# POINCARÉ INVARIANCE IN SIX-SPACE

By E. Y. Hsu*

Dapartment of Physics, University of Michigan

*(Received June 11, 1970)*

Implications of Poincaré invariance in six-space are considered. It is shown that isospin space emerges in a natural way. The usual space-time is defined by electromagnetic interactions. The one meson exchange amplitude is naturally modified to have a sharp forward peak. The corresponding potential is derived.

## 1. Introduction

The fundamental nondynamical symmetry group is here assumed to be the Poincaré group in six-space. The rotational symmetry is expressed by the constancy of the metric $g_{\xi\eta}(\xi, \eta = 1, 2, 3, 4, 5, 6)$; $g_{11} = g_{22} = g_{33} = -1$, $g_{44} = g_{55} = g_{66} = 1$, $g_{\xi\eta} = 0$ $(\xi \neq \eta)$. The translational symmetry gives rise to the six-vector $p^{\xi}$. Four of its components $p^{\mu}(\mu = 1\ 2, 3, 4)$ are identified with momentum and energy. The physical meaning of the extra components is found in Section 2. The special role of the photon as the mediator of electromagnetic interaction and therefore macroscopic measurement is discussed in Section 3. As an application, a generalization of the one meson exchange amplitude and the nonrelativistic potential corresponding to it is given in Section 4.

## 2. Isospin space

The Clifford algebra for six-space is

$$\Gamma_{\xi}\Gamma_{\eta}+\Gamma_{\eta}\Gamma_{\xi} = 2g_{\xi\eta} \tag{1}$$

An irreducible 8×8 matrix representation using the Dirac and Pauli matrices $\gamma_{\mu}$ and $\tau$; is

$$\Gamma_{\xi} = \gamma_{\mu}\times\tau_{3}, \quad 1\times\tau_{1}, \quad 1\times\tau_{2} \tag{2}$$

where $x$ denotes direct product.

The Dirac equation is generalized to

$$\Gamma_{\xi}p^{\xi}\psi(p) = \mu\psi(p) \tag{3}$$

* Postal address: c/o Andrew J. Salthouse, 436 Physics-Astronomy Bldg., Department of Physics, University of Michigan, Ann Arbor, Michigan 48104, U.S.A.

where $\mu$ is an invariant parameter of a particle. Writing the eight component spinor in the form

$$\psi = N \begin{pmatrix} A \\ B \end{pmatrix}$$

where $A$ and $B$ each is a four component spinor and $N$ is a normalization factor, (3) can be rewritten as

$$(\mu - \gamma_\mu p^\mu)\, A = f^* B \tag{4a}$$

$$(\mu + \gamma_\mu p^\mu)\, B = f A \tag{4b}$$

where $f = p^5 + ip^6$. $B$ is therefore determined given $A$.
$A$ is specified to be the solution of

$$\gamma_\mu p^\mu A = mA \tag{5}$$

where $m$ is defined to be the positive root of

$$m^2 \equiv -\vec{p}^{\,2} + (p^4)^2 = \mu^2 - |f|^2 \tag{6}$$

From (4a), $B$ must satisfy

$$\gamma_\mu p^\mu B = mB \tag{7}$$

An explicit solution of (3) is

$$\psi = \sqrt{\frac{\mu + m}{2\mu}} \begin{pmatrix} A \\ \dfrac{f}{\mu + m}\, A \end{pmatrix} \tag{8}$$

where the normalization $\psi^+\psi = A^+A = P^4/m$ is used.

In case $f = 0$, (4a) becomes $(\mu - \mu)A = 0$ which puts no further restriction on $A$; (4b) becomes $2\mu B = 0$ which means $B = 0$ if $\mu = m \neq 0$.

Since $A$ and $B$ satisfy the same Dirac equation as in (5) and (7), they are identified as isospin components. A graphic interpretation can be obtained by writing $\psi$ in the form for a spin $\frac{1}{2}$ spinor and spin axis making angles $\theta$ and $\varphi$ with the third axis:

$$\psi = A \begin{pmatrix} \cos\dfrac{\theta}{2} \\ e^{i\varphi} \sin\dfrac{\theta}{2} \end{pmatrix} \tag{9}$$

The angles are given by

$$\theta = \arccos \frac{m}{\mu}, \qquad \varphi = \arg f. \tag{10}$$

The picture then emerges that the three-space spanned by $mp^5p^6$ or equivalently $\tau x^5x^6$ where $\tau$ is the proper time is the isospin space. The direction of isospin is the direction $p^\xi$ makes in the $mp^5p^6$ space (Fig. 1).

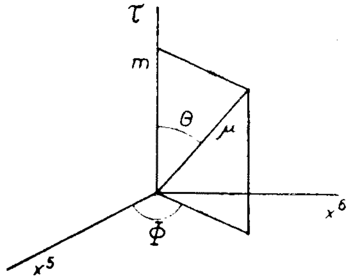


**Fig. 1. The direction of isospin in the $mp^5p^6$ space**

## *3. The photon*

Relation (6) is assumed to hold for any real particle. If a particle has $\mu = 0$ then $m^2$ and $|f|^2$ being positive must also be zero. So the photon may be assigned $\mu = m = f = 0$. For a virtual photon that is being exchanged in an electromagnetic process it is plausible as shown in the next section that the relation $0 = \mu \geqslant \sup |\Delta f|$, where $\Delta f$ is the amount of 5-and 6-momentum carried by the virtual photon, holds. Then $\Delta f = 0$ too. This reduces the lowest order form of current conservation

$$q^\xi \bar{\psi}(p+q)\, \Gamma_\xi \psi(p) = 0 \tag{11}$$

which can be derived from (3), to the well-known form

$$q^\mu \bar{\psi}(p+q)\, \gamma_\mu \varkappa \tau_3 \psi(p) = 0 \tag{12}$$

which singles out an axis of isospin space.

## *4. One meson exchange amplitude*

Consider a one meson exchange process. The Green function is

$$\frac{1}{q_\xi q^\xi - \varkappa^2} = \frac{1}{q^2 + |\Delta f|^2 - \varkappa^2} \tag{13}$$

where $q^\xi$ is the six-momentum carried by the meson, $\varkappa$ is its invariant parameter, $q^2 = q_\mu q^\mu$, $\Delta f = q^5 + iq^6$. For simplicity let the external particle lines all have identical invariant parameter $\mu$ and mass $m$. Then they all have the same $|f|$. The range of $|\Delta f|^2$ is then

$$0 \leqslant |\Delta f|^2 \leqslant 4|f|^2 = 4(\mu^2 - m^2) \tag{14}$$

Since $|\Delta f|^2$ is not measured in experiment the physical amplitude must be averaged

$$M = \int_0^{4(\mu^2 - m^2)} \frac{\varrho(|\Delta f|^2)\, d|\Delta f|^2}{q^2 + |\Delta f|^2 - \varkappa^2} \tag{15}$$

where $\varrho$ is a weighting function. A simple choice for $\varrho$ is the constant

$$\varrho = \frac{1}{4(\mu^2 - m^2)}. \tag{16}$$

This is justified on the ground that the integral over $|\Delta f|^2$ is really over $dq^5 dq^6 \alpha |\Delta f| d |\Delta f| \alpha d |\Delta f|^2$ which has no weighting as in an usual Feynman integral.

Using (16) in (15).

$$M = \frac{1}{b} \ln \left(1 + \frac{b}{q^2 - \varkappa^2}\right). \tag{17}$$

where $b \equiv 4(\mu^2 - m^2)$. (17) has the limit

$$\lim_{b \to 0} M = \frac{1}{q^2 - \varkappa^2}.$$

$M$ is analytic in $q^2$ except on the cut from $\varkappa^2 - \mathrm{b}$ to $\varkappa^2$. The restriction $\varkappa^2 \geqslant b$ is imposed so that the cut does not extent to the physical region. (This is the basis of $\mu^2 \geqslant \sup |\Delta f|^2$ in the last section). The discontinuity across the cut is $-2\pi i/b$. The cut reduces to a pole as

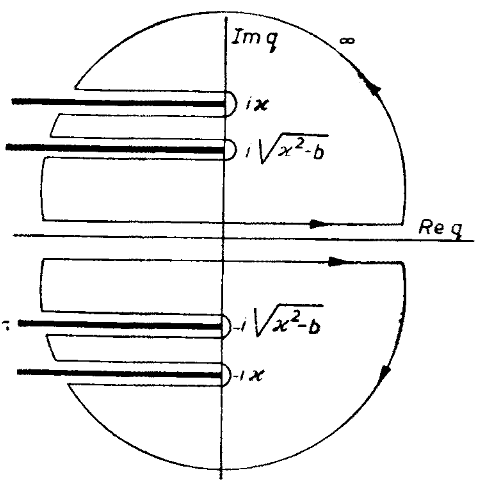


Fig. 2. The contour for evaluating the integral in Eq. (18)

$b \to 0$. For $b$ very near $\varkappa^2$, the amplitude has a sharp peak at small negative $q^2$. This does not disagree with the experimentally observed feature of forward peaking in many two body and quasi-two body processes.

A scattering potential can be derived from $M$. It is defined as the Fourier transform of the static form of $M$.

$$V(r) = \int \frac{d^3 q}{(2\pi)^3} e^{i q \cdot r} \frac{1}{b} \ln \frac{|q|^2 + \varkappa^2 - b}{|q|^2 + \varkappa^2}. \tag{18}$$

The $|q|$ integral is evaluated using the contours in Fig. 2. The result is

$$V(r) = \frac{1}{2\pi br}\left[\frac{\sqrt{\varkappa^2-b}\exp\left(-\sqrt{\varkappa^2-b}\,r\right)-\varkappa\exp(-\varkappa r)}{r}+\right.$$
$$\left.+\frac{\exp\left(-\sqrt{\varkappa^2-b}\,r\right)-\exp(-\varkappa r)}{r^2}\right]. \tag{19}$$

It has the limit

$$\lim_{b\to 0} V = -\frac{\exp(-\varkappa r)}{4\pi r}$$

*i. e.*, the Yukawa potential.

REFERENCES

[1] Similiar ideas are found in J. Rayski, *Acta Phys. Polon.*, **27**, 947 (1965); **28**, 87 (1965); **29**, 519 (1965).