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GENERALIZED STATISTICAL TENSORS AND THEIR APPLICATIONS

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Measurable quantities, called generalized statistical tensors, which are functions of the reaction amplitudes, are introduced and investigated. Their role in the full experimental check of theoretical predictions between the amplitudes is discussed. As an example the polarizations predictions following from the additivity assumption in the quark model are investigated. It is shown that the full experimental check of the quark model predictions for the reaction $\frac{1}{2}^+ + 0^- \to \frac{3}{2}^+ + 0^-$ cannot be made from knowledge of the angular distribution of the linear momenta of the decay products alone.

1. *Introduction*

As the volume of data on scattering of polarized primary beams on polarized targets increases, the most convenient formalism useful for comparison of theoretical predictions with experimental data is required.

Let us consider the two body reaction of the type

$$1+2 \to 3+4. \tag{1}$$

If one is concerned with polarization experiments, the initial and final states of the interacting particles are determined by the initial and final density matrices $\hat{\varrho}^i$ and $\hat{\varrho}^f$, respectively. Their matrix elements are related in the following way [1]:

$$\varrho^f_{k_3k_4l_3l_4} = \frac{1}{N} \sum_{k_1,k_2} \sum_{l_1,l_2} f_{k_3k_4k_1k_2} \varrho^i_{k_1k_2l_1l_2} f^*_{l_3l_4l_1l_2} \tag{2}$$

where k_i and l_i enumerate the spin states of the particles (they can denote helicities, transversities or spin states in any other representation); N is the normalization factor:

$$N = \sum_{ki,li} f_{k_3k_2k_1k_2} \varrho^i_{k_1k_2l_1l_2} f^*_{k_3k_4l_1}, \tag{3}$$

$f_{k_3k_4k_1k_2}$ denote the amplitudes of the reaction (1).

Conservation laws or dynamical models link some amplitudes with the others and thus imply some relations between the matrix elements $\varrho^f_{k_3k_4l_3l_4}$. However the determination of

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$\varrho^f_{k_3k_4l_3l_4}$ from the experimental data is quite cumbersome; it was proposed [5, 6], therefore, to derive theoretical predictions for the quantities simply connected with the measured data, namely, for the expansion coefficients $q^{L_3L_4}_{M_3M_4}$ of the $\hat{\varrho}^f$ into the direct products of the operators $\hat{Q}^{L_3}_{M_3} \otimes \hat{Q}^{L_4}_{M_4}$:

$$\hat{\varrho}^f = \frac{1}{(2s_3+1)\,(2s_4+1)} \sum_{L_3,L_4} \sum_{M_3,M_4} q^{L_3L_4}_{M_3M_4} \hat{Q}^{L_3}_{M_3} \otimes \hat{Q}^{L_4}_{M_4}. \tag{4}$$

The explicit form of the coefficients $q^{L_3L_4}_{M_3M_4}$ is the following [1]:

$$q^{L_3L_4}_{M_3M_4} = \sqrt{2s_3+1)(2s_4+1)} \sum_{k_i,l_i} (-1)^{s_3+s_4-l_3-l_4} \times$$

$$\times \langle s_3 s_3 k_3 - l_3 L_3 | M_3 \rangle \langle s_4 s_4 k_4 - l_4 | L_4 M_4 \rangle \varrho^f_{k_3k_4l_3l_4}. \tag{5}$$

The q's can be determined from the knowledge of the probability distribution of linear momenta and/or polarizations of decay products or from double scattering. If the final particles are unstable the q's are simply expressed by the average values $\langle Y^{L_3 *}_{M_3} Y^{L_4 *}_{M_4} \rangle$ [2]:

$$\langle Y^{L_3}_{M_3} Y^{L_4 *}_{M_4} \rangle = q^{L_3L_4}_{M_3M_4 4} b_{L_3} b_{L_4}, \tag{6}$$

where

$$\langle Y^{L_3}_{M_4} Y^{L_4}_{M_4} \rangle$$

$$= \int_0^\pi \sin\vartheta_3 d\vartheta_3 \int_0^{2\pi} d\varphi_3 \int_0^\pi \sin\vartheta_4 d\vartheta_4 \int_0^{2\pi} d\varphi_4 W(\vartheta_3, \varphi_3, \vartheta_4, \varphi_4) Y^{L_3}_{M_3}(\vartheta_3, \varphi_3) Y^{L_4}_{M_4}(\vartheta_4, \varphi_4). \tag{7}$$

Here, $W(\vartheta_3, \varphi_3, \vartheta_4, \varphi_4)$ denotes the normalized angular distribution function of momenta or polarization of the decay products, and the coefficients b_{L_3} and b_{L_4} depend on the decay amplitudes only. A more detailed discussion of the measurability of the q's will be given later.

In spite of the fact that the predictions for the q's can be easily checked they are not very convenient for theoretical analysis, since they depend explicitly on the initial density matrix.

The theoretical models always give relations between the reaction amplitudes. Very often they cannot be checked in one experiment; usually, to check them one has to perform experiments with different initial polarizations. The problem of the choice of the minimal number of experiments which can provide a complete test of the theoretical model is quite important and cannot be solved in a simple way by the known methods.

In this paper we propose to describe the theoretical predictions in terms of the coefficients $A^{L_1L_2L_3L_4}_{M_1M_2M_3M_4}$ (called by us generalized statistical tensors) which are defined as follows:

$$q^{L_3L_4}_{M_3M_4} = \frac{1}{N} \sqrt{\frac{(2s_3+1)\,(2s_4+1)}{2s_1+1)\,(2s_2+1)}} \sum_{L_1,L_2} \sum_{M_1,M_2} A^{L_1L_2L_3L_4}_{M_1M_2M_3M_4}\, q^{L_1L_2}_{M_1M_2}, \tag{8}$$

where the $q^{L_1L_2}_{M_1M_2}$ are coefficients analogous to the $q^{L_3L_4}_{M_3M_4}$ but for the initial density matrix.

The advantages of using the A's are as follows:

(*i*) The A's can be easily obtained from the linear Eqs (8), so they are closely related to the measured quantities.

(*ii*) The A's are simple functions of the bilinear products of the reaction amplitudes and are independent of the initial polarization, so the symmetries of the A's can be easily derived from the symmetries of the reaction amplitudes. Also the minimal set of equations for the A's which is equivalent to a given set of equations for the amplitudes can be easily found. From this minimal set of equations one can easily see how many experiments and with what initial polarization are needed to check them.

(*iii*) If the relations between the reaction amplitudes are linear then the derivation of relations between the A's is almost immediate and can be performed in the same way for any reaction of the type (1). Having linear relations between the A's one can derive from them the relations for the q's obtained in the experiments with different initial polarizations.

In Section 2 we find the explicit form of the A's and we investigate the problem of their independence.

In Sections 3 and 4 we find the relations between the A's stemming from the parity conservation and from the quark model predictions for the transversity amplitudes.

In Section 5 we discuss the problem of derivation of the relations between the q's from linear relations between the A's and the problem of "measurability" of the q's by means of measuring $\langle Y^{L_3 *}_{M_3} Y^{L_4 *}_{M_4} \rangle$.

In Section 6 we discuss the problem of obtaining predictions for the "measurable" A's on the example of the quark model predictions for the reactions $\frac{1}{2}^+ + 0^- \to \frac{3}{2}^+ + 0^-$. We also investigate the problem of full testability of theoretical relations between the amplitudes by the use of "measurable" A's. It is shown that the quark model predictions for these reactions cannot be completely tested. It is also found that the quark model implies that the cross-section for the reactions of this type is independent of the initial polarization.

2. *General formalism*

Starting from Eq. (5), using (2) and expressing $\hat{\varrho}^i$ by the coefficients $q^{L_1 L_2}_{M_1 M_2}$ in the usual manner [1],

$$\hat{\varrho}^i_{k_1 k_2 l_1 l_2} = \frac{1}{\sqrt{(2s_1+1)\,(2s_2+1)}} (-1)^{s_1+s_2-l_1-l_2} \times$$

$$\times \sum_{L_1, L_2} \langle s_1 s_1 k_1, -l_1 | L_1 M_1 \rangle \langle s_2 s_2 k_2, -l_2 | L_2 M_2 \rangle q^{L_1 L_2}_{M_1 M_2} \tag{9}$$

we obtain the following formula:

$$q^{L_3 L_4}_{M_3 M_4} = \frac{1}{N} \sqrt{\frac{(2s_3+1)\,(2s_4+1)}{(2s_1+1)\,(2s_2+1)}} \sum_{L_1, L_2} \sum_{M_1, M_2} A^{L_1 L_2 L_3 L_4}_{M_1 M_2 M_3 M_4}\, q^{L_1 L_2}_{M_1 M_2}, \tag{10}$$

where the coefficients $A^{L_1L_2L_3L_4}_{M_1M_2M_3M_4}$ (denoted in the abbreviated notation by A^L_M or A's)[1] are defined as

$$A^L_M = \sum_l \alpha^l_L(s, M) f_{M+l} f_l^* . \tag{11}$$

Here L, M, s, l denote the following four-indices:

$$L = (L_1L_2L_3L_4),\ M = (M_1M_2M_3M_4),\ s = (s_1s_2s_3s_4),\ l = (l_1l_2l_3l_4),\ (M+l)_i = M_i + l_i$$

and

$$\alpha^l_L(s, M) = \prod_{i=1}^{4} (-1)^{s_i - l_i} \langle s_i s_i M_i + l_i, \ -l_i | L_i M_i \rangle . \tag{12}$$

One can notice that

$$A^{0000}_{0000} = \sqrt{\frac{(2s_1+1)\,(2s_2+1)}{(2s_3+1)\,(2s_4+1)}}\ \sigma_0, \tag{13}$$

where σ_0 is the differential cross-section for the reaction (1) with unpolarized initial particles.

From the properties of the Clebsch–Gordan coefficients we easily derive that

$$\sum_l \alpha^l_L(s, M) \alpha^l_{L'}(s, M') = \delta_{LL'}\, \delta_{MM'} \tag{14}$$

and

$$\sum_L \alpha^l_L(s, M) \alpha^{l'}_L(s, M) = \delta_{ll'} . \tag{15}$$

Using Eq. (15) we obtain the inverse formula to Eq. (10) which reads:

$$f_k f_l^* = \sum_L \alpha^l_L(s, k-l) A^L_{k-l} . \tag{16}$$

Equation (16) shows that any product of amplitudes of the type $f_k f_l^*$ can be expressed in terms of the A's. It is clear, therefore, that the set of A's contains the same physical information as the set of reaction amplitudes.

It is easily seen that the A's transform under the rotation of the reference frame for spin quantization in the following way[2]:

$$'A^{L_1L_2L_3L_4}_{M_1M_2M_3M_4} = \sum_{M_i'} \left(D^{L_1}_{M_1M_1'}(\alpha_1, \beta_1, \gamma_1)\, D^{L_2}_{M_2M_2'}(\alpha_2, \beta_2, \gamma_2)\right)^* \times \tag{17}$$

$$\times D^{L_3}_{M_3M_3'}(\alpha_3, \beta_3, \gamma_3)\, D^{L_4}_{M_4M_4'}(\alpha_4, \beta_4, \gamma_4)\, A^{L_1L_2L_3L_4}_{M_1M_2M_3M_4},$$

where α_i, β_i, γ_i are the Euler angles of rotation of the i-th particle reference frame for spin quantization.

Now we shall discuss the problem of independence of the A's. There exist many relations between the A's following from the construction of the A's from the amplitudes of

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[1] The A's are functions of energy E and the scattering angle ϑ, but we shall not write out this dependence, unless necessary to avoid confusion.

[2] Notice that we rotate the reference frame for spin quantization for each particle separately.

the reaction. The most simple ones are readily obtained from the definition (11):

$$(A^L_M)^* = (-1)^{M_1+M_2+M_3+M_4} A^L_{-M}. \tag{18}$$

However, in general, construction of the maximal set of independent A's seems to be quite difficult. Therefore, we shall find the maximal set of independent equations which have to be satisfied by the A's. At first we construct the set of independent products of amplitudes for the reaction (1). If nothing is assumed about this reaction we have $n=(2s_1+1)\times\times(2s_2+1)\,(2s_3+1)\,(2s_4+1)$ independent, complex amplitudes. We write them as elements of an n-dimensional vector $(f_1, f_2, \dots, f_n)$. We can represent all products of these amplitudes as elements of an $(n\times n)$ hermitian matrix $\hat{B}$ obtained in the following way:

$$\hat{B} = \left\| \begin{matrix} f_1^* \\ \\ f_n^* \end{matrix} \right\| (f_1, \dots, f_n) = \left\| \begin{matrix} f_1 f_1^*, \dots, f_n f_1^* \\ \vdots \\ f_1 f_n^*, \dots, f_n f_n^* \end{matrix} \right\|. \tag{19}$$

The matrix $\hat{B}$ is of rank one [3], since all its two dimensional minors vanish, so the matrix $\hat{B}$ is determined by any non vanishing row or column. If we assume that p-th row is given and that $B_{pp} \neq 0$, then all other matrix elements can be calculated with the help of the equations

$$B_{ij} = \frac{B_{ip}B_{jp}^*}{B_{pp}}, \quad 1 \leqslant i, j \leqslant n, \quad j \neq p. \tag{20}$$

TABLE I

Table of the coefficients $a^L_{L'}(s, M)$

$s=0$

$a^0_0\,(0,0)=1$

$s=\frac{1}{2}$

$L'\backslash LM$	1,1	1,0	1,-1	0,0
1	1	0	-1	1
0	0	1	0	0

$s=1$

$L'\backslash LM$	2,2	2,1	2,0	2,-1	2,-2	1,1	1,0	1,-1	00
2	1	0	$-\frac{1}{3}$	0	1	1	0	-1	$\frac{4}{3\sqrt{2}}$
1	0	1	0	-1	0	0	1	0	0
0	0	0	$\frac{4}{3\sqrt{2}}$	0	0	0	0	0	$\frac{1}{3}$

$s=3/2$

$L'\backslash LM$	3,3	3,2	3,1	3,0	3,-1	3,-2	3,-3	2,2	2,1	2,0	2,-1	2,-2	1,1	1,0	1,1	00
3	1	0	$-\frac{1}{5}$	0	$\frac{1}{5}$	0	-1	1	0	$-\frac{1}{\sqrt{5}}$	0	1	$\frac{12}{5\sqrt{6}}$	0	$-\frac{12}{5\sqrt{6}}$	$\frac{2}{\sqrt{5}}$
2	0	1	0	$-\frac{1}{\sqrt{5}}$	0	1	0	0	1	0	-1	0	0	$\frac{2}{\sqrt{5}}$	0	0
1	0	0	$\frac{12}{5\sqrt{5}}$	0	$-\frac{12}{5\sqrt{6}}$	0	0	0	0	$\frac{2}{\sqrt{5}}$	0	0	$\frac{1}{5}$	0	$-\frac{1}{5}$	$\frac{1}{\sqrt{5}}$
0	0	0	0	$\frac{2}{\sqrt{5}}$	0	0	0	0	0	0	0	0	0	$\frac{1}{\sqrt{5}}$	0	0

Equation (20) and the condition $\mathrm{Im}(B_{pp}) = 0$ written in terms of the A's form the maximal set of independent equations to be fulfiled stemming only from the fact that the amplitudes are complex numbers. From Eqs (20) also follows the equality

$$A_0^0 A_M^L = \prod_{i=1}^{4} \frac{1}{\sqrt{2s_i+1}} \sum_{L_i', L_i''} \sum_{M_i'} (-1)^{L_i'} \sqrt{2L_i'+1}\, \sqrt{2L_i''+1}$$
$$\times \langle L_i' L_i'' M-M', M' | LM \rangle W(L_i' s_i L_i s_i;\, s_i L_i'') A_{M-M'}^{L'} A_{M'}^{L''}, \tag{21}$$

where $W(L_i' s_i L_i s_i;\, s_i L_i'')$ are standard Racah coefficients. Since Eqs (20) written in terms of the A's and the Eqs (21) are rather complicated it is difficult to deduce from them the complete set of independent A's. The easiest way to find such a set is to choose the set of A's (the easiest to be measured) and then to check whether they form the complete set of independent functions of all independent amplitudes. The simplest way to perform this is to check whether the appropriate Jacobi determinant does not vanish.

3. *Symmetries of the coefficients A*

By the symmetries of the A's we shall understand relations between them following from the symmetries of the amplitudes of the reaction (1). To derive some the most important symmetries we have to use some representation for numerating the spin states of the particles participating in the reaction (1). In this paper we shall use the transversity representation, we choose the normal to the reaction plane as the quantization axis of spin of the particles in the reaction. We choose also the z-axis for each particle in the direction of its momentum. To write our amplitudes explicitly we define the states:

$$|0\vartheta E l_i l_j\rangle = \sqrt{\frac{p}{E}}\Big\rangle R_y(\vartheta)\, (L_z(v_i)|000\, l_i\,\rangle L_{-z}(v_j)|000\, l_j\,\rangle), \tag{22}$$

where $R_y(\vartheta)$ is the operator of rotation through the angle ϑ around the y-axis and $L_z(v)$ is the operator of the Lorentz transformation in the direction of the z-axis with velocity v. The states $|000\, l_i\rangle$ and $|000\, l_j\rangle$ are the spin states of the particles i and j in their rest frames with the y-axis as the quantization axis. The velocities v_i and v_j are chosen in such a way that the total three-momentum of the two-particle system vanishes. The transversity amplitude in the c. m. system is written with the help of the states (22) in the following way:

$$f_{l_3 l_4 l_1 l_2}(E, \vartheta) = \frac{2\pi}{p} \langle 0\, \vartheta E l_3 l_4 | T | 00\, E l_1 l_2 \rangle, \tag{23}$$

where E is the total centre of mass energy, ϑ is the scattering angle and p is the relative momentum in the c. m. system.

If parity conservation and rotational invariance are assumed, the following relations between the amplitudes (23) are obtained:

$$f_{l_3 l_4 l_1 l_2} = \eta(-1)^{l_3+l_4-l_1-l_2} f_{l_3 l_4 l_1 l_2} \tag{24}$$

where $\eta = \eta_1\eta_2\eta_3\eta_4$ is the product of intrinsic parities of the interacting particles. Now starting from Eq. (11) and using Eq. (24) we easily obtain the following equations:

$$A_M^L = \sum_l \alpha_L^l(s, M) f_{M+l}\, f_l^*(-1)^{M_1+M_2+M_3+M_4} \tag{25}$$

and

$$A^L_M = \eta \sum_l (-1)^{l_1+l_2-l_3-l_4} \alpha^l_L(s, M) f_{M+l}\, f^*_l. \tag{26}$$

Using Eqs (14) and (16) we can write Eqs (25) and (26) in the following form

$$A^L_M = (-1)^{M_1+M_2+M_3+M_4} A^L_M \tag{27}$$

and

$$A^L_M = \eta(-1)^{s_1+s_2-s_3-s_4} \sum_{L'} a^L_{L'}(s, M)\, A^{L'}_M, \tag{28}$$

where

$$a^L_{L'}(s, M) = \sum_{l_i} (-1)^{s_1+s_2+s_3+s_4-l_1-l_2-l_3-l_4} \alpha^l_{L'}(s, M)\, \alpha^l_L(s, M). \tag{29}$$

The coefficients $a^L_{L'}(s, M)$ factorize:

$$a^L_{L'}(s, M) = \prod_{i=1}^{4} a^{Li}_{Li'}(s_i, M_i), \tag{30}$$

where

$$a^{Li}_{Li'}(s_i, M_i) = \sum_{l_i} (-1)^{si-li} \langle s_i s_i M_i + l_i, -l_i | L_i M_i \rangle \langle s_i s_i M_i + l_i, -l_i | L'_i M_i \rangle. \tag{31}$$

The coefficients $a^{Li}_{Li'}(s_i, M_i)$ fulfil the following relations:

$$a^{Li}_{Li'}(s_i, M_i) = 0 \text{ for } L_i < |M_i| \text{ or } L'_i < |M_i|, \tag{32}$$

$$a^{Li}_{Li'}(s_i, M_i) = \tfrac{1}{2}[1+(-1)^{2si+Li+L'i+Mi}] a^{Li}_{L'i}(s_i, M_i), \tag{33}$$

$$a^{Li}_{Li'}(s_i, M_i) = a^{Li'}_{Li}(s_i, M_i) \tag{34}$$

and

$$a^{Li}_{Li'}(s_i, M_i) = (-1)^{Mi} a^{Li}_{Li'}(s_i, -M_i). \tag{35}$$

From the properties of the Clebsch-Gordan coefficients it is easy to show that the $a^{Li}_{Li'}(s_i, M_i)$ are orthogonal:

$$\sum_{Li''} a^{Li}_{Li''}(s_i, M_i) a^{Li'}_{Li''}(s_i, M_i) = \delta_{Li,Li'}. \tag{36}$$

The transversity amplitudes $f_{l_3l_4l_1l_2}$ of the reaction (1) and the amplitudes $g_{l_1l_2l_3l_4}$ of the "time-reversed" reaction

$$3+4 \to 1+2 \tag{37}$$

are related in the following way:

$$f_{l_3l_4l_1l_2} = \frac{p'}{p} g_{l_1l_2l_3l_4}, \tag{38}$$

where p and p' are the initial and the final relative momenta for the reaction (1) in the c. m. system. Taking into account the definition of the A's the following relations among the generalized statistical tensors for these two reactions are obtained:

$$A^{L_1L_2L_3L_4}_{M_1M_2M_3M_4} = {}'A^{L_3L_4L_1L_2}_{M_3M_4M_1M_2} \tag{39}$$

where A^L_M and $'A^L_M$ are the A's for the reactions (1) and (37) respectively. If the reaction (1) is elastic then the relations (39) reduce the number of independent A's.

If the reaction (1) is of the type

$$A+\bar{A} \to B+\bar{B} \tag{40}$$

then it is invariant under charge conjugation, and the amplitudes of the reaction (40) fulfil the relations

$$f_{l_3l_4l_1l_2} = (-1)^{l_3+l_4-l_1-l_2} f_{l_4l_3l_1l_2}. \tag{41}$$

If we combine the relations (41) and (20) we derive the following relations:

$$f_{l_3l_4l_1l_2} = \eta f_{l_4l_3l_1l_2} \tag{42}$$

which rewritten in terms of the A's takes the form

$$A^{L_1L_2L_3L_4}_{M_1M_2M_3M_4} = A^{L_2L_1L_4L_3}_{M_2M_1M_4M_3}. \tag{43}$$

4. *Quark model predictions*

We shall present here the symmetries of the A's following from the quark model. We shall consider the reactions of the type:

$$\frac{1}{2}^+ + 0^- \to \frac{3}{2}^+ + 0^-, \tag{44}$$

$$\frac{1}{2}^+ + 0^- \to \frac{3}{2}^+ + 1^-, \tag{45}$$

$$\frac{1}{2}^+ + \frac{1}{2}^+ \to \frac{3}{2}^+ + \frac{1}{2}^+, \tag{46}$$

$$\frac{1}{2}^+ + \frac{1}{2}^+ \to \frac{3}{2}^+ + \frac{3}{2}^+. \tag{47}$$

The following relations among the reaction amplitudes hold [7]:

$$\begin{aligned} f_{l_3l_4l_1l_2} = {} & N_{s_1}(l_1)\, N_{s_3}(l_3)\, f_{l_3+1,l_4,l_1+1,l_2} + \\ & + N_{s_1}(-l_1)\, N_{s_3}(-l_3)\, f_{l_3-1,l_4,l_1-1,l_2}. \end{aligned} \tag{48}$$

Amplitudes of the reaction (47) fulfil additionally two other classes of relations

$$\begin{aligned} f_{l_3l_4l_1l_2} = {} & N_{s_2}(l_2)\, N_{s_4}(l_4)\, f_{l_3,l_4+1,l_1,l_2+1} + \\ & + N_{s_2}(-l_2)\, N_{s_4}(-l_4)\, f_{l_3,l_4-1,l_1,l_2-1} \end{aligned} \tag{49}$$

and

$$\begin{aligned} f_{l_3 l_4 l_1 l_2} &= N_{s_1}(l_1)\, N_{s_2}(l_2)\, N_{s_3}(l_3)\, N_{s_4}(l_4)\, f_{l_3+1, l_4+1, l_1+1, l_2+1} + \\ &+ N_{s_1}(l_1)\, N_{s_2}(-l_2)\, N_{s_3}(l_3)\, N_{s_4}(-l_4)\, f_{l_3+1, l_4-1, l_1+1, l_2-1} + \\ &+ N_{s_1}(-l_1)\, N_{s_2}(l_2)\, N_{s_3}(-l_3)\, N_{s_4}(l_4)\, f_{l_3-1, l_4+1, l_1-1, l_2+1} + \\ &+ N_{s_1}(-l_1)\, N_{s_2}(-l_2)\, N_{s_3}(-l_3)\, N_{s_4}(-l_4)\, f_{l_3-1, l_4-1, l_1-1, l_2-1} \end{aligned} \tag{50}$$

where Eq. (50) is obtained by iteration of Eqs (48) and (49). The coefficients $N_s(l)$ are defined as follows:

$$N_s(l) = \begin{cases} 1 & \text{for } l = -\tfrac{1}{2} \\ 0 & \text{for all other cases} \end{cases} \tag{51}$$

and

$$N_{3/2}(l) = \begin{cases} \dfrac{1}{\sqrt{3}} & \text{for} \quad l = \dfrac{1}{2} \\ 1 & \text{for} \quad l = -\dfrac{1}{2} \\ \sqrt{3} & \text{for} \quad l = -\dfrac{3}{2} \\ 0 & \text{for all other cases} \end{cases} \tag{52}$$

Relations (48) put into the definition (11) of the A's give for them the following relations:

$$\begin{aligned} A^{L_1L_2L_3L_4}_{M_1M_2M_3M_4} &= \sum_{L_1', L_3'} (\beta^{L_1}_{L_1'}(s_1, M_1)\beta^{L_3}_{L_3'}(s_3, M_3) A^{L_1'L_2L_3'L_4}_{M_1+1, M_2, M_3+1, M_4} + \\ &+ (-1)^{L_1+L_1'+L_3+L_3'} \beta^{L_1}_{L_1'}(s_1, -M_1)\beta^{L_3}_{L_3'}(s_3, -M_3) A^{L_1'L_2L_3'L_4}_{M_1-1, M_2, M_3-1, M_4} + \end{aligned} \tag{53}$$

where the coefficients $\beta^{L}_{L'}(s, M)$ are defined as

$$\beta^{L}_{L'}(s, M) = \sum_{l_1, l_2} N_s(l_1) \langle s s l_1,\ -l_2 | L M \rangle \langle s s l_1 + 1,\ -l_2 | L' M + 1 \rangle. \tag{54}$$

The coefficients $\beta^{L}_{L'}(s, M)$ for $s = 1/2$ and $s = 3/2$ are displayed in Table II.

If we take the complex conjugate of Eq. (53), exchange $M_i \to -M_i$ and apply Eq. (18) we come to the relation

$$\begin{aligned} A^{L_1L_2L_3L_4}_{M_1M_2M_3M_4} &= \sum_{L_1', L_3'} (-1)^{L_1+L_1'+L_3+L_3'} (\beta^{L_1}_{L_1'}(s_1, M_1)\beta^{L_3}_{L_3'}(s_3, M_3) A^{L_1'L_2L_3'L_4}_{M_1+1, M_2, M_3+1, M_4} + \\ &+ (-1)^{L_1+L_1'+L_3+L_3'} (\beta^{L_1}_{L_1'}(s_1, -M_1)\beta^{L_3}_{L_3'}(s_3, -M_3) A^{L_1'L_2L_3'L_4}_{M_1-1, M_2, M_3-1, M_4}). \end{aligned} \tag{55}$$

Now combining the relations (53) and (55) we derive

$$\begin{aligned} 0 = \sum_{L_1', L_3'} (1-(-1)^{L_1+L_1'+L_3+L_3'}) (\beta^{L_1}_{L_1'}(s_1, M_1)\beta^{L_3}_{L_3'}(s_3, M_3) A^{L_1'L_2L_3'L_4}_{M_1+1, M_2, M_3+1, M_4} - \\ - \beta^{L_1}_{L_1'}(s_1, -M_1)\beta^{L_3}_{L_3'}(s_3, -M_3) A^{L_1'L_2L_3'L_4}_{M_1-1, M_2, M_3-1, M_4}) \end{aligned} \tag{56}$$

TABLE II

Table of the coefficients $\beta^{L}_{L'}(s, M)$

$s = \frac{1}{2}$

$L' = 1$

$M \backslash L$	1	0
1	0	
0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
−1	$\frac{1}{\sqrt{2}}$	

$L' = 0$

$M \backslash L$	1	0
1	0	
0	0	0
−1	$\frac{1}{\sqrt{2}}$	

$s = 3/2$

$L' = 3$

$M \backslash L$	3	2	1	0
3	0			
2	$\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{6}}$		
1	$\frac{2}{\sqrt{10}}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{15}}$	
0	$\frac{3}{2\sqrt{3}}$	$-\frac{5}{2\sqrt{15}}$	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{15}}$
−1	$\frac{19}{10\sqrt{3}}$	$-\frac{4}{\sqrt{30}}$	$-\frac{2}{5\sqrt{2}}$	
−2	$\frac{4}{\sqrt{10}}$	$-\frac{2}{\sqrt{10}}$		
−3	$\frac{3}{\sqrt{6}}$			

$L' = 2$

$M \backslash L$	3	2	1	0
3	0			
2	0	0		
1	0	$\frac{1}{2}$	$-\frac{5}{2\sqrt{15}}$	
0	0	$\frac{2}{\sqrt{6}}$	$-\frac{5}{\sqrt{30}}$	$\frac{1}{\sqrt{6}}$
−1	$\frac{1}{2\sqrt{15}}$	$\frac{2}{\sqrt{6}}$	$-\frac{2}{\sqrt{10}}$	
−2	$\frac{1}{2}$	$\frac{1}{2}$		
−3	$\frac{3}{\sqrt{6}}$			

$L' = 1$

$M \backslash L$	3	2	1	0
3	0			
2	0	0		
1	0	0	0	
0	0	0	$\frac{1}{\sqrt{2}}$	$-\frac{3}{\sqrt{10}}$
−1	$-\frac{3}{10\sqrt{3}}$	$\frac{3}{\sqrt{30}}$	$\frac{1}{5\sqrt{2}}$	
−2	$-\frac{3}{2\sqrt{15}}$	$\frac{9}{2\sqrt{15}}$		
−3	0			

$L' = 0$

$M \backslash L$	3	2	1	0
3	0			
2	0	0		
1	0	0	0	
0	0	0	0	0
−1	$\frac{1}{2\sqrt{15}}$	$-\frac{1}{\sqrt{6}}$	$\frac{3}{\sqrt{10}}$	
−2	0	0		
−3	0			

and

$$A^{L_1L_2L_3L_4}_{M_1M_2M_3M_4} = \sum_{L_1',L_3'} \tfrac{1}{2}\left(1+(-1)^{L_1+L_1'+L_3+L_3'}\right)\left(\beta^{L_1}_{L_1'}(s_1, M_1)\beta^{L_3}_{L_3'}(s_3, M_3)\times\right.$$
$$\left.\times A^{L_1'L_2L_3'L_4}_{M_1+1,M_2,M_3+1,M_4}+\beta^{L_1}_{L_1'}(s_1, -M_1)\beta^{L_3}_{L_3'}(s_3, -M_3)A^{L_1'L_2L_3'L_4}_{M_1-1,M_2,M_3-1,M_4}\right). \tag{57}$$

For applications iteration of Eq. (57) is also very useful:

$$A^{L_1L_2L_3L_4}_{M_1M_2M_3M_4} = \sum_{L_1',L_3'}\ \sum_{L_1'',L_3''} \tfrac{1}{4}\left(1+(-1)^{L_1+L_1'L_3+L_3'}\right)\left(1+(-1)^{L_1'+L_1''+L_3'+L_3''}\right)\times$$
$$\times\left[\beta^{L_1}_{L_1'}(s_1, M_1)\beta^{L_3}_{L_3'}(s_3, M_3)\beta^{L_1'}_{L_1''}(s_1, M_1+1)\beta^{L_3'}_{L_3''}(s_3, M_3+1)A^{L_1''L_2L_3''L_4}_{M_1+2,M_2,M_3+2,M_4}+\right.$$
$$+\beta^{L_1}_{L_1'}(s_1, -M_1)\beta^{L_3}_{L_3'}(s_3, -M_3)\beta^{L_1'}_{L_1''}(s_1, -M_1+1)\beta^{L_3'}_{L_3''}(s_3, -M_3+1)A^{L_1''L_2L_3''L_4}_{M_1-2,M_2,M_3-2,M_4}+$$
$$+\beta^{L}_{L_1'}(s_1, M_1)\beta^{L_3}_{L_3'}(s_3, M_3)\beta^{L_1'}_{L_1''}(s_1, -M_1-1)\beta^{L_3'}_{L_3''}(s_3, -M_3-1)A^{L_1''L_2L_3''L_4}_{M_1M_2M_3M_4}+$$
$$\left.+\beta^{L_1}_{L_1'}(s_1, -M_1)\beta^{L_3}_{L_3'}(s_3, -M_3)\beta^{L_1'}_{L_1''}(s_1, M_1-1)\beta^{L_3'}_{L_3''}(s_3, M_3-1)A^{L_1''L_2L_3''L_4}_{M_1M_2M_3M_4}\right]. \tag{58}$$

Using Eqs (49) and (11) we can derive analogous equations for the A's of reaction (47). These equations are obtained from Eqs (53), (55), (56), (57) and (58) by simple exchange of indices: $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. The relations for the A's implied by Eq. (43) are obtained in a similar way as the relations for the A's following from Eqs (48) and (49).

With the help of Eqs (18) and (28) one can easily derive the relations for the statistical tensors given in [5] following from the quark model. It has to be noticed that the connection between the A coefficients and the statistical tensors of [5] is the following:

$$\sqrt{(2s_1+1)(2s_2+1)}\, T^{L_3L_4}_{M_3M_4} = \frac{1}{N} \left(A^{00L_3L_4}_{00M_3M_4}\right)^*. \tag{59}$$

5. *Relations among the q's implied by the relations between the A's*

We shall now consider the problem of derivation of relations for the coeffficients $q^{L_3L_4}_{M_3M_4}$ stemming from the relations among the A's. Let us suppose that we have linear relations for the A's. Any set of linear relations for the A's can be written in the form:

$$A^{L_1L_2L_3L_4}_{M_1M_2M_3M_4} = \sum_{k=1}^{m_0} \sum_{L_i', M_i'} c^{L_1L_2L_3'L_4'}_{kM_3M_4M_3'M_4'} b^{L_1L_2L_1'L_2'}_{kM_1M_2M_1'M_2'} A^{L_1'L_2'L_3'L_4'}_{M_1'M_2'M_3'M_4'}. \tag{60}$$

Let us now suppose that we perform the polarization experiment with the initial polarization described by $q^{iL_1L_2}_{M_1M_2}$; if we contract both sides of Eq. (60) with $q^{iL_1L_2}_{M_1M_2}$, then using Eq. (10) we obtain

$$q^{L_3L_4}_{M_3M_4} = \frac{1}{N_0} \sqrt{\frac{(2s_3+1)(2s_4+1)}{(2s_1+1)(2s_2+1)}} \sum_{k=1}^{m_0} \sum_{L_3', L_4'} \sum_{M_3', M_4'} c^{L_3L_4L_3'L_4'}_{kM_3M_4M_3'M_4'} \times$$
$$\times \sum_{L_1', L_2'} \sum_{M_1', M_2'} \Big(\sum_{L_1, L_2} \sum_{M_1, M_2} q^{iL_1L_2}_{M_1M_2} b^{L_1L_2L_1'L_2'}_{kM_1M_2M_1'M_2'} \Big) A^{L_1'L_2'L_3'L_4'}_{M_1'M_2'M_3'M_4'}. \tag{61}$$

N_0 is the normalization factor calculated from Eq. (3). From the expression in the bracket we can form the $n \times n$ matrix $(n = (2s_1+1)(2s_2+1))$:

$$\hat{M}_k = \sum_{L_1', L_2'} \sum_{M_1', M_2'} \sum_{L_1, L_2} \sum_{M_1, M_2} q^{iL_1L_2}_{M_1M_2} b^{L_1L_2L_1'L_2'}_{kM_1M_2M_1'M_2'} \hat{Q}^{L_1'}_{M_1'} \otimes \hat{Q}^{L_2'}_{M_2'}. \tag{62}$$

In the space of the $n \times n$ matrices we can always form a basis $(\hat{\varrho}^l)$ consisting of n^2 positively definite, hermitian matrices with trace equal one (each of the matrices $\hat{\varrho}^l$ is a spin density matrix). If we now expand the matrix $\hat{M}_k$ in our basis we obtain

$$\hat{M}_k = \sum_l p_{kl} \hat{\varrho}^l. \tag{63}$$

Equation (56) immediately yields

$$\sum_{L_1, L_2} \sum_{M_1, M_2} q^{iL_1L_2}_{M_1M_2} b^{L_1L_2L_1'L_2'}_{kM_1M_2M_1'M_2'} = \sum_l p_{kl} q^{lL_1'L_2'}_{M_1'M_2'}. \tag{64}$$

Substitution of Eq. (64) into Eq. (61) gives

$$q^{L_3L_4}_{M_3M_4} = \frac{1}{N_0} \sum_{k=1}^{m_0} \sum_l \sum_{L_3', L_4'} \sum_{M_3', M_4'} N_l c^{L_3L_4L_3'L_4'}_{kM_3M_4M_3'M_4'} p^{kl} q^{lL_3'L_4'}_{M_3'M_4'}, \tag{65}$$

where $q^{lL_3'L_4'}_{M_3'M_4'}$, are the q's of the final density matrix when the initial polarization is described by the matrix $\hat{\varrho}^l$, and N_l are the appropriate normalization factors. The choice of our basic polarizations (ϱ^l) is quite arbitrary. In any particular case we choose such a basis that in the Eqs (63) the smallest possible number of basic matrices appear, because in this case we have to compare the q's of the final density matrix measured in the smallest number of experiments.

As an example of this procedure we shall consider the Eqs (53) for the reaction (44). In our case we have

$$A^{L_1 0 L_3 0}_{M_1 0 M_3 0} = \sum_{L_1', L_3'} \sum_{M_1', M_3'} (b^{L_3 L_3'}_{1 M_3 M_3'} b^{L_1 L_1'}_{1 M_1 M_1'} + b^{L_3 L_3'}_{2 M_3 M_3'} b^{L_1 L_1'}_{2 M_1 M_1'}) A^{L_1' 0 L_3' 0}_{M_1' 0 M_3' 0}, \tag{66}$$

where

$$\begin{aligned} b^{L_i L_i'}_{1 M_i M_i'} &= \beta^{L_i}_{L_i'}(M_i)\ \delta_{M_i', M_i+1}, \\ b^{L_i L_i'}_{2 M_i M_i'} &= (-1)^{L_i + L_i'} \beta^{L_i}_{L_i'}(-M_i)\ \delta_{M_i', M_i-1}. \end{aligned} \tag{67}$$

The matrices M_k ($k = 1, 2$) are

$$\hat{M}_k = \sum_{L_1, L_1'} \sum_{M_1, M_1'} b^{L_1 L_1'}_{k M_1 M_1'} q^{i L_1 0}_{M_1 0} \hat{Q}^{L_1'}_{M_1'} \otimes \hat{Q}^0_0. \tag{68}$$

One should notice that $\hat{Q}^0_0$ is the identity matrix. If we assume that the initial particles are unpolarized we obtain

$$\hat{M}_k = \sum_{L_1'} \sum_{M_1'} b^{0 L_1'}_{k 0 M_1'} \hat{Q}^{L_1'}_{M_1'} \otimes \hat{Q}^0_0. \tag{69}$$

Thus,

$$\begin{aligned} \hat{M}_1 &= \beta^0_1(0)\ \hat{Q}^1_1 \otimes \hat{Q}^0_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \hat{M}_2 &= -\beta^0_1(0) \hat{Q}^1_{-1} \otimes \hat{Q}^0_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{70}$$

If the following matrices are chosen as our basis:

$$\begin{aligned} \hat{\varrho}^0 &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \hat{\varrho}^1 &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\ \hat{\varrho}^2 &= \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, & \hat{\varrho}^3 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{71}$$

then we have:

$$\begin{aligned} \hat{M}_1 &= -(1+i)\ \hat{\varrho}_0 + \hat{\varrho}^1 + i\hat{\varrho}^2, \\ \hat{M}_2 &= -(1-i)\hat{\varrho}_0 + \hat{\varrho}^1 - i\hat{\varrho}^2 \end{aligned} \tag{72}$$

and this immediately gives

$$q^{0L_3\,0}_{M_3 0} = \frac{1}{2N_0} \sum_{L_3'=0}^{3} \{\beta^{L_3}_{L_3'}(M_3)\,[-N_0(1+i)\,q^{0L_3'\,0}_{M_3+1\,0} +$$

$$+N_1 q^{1\,L_3'\,0}_{M_3+1\,0} + iN_2 q^{2\,L_3'\,0}_{M_3+1\,0}] + (-1)^{L_3+L_3'}\beta^{L_3}_{L_3'}(-M_3)\times$$

$$\times[-N_0(1-i)q^{0\,L_3'\,0}_{M_1-1\,0} + N_1 q^{1\,L_3'\,0}_{M_3-1\,0} - iN_3 q^{2\,L_3'\,0}_{M_3-1\,0}]\}. \tag{73}$$

Thus, we obtained the quark model predictions between the q's. To check these predictions one has to measure all the q's in three experiments with initial states determined by the density matrices $\hat{\varrho}^0$, $\hat{\varrho}^1$, $\hat{\varrho}^2$. The matrix $\hat{\varrho}^0$ describes the unpolarized initial particles and $\hat{\varrho}^1$ and $\hat{\varrho}^2$ describe the polarization of the initial fermion in the reaction plane. The use of our basis (71) enables similar derivations for any equation between the A's in the form (60) and this was the reason for the detailed calculations given above. The analogous, however more complicated, reasoning can be performed for other reactions. We are not going to discuss this problem any longer, because there are experimental difficulties which make the relations (73) useful only in cases when all the q's involved in the relation of this type are measurable. However, not every coefficient q can be easily measured. The coefficients $q^{L_3L_4}_{M_3M_4}$ can be calculated with the help of Eq. (6), but unfortunately for parity conserving decays the b_L coefficients with L odd vanish for the angular distribution of momenta and thus in this case the measurement of $q^{L_3L_4}_{M_3M_4}$ with L_3 and L_4 odd would require the cumbersome observation of polarization of the decay products and at present technical reasons make this impossible. For reasons given we shall call the $q^{L_3L_4}_{M_3M_4}$ with L_3 and L_4 odd the "non-measurable" q's.

Since we are interested in theoretical results which can be tested by the existing methods, we shall solve the problem of this section in another way. We shall consider in the next section the set of "measurable" A's (which can be calculated from the "measurable" q's) and relations between them stemming from the relations (48) between the reaction amplitudes. It will be seen on the example of the reaction (44) that the relations among the "measurable" A's do not check the relations (48) completely. Therefore, at present a thorough check of the additivity assumption in the quark model for that reaction is not possible.

6. *Full testability and quark model predictions*

At first we shall give a precise definition of the notion of the "measurable" A's. By "measurable" A's we mean those (non-vanishing from parity conservation) A's which are coefficients in the expansion (8) of the measurable q's. To calculate them one must have all measurable q's measured in $(2s_1+1)^2(2s_2+1)^2$ experiments with different initial polarizations of the initial particles. The initial polarizations should be chosen in such a way that the obtained system of linear equations for the A's:

$$q^{L_3L_4}_{M_3M_4}(i) = \frac{1}{N_i}\sqrt{\frac{(2s_3+1)\,(2s_4+1)}{(2s_1+1)\,(2s_2+1)}} \sum_{L_1,L_2} \sum_{M_1,M_2} A^{L_1L_2L_3L_4}_{M_1M_2M_3M_4} q^{L_1L_2}_{M_1M_2}(i) \tag{74}$$

could be solved. Throughout this section by "A" we shall understand the "measurable" A.

Now we shall discuss the problem of derivation of the relations between the A's from the relations between the amplitudes predicted by some model. The relations for the amplitudes can be divided into two classes:

i) vanishing of some amplitudes

ii) some amplitudes expressed as the functions of the other amplitudes.

To check the relations (*i*) we write down explicitly all the A's with all $M_i = 0$; they are the linear combinations of the squares of the moduli of the amplitudes. Next, we try to build a linear combination $L(A)$ from the mentioned A's equal to the linear combination with positive coefficients of the squares of the moduli of only those amplitudes which have to vanish. The vanishing of $L(A)$ implies the relations (*i*).

In the case of the relations (*ii*) the situation is more involved. Now the procedure is the following:

a. assuming that the relations (*i*) hold we write down all the A's in terms of the non-vanishing amplitudes,

b. we write down the maximal set of independent relations between the A's stemming from the relations (*ii*),

c. we derive all the necessary equations for the amplitudes which also lead to the relations (b) for the A's,

d. from the equations (c) we choose, if possible, such a set of equations which is equivalent to the equations (*ii*) and then the set of the equations for the A's following from the chosen equations for the amplitudes is the required set of equations for the A's equivalent to the relations (*ii*).

As an example of this procedure we shall discuss the reaction (44) once more and quark model predictions for this reaction. In our case we have the following amplitudes f_i:

$$f_1 = f_{{}^3/_2\,0\,-{}^1/_2\,0}, \quad f_2 = f_{{}^1/_2\,0\,{}^1/_2\,0}, \tag{75}$$

$$f_3 = f_{-{}^1/_2\,0\,-{}^1/_2\,0}, \quad f_4 = f_{-{}^3/_2\,0\,{}^1/_2\,0}.$$

Equations (53) imply the following relations:

i)
$$f_1 = 0,\ f_4 = 0 \tag{76}$$

and

ii)
$$f_2 = f_3. \tag{77}$$

Let us notice that Eqs (76) are implied also by much weaker assumption that the value of transversity in the reaction can be changed by 1, 0 or −1. To check the validity of the quark model for that reaction one has to check the relations (76) and (77).

For simplicity's sake we denote:

$$A^{L_1 0 L_3 0}_{M_1 0 M_3 0} = A^{L_1 L_3}_{M_1 M_3}. \tag{78}$$

Now we write explicitly the "measurable" A's with $M_i = 0$:

$$A^{00}_{00} = \frac{1}{2\sqrt{2}}\left(|f_1|^2 + |f_2|^2 + |f_3|^2 + |f_4|^2\right),$$

$$A^{02}_{00} = \frac{1}{2\sqrt{2}}\left(|f_1|^2 - |f_2|^2 - |f_3|^2 + |f_4|^2\right),$$

$$A^{10}_{00} = \frac{1}{2\sqrt{2}}\left(-|f_1|^2 + |f_2|^2 - |f_3|^2 + |f_4|^2\right),$$

$$A^{12}_{00} = \frac{1}{2\sqrt{2}}\left(-|f_1|^2 - |f_2|^2 + |f_3|^2 + |f_4|^2\right). \tag{79}$$

For $L(A)$ we have

$$L(A) = A^{00}_{00} + A^{02}_{00} = \frac{1}{\sqrt{2}}\left(|f_1|^2 + |f_4|^2\right), \tag{80}$$

so the relation

$$A^{00}_{00} + A^{02}_{00} = 0 \tag{81}$$

checks the relations (76).

If Eqs (76) hold then the only non-vanishing A's are those from Eq. (79). In our case the relations (*ii*) imply only one independent equation

$$A^{10}_{00} = 0. \tag{82}$$

The necessary equation for the amplitudes leading to Eq. (82) is

$$|f_2| = |f_3| \tag{83}$$

and thus the quark model cannot be fully checked in this case.

Other relations among the A's which follow from the relations (76) and (77) are:

$$A^{12}_{00} = 0, \quad A^{02}_{02} = 0,$$

$$A^{12}_{02} = 0, \quad A^{12}_{-11} = 0 \tag{84}$$

and they can be treated as a complementary test of the relations (76) and (77).

Let us notice that from the relation (84) and from parity conservation it follows that the differential cross-section for the reaction (44), with polarization of the initial fermion equal to P, is independent of the polarization of the initial fermion:

$$\sigma(P) = \sigma_0. \tag{85}$$

The reactions (45), (46) and (47) can be considered in a similar way. Perhaps for these reactions the quark model predictions can be experimentally checked to full extent. The investigation of this and other problems is continued and the results will be published elsewhere.

7. *Discussion*

Since we discussed results in each section we shall only add some comments here. At first we would like to compare two methods of derivation of the relations between the A's stemming from the relations between the amplitudes. The method presented in Section 5 gives quite general results, but checking them requires knowledge of the "non-measurable" A's and q's. The method of Section 6, being less elegant, gives predictions between the "measurable" A's only. This method has yet one more advantage, namely, direct calculation gives a clear view of the changes in results caused by the use of weaker or stronger assumptions. The second method gives a set of independent relations, while for the first method this problem is quite involved.

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REFERENCES

[1] J. Werle, *Relativistic Theory of Reactions*, North-Holland 1966, Sects 26 and 32.
[2] A. Barut, *General Scattering Theory*, Gordon and Breach, New York 1969, pp. 85–126.
[3] M. Ademollo, R. Gatto, *Phys. Rev.*, **133**, B531 (1964); M. Ademollo, R. Gatto, G. Preparata, *Phys. Rev.*, **140**, B192 (1965).
[4] A. E. Edmonds, *Angular Momentum in Quantum Mechanics*, Princeton University Press 1957.
[5] A. Kotański, K. Zalewski, *Nuclear Phys.*, **B4**, 559 (1968).
[6] A. Białas, K. Zalewski, *Nuclear Phys.*, **B6**, 449 (1968).
[7] M. Kupczyński, *Acta Phys. Polon.*, **B1**, 301 (1970).