 \end{cases} \tag{15}$$

With $\varrho(T)$ and $\mu(T)$ given as solutions of (9, 12), the scalar field S is determined by (10) (S_1 constant of integration):

$$S = \frac{\varkappa_0 A}{2\varrho_0^2}\varrho - \frac{S_1 \varrho_0}{\varrho} = \frac{1}{\varrho_0 \sqrt{2\varrho_0 T}}(\varkappa_0 A T - S_1 \varrho_0^2) \qquad \varepsilon = 0, \ \varrho \neq \varrho_0 \tag{16a}$$

$$S = \varepsilon \frac{\varkappa_0 A + S_1 T}{\sqrt{\Delta\varrho_0^2 - \varepsilon T^2}} = \frac{\varepsilon \varkappa_0 A}{\varrho} + S_1 \sqrt{\varepsilon \frac{\Delta\varrho_0^2 - \varrho^2}{\varrho^2}} \qquad \varepsilon = \pm 1 \tag{16b}$$

Testing the cosmological model by astrophysical experience the assumption of isotropy and homogeneity of space should be checked critically as well as the negligibility of pressure and electromagnetism.

Theory and nature are compared by substituting the empirical values of the physical quantities into the theoretical relations, and by investigating whether or how far they are fulfilled. Moreover, quantities that are difficult to find (*e. g.* ε, ϱ) can be calculated.

First of all we have as measurable physical quantities the expansion parameters H and q, which can be obtained from the cosmological red-shift:

$$cH = \frac{c\dot{\varrho}}{\varrho} = 0.8 \cdot 10^{-10}\, a^{-1} \tag{17}$$

$$qH^2 = -\frac{\ddot{\varrho}}{\varrho},\ 0.5 \leqslant q \leqslant 5.0. \tag{18}$$

The cosmic medium is characterized by

$$p = 0,\ 10^{-31}\,\mathrm{gcm}^{-3} \leqslant \mu \leqslant 10^{-27}\,\mathrm{gcm}^{-3}. \tag{19}$$

To estimate the world-radius we consider cosmic objects with a maximal distance of 10^{10} light-years.

Furthermore we introduce the variability-parameter of the gravitational coefficient respectively of the scalar field

$$G = -\frac{\dot{\varkappa}}{\varkappa} = \frac{\dot{S}}{S}\,. \tag{20}$$

Up to now there exists no useful, convenient method to measure G, but in agreement with several astro- and geophysical facts it could be positive [3].

In order to evaluate the cosmological and generalized Friedman-equations we rewrite them by using the above introduced quantities and isolating the terms with ε — where it is suitable to oppose the projective (P) and Einsteinian (E) theories:

$$\frac{\varepsilon}{\varrho^2} = \frac{\varkappa\mu c^2}{3} - H^2 - GH \tag{P21}$$

$$\frac{\varepsilon}{\varrho^2} = \frac{\varkappa\mu c^2}{3} - H^2 \tag{E21}$$

$$\frac{\varepsilon}{\varrho^2} = H^2(q-1) \tag{P22}$$

$$\frac{\varepsilon}{\varrho^2} = H^2(2q-1). \tag{E22}$$

The determination of ε from (21) is, as compared with (22), more difficult, because in (21) we have to consider apart from the not too reliable value of Hubble's coefficient H and the value of the material density, which is uncertain by orders of magnitude, the empirically still not precisely given variability-parameter of the gravitational coefficient.

Under these circumstances it is more convenient to take equations (22), in which we have to know only the kinematic parameters q and H. Because of the variation of q(19) all signs of curvature ($\varepsilon = +1{,}0, -1$) are compatible with (22) — in spite of the tendency towards a closed model that meets theoretical considerations with respect to the cosmological principle of homogeneity and to boundary conditions. But a precise q can determine the sign of curvature alone. Opposing formulas (E22) and (P22) we see, that in the projective theory a closed model is less favoured.

After having discussed some consequences directly from the basic equations we investigate the physical significance of the solutions. Because of the well-known increase of ϱ (positive Hubble's coefficient, cosmological red shift) and the hypothetical decrease of $\varkappa$ at the present time several types and branches of the solution have to be abandoned as useless right from the start.

The singularities of the remaining solutions of $\varrho(T)$ or $S(T)$ can fix the beginning of the universe. But we will join these both definitions of the start of the world, and by this we obtain additional information about the constants of integration:

$$\begin{aligned} S_1 &= 0 \quad \varepsilon = 0 \\ \varkappa_0 A &= S_1 \varrho_0 \quad \varepsilon = +1 \\ -\varkappa_0 A &= S_1 \varrho_0 \qquad \varepsilon = -1, \Delta = -1. \end{aligned} \tag{23}$$

In the case of $\varepsilon = -1$, $\Delta = 0, +1$ such a connection is not possible. If one introduces the age of the universe into Hubble's coefficient, one obtains in the case of $\varepsilon = 0$ ($\varepsilon = \pm 1$ give similar results)

$$H = \frac{1}{2T_w}. \tag{24}$$

With the settlement (23) we can deduce the remarkable relation

$$G = H. \tag{25}$$

This renders possible an interesting estimation:

For a relative change of the gravitational coefficient $\varkappa$ we have a corresponding change of the radius R of the earth

$$\frac{\delta R}{R} = -K\frac{\delta \varkappa}{\varkappa} = K\frac{\delta S}{S} = K \cdot cG = K \cdot cH. \tag{26}$$

K is a number of order of magnitude one. Because of the (rough) estimate (26) we expect a relative change of 10^{-10} per year. That means an annual increase of the radius of the earth of about half a millimeter. This value agrees with Egyed's results [4] obtained from geological facts.

Investigating the cosmological consequences we seize the opportunity to point out that because of the conservation law (4), (11) Jordan's hypothesis of creation of matter is not compatible with the variant presented here of the projective theory. By the way, it is just the Einsteinian law (11) that agrees with astrophysical observation concerning the 3°K-radiation.

On the other hand the formation of organic life can be understood in spite of a formerly greater gravitational coefficient (Teller-Gamov's objection). The rough estimation [5]

$$\varkappa(1/2\ T_w) < 1.1\ \varkappa(T_w) \tag{27}$$

can be realized by the explicitly given solutions (16) if one regards the conditions (23) and makes a suitable choice of ϱ_0.

A special problem that is also connected with the scalar field is posed by the examination of the following three criteria:

1. Simultaneous singularity of $S(T)$ and $\varrho(T)$

2. Reversibility of the cosmos in the meaning, that the phase of expansion is just the inverse of the phase of contraction,

3. Decrease of the gravitational coefficient and simultaneous expansion of the universe.

It turns out, that the conditions *1.*, *2.* and *3.* cannot be realized simultaneously for any model. *1.* and *2.*, or *1.* and *3.* can be fulfilled together, whereas decreasing gravitational coefficient and the above commented reversibility exclude each other.

Because of $T^{ik}_{;k} = 0$ the thermodynamic reversibility is given and does not depend on the sign of G.

Finally another remark:

If in the projective theory one investigates the embedding of a radially symmetric vacuum-solution (*e. g.* Fricke-Heckmann-Jordan- solution [2]) in the expanding universe, one finds out, that the radially symmetric vacuum-solution cannot be non-static only by a time-scale-function as it is in the Einstein-Strauss-solution [6] but it must be essentially non-static.

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