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A THEOREM ON PERIODS IN THE TWO-DIMENSIONAL RELATIVISTIC DYNAMICS

By A. Staruszkiewicz

Institute of Physics, Jagellonian University, Cracow*

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We consider two invariant dynamical problems on the Minkowski plane: $\delta\int L(\dot{u}_1, \dot{u}_2, u_1-u_2)dv = 0$ and $\delta\int L(\dot{v}_1, \dot{v}_2, v_1-v_2)du = 0$. We show that for an oscillatory motion, u-period and v-period in the first case are equal respectively to u-period and v-period in the second case, provided the total energy and momentum in both cases are the same. We discuss also some general features of relativistic dynamical problems which are solvable by means of ordinary differential equations.

Introduction

Hill and Rudd [1] and the author [2], [3], [4], have discovered recently that some dynamical problems in the special theory of relativity are solvable by means of ordinary differential equations. All the solvable problems have some common characteristic features. We should like to analyse those features by considering a formal structure which might be called a "two-dimensional relativistic dynamics".

The structure of the Newtonian dynamics

Let us denote by t and x respectively the time and space coordinate of an event in some inertial system of reference. Two events $a_1 = (t_1, x_1)$ and $a_2 = (t_2, x_2)$ are said to be simultaneous if $t_1 = t_2$. The relation of simultaneity is symmetric, reflexive and transitive. Moreover, it is invariant with respect to the Galileo group. Consequently, the space of events in the classical dynamics is divided by the relation of simultaneity on disjoint equivalence classes; the division is invariant with respect to the Galileo group. Let us consider now n world lines $l_1, l_2, \ldots, l_n$ representing n particles. It is assumed in the Newtonian dynamics that en event $a_i \in l_i$ interacts with an event $a_k \in l_k$ if and only if a_i and a_k belong to the same equivalence class. We may associate with each equivalence class a real number T; it may

* Address: Instytut Fizyki UJ, Kraków 16, Reymonta 4, Poland.

be the time t or any increasing function of t. The Newtonian equations of motion are differential equations in which T is a parameter while spatial coordinates of events belonging to the same equivalence class are functions to be determined.

The structure of the Minkowski plane

In the special theory of relativity the relation of simultaneity is no longer invariant. However, let us introduce new variables $u = t-x$ and $v = t+x$. Two events $a_1 = (u_1, v_1)$ and $a_2 = (u_2, v_2)$ are said to be u-simultaneous of $u_1 = u_2$ and v-simultaneous if $v_1 = v_2$. The relations of u-simultaneity and v-simultaneity are easily seen to be reflexive, symmetric, transitive and invariant. Consequently, for each of them we can repeat construction of dynamics which we have described previously.

Examples of dynamical laws

We shall assume that a particle is represented by two world lines corresponding to two histories of a particle: history a in which u is a coordinate while v is a parameter and history b in which v is a coordinate while u is a parameter. The dynamical law is most conveniently given in the form of an action principle: $\delta S_a = 0$ for the history a and $\delta S_b = 0$ for the history b. It might be instructive to write down several examples; we shall write the action for two particles with masses m_1 and m_2; it is trivial to generalize it to any number of particles.

$$S_a = -\int m_1 \sqrt{\dot{u}_1} + m_2 \sqrt{\dot{u}_2} + k|u_1 - u_2| dv, \tag{1}$$

$$S_b = -\int m_1 \sqrt{\dot{v}_1} + m_2 \sqrt{\dot{v}_2} + k|v_1 - v_2| du.$$

The action describes two particles interacting by means of a constant force. The equations of motion were solved in [5].

$$S_a = -\int m_1 \sqrt{\dot{u}_1} + m_2 \sqrt{\dot{u}_2} + e_1 e_2 \frac{\dot{u}_1 + \dot{u}_2}{|u_1 - u_2|} dv,$$

$$S_b = -\int m_1 \sqrt{\dot{v}_1} + m_2 \sqrt{\dot{v}_2} + e_1 e_2 \frac{\dot{v}_1 + \dot{v}_2}{|v_1 - v_2|} du. \tag{2}$$

The action describes two particles interacting by means of the electromagnetic forces. The equations of motion were solved by Hill and Rudd [1] and the author [2].

$$S_a = -\int m_1 \sqrt{\dot{u}_1 + k(u_1 - u_2)^2} + m_2 \sqrt{\dot{u}_2 + k(u_1 - u_2)^2} dv,$$

$$S_b = -\int m_1 \sqrt{\dot{v}_1 + k(v_1 - v_2)^2} + m_2 \sqrt{\dot{v}_2 + k(v_1 - v_2)^2} du. \tag{3}$$

The action describes two particles bound by an anharmonic force. The equations of motion can be solved by means of elliptic integrals.

In general we have

$$S_a = \int L(\dot{u}_1, \dot{u}_2, u_1-u_2)dv,$$
$$S_b = \int L(\dot{v}_1, \dot{v}_2, v_1-v_2)du, \tag{4}$$

where the Lorentz invariance implies that for every real number φ

$$L(e^{2\varphi}\dot{u}_1, e^{2\varphi}\dot{u}_2, e^{\varphi}u_1-e^{\varphi}u_2) = e^{\varphi}L(\dot{u}_1, \dot{u}_2, u_1-u_2). \tag{5}$$

Conservation laws of energy and momentum in the case a take on the form

$$\dot{u}_1 \frac{\partial L}{\partial \dot{u}_1} + \dot{u}_2 \frac{\partial L}{\partial \dot{u}_2} - L = \frac{1}{2}(E-P), \tag{6}$$

$$\frac{\partial L}{\partial \dot{u}_1} + \frac{\partial L}{\partial \dot{u}_2} = -\frac{1}{2}(E+P), \tag{7}$$

where E and P denote respectively the total energy and momentum. For the case b we have to make in (6) and (7) substitutions $u \rightleftarrows v$, $P \to -P$.

Relationship between the two histories

It might happen that the history a is identical with the history b; for example this is the case for the attractive force ($k < 0$) in the action integral (1). The interaction part of this integral represents an area between two world lines. Since

$$\int |u_1-u_2|dv = \int |v_1-v_2|du, \tag{8}$$

the integrals S_a and S_b are in this case identical. In general we shall call an action integral reducible, if $S_a \equiv S_b$. There are, however, irreducible integrals, for example the electromagnetic action is certainly irreducible.

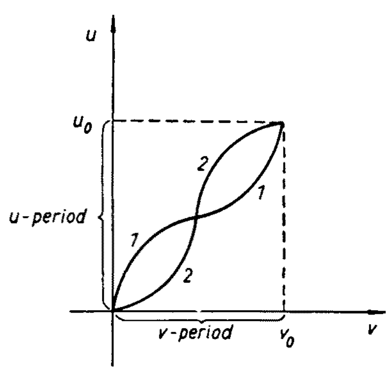

Fig. 1. An oscillatory motion

Suppose that two particles attract each other and as a result perform an oscillatory motion. An increment of the coordinate u in each period will be called u-period; similarly an increment of the coordinate v in each period will be called v-period (Fig. 1). We shall prove the theorem: if the total energy and momentum in the case a are equal to the total energy and momentum in the case b, then u-period in the case a is equal to u-period in the case b and v-period in the case a is equal to v-period in the case b.

Let us multiply the conservation law (7) by $\dot{u}_2$ and substract from (6). We obtain

$$\frac{1}{2}(E-P)+\frac{1}{2}(E+P)\dot{u}_2=(\dot{u}_1-\dot{u}_2)\frac{\partial L}{\partial \dot{u}_1}-L. \tag{9}$$

Let us integrate this equation over the whole v-period $v_0(a)$. Integrating by parts and taking into account that u_1-u_2 vanishes for $v=0$ and $v=v_0(a)$ we obtain

$$\frac{1}{2}(E-P)\,v_0(a)+\frac{1}{2}(E+P)u_0(a)=-\int\limits_0^{v_0(a)}(u_1-u_2)\frac{\partial L}{\partial u_1}+Ldv. \tag{10}$$

In general this is all we can say about periods. However, for Lorentz invariant Lagrangians the right-hand side of (10) can be calculated. In fact, let us differentiate (5) with respect to φ and put $\varphi=0$. We obtain

$$2\dot{u}_1\frac{\partial L}{\partial \dot{u}_1}+2\dot{u}_2\frac{\partial L}{\partial \dot{u}_2}+(u_1-u_2)\frac{\partial L}{\partial u_1}=L. \tag{11}$$

Eliminating velocities by means of the energy integral we find

$$(u_1-u_2)\frac{\partial L}{\partial u_1}+L=-(E-P). \tag{12}$$

Hence, the right-hand side of (10) is equal to $(E-P)v_0(a)$ and consequently

$$(E-P)v_0(a)=(E+P)u_0(a). \tag{13}$$

Now, if for the case a $u_0(a)=f(E,P)$ then for the case b $v_0(b)=f(E,-P)$ because the case b arises from the case a by substitution $u\rightleftarrows v$, $P\to -P$. Hence

$$(E+P)u_0(a)=(E-P)v_0(b),$$

$$(E-P)v_0(a)=(E+P)u_0(b) \tag{14}$$

and consequently from (14) and (13)

$$u_0(a)=u_0(b),\ v_0(a)=v_0(b). \tag{15}$$

Looking through the proof we can see that the theorem holds also for half-periods.

The electromagnetic case

Unfortunately, the theorem does not hold in the case of electromagnetic forces because integrating by parts and using equations of motion we have assumed equations of motion to hold everywhere. This is not true for the electromagnetic forces which are singular for $u_1=u_2$. But even for the electromagnetic forces there exists a simple relationship between periods. Let us denote by v^* the internal point in which $u_1 = u_2$ and integrate again equation (10) over two segments

$$\varepsilon \leqslant v \leqslant v^* - \varepsilon, \quad v^* + \varepsilon \leqslant v \leqslant v_0(a) - \varepsilon, \tag{16}$$

where ε is a small positive number. Assuming that the solutions u_1 and u_2 are continuous we find

$$\frac{1}{2}(E+P)u_0(a) - \frac{1}{2}(E-P)v_0(a) = \lim_{\varepsilon \to 0} \{F(v_0(a)-\varepsilon) - F(v^*+\varepsilon) + F(v^*-\varepsilon) - F(\varepsilon)\}, \tag{17}$$

where

$$F = (u_1 - u_2)\frac{\partial L}{\partial \dot{u}_1}. \tag{18}$$

The limit may be calculated by means of the conservation laws of energy and momentum. In this way we obtain

$$\frac{1}{2}(E+P)u_0(a) - \frac{1}{2}(E-P)v_0(a) = -4e^2. \tag{19}$$

Hence, we have in the electromagnetic case

$$u_0(a) - u_0(b) = -\frac{8e^2}{E+P},$$

$$v_0(a) - v_0(b) = \frac{8e^2}{E-P}. \tag{20}$$

Generalized solutions in the electromagnetic case

We find that the theorem on periods, which is a general integral theorem based on conservation laws, does not hold in the electromagnetic case because equations of motion are not valid for $u_1 = u_2$. Assuming that the solutions are continuous for $u_1 = u_2$ we choose a principle of continuation in the point in which we have none because equations of motion do not hold. It is possible to take another point of view, namely to assume that the conservation laws hold always and to use this assumption as a principle of continuation in the points in which equations of motion break down. Such a procedure is used for example in the theory of shock waves. In our case the theorem on periods can be made valid if solutions are discontinuous; for example in the case a both solutions u_1 and u_2 have to make a jump $[u_1] = [u_2] = 4e^2(E+P)^{-1}$ after each half-period.

The case of repulsive forces

It is very likely that a general relationship between the two histories exists also for repulsive forces, when the distance between particles tends to infinity. For the electromagnetic forces such a relationship can be derived from the exact solutions [2], but we have not found as yet a general theorem similar to the theorem about periods.

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