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QUANTUM MARKOVIAN PROCESS FOR A PARTICLE WITH ADDITIONAL DISCRETE DEGREES OF FREEDOM

By W. Garczyński and B. Jancewicz

Institute for Theoretical Physics, University of Wrocław*

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A quantum Markovian process within a set od states labelled by continuous and discrete parameters is constructed. While the scheme is a general one, special attention is paid to a process in Euclidean three-dimensional space which corresponds to a single nonrelativistic particle with internal degrees of freedom, *e. g.*, spin.

1. Introduction

In this paper we shall expand the theory of a particle with spin developed earlier by one of us [1]. Namely, we shall consider here a theory of quantum stochastic processes [2], [3] in a space of states characterized by one continuous parameter varying over some domain Ω in the space $\mathscr{R}^3$ and one discrete parameter ranging over some subset I of the set Z of natural numbers. Physically speaking, this theory corresponds to a particle with additional degrees of freedom, *e.g.*, spin.

In comparison with paper [1], more attention is devoted to the electromagnetic structure of the Schrödinger equations and to the theory of a particle with arbitrary spin.

2. Basic assumptions and equations

Let $(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda)$ be a probability density amplitude of finding the particle at the time t at the point $x \in \Omega$ and with the additional degree of freedom $\lambda \in I$, when it is known that at the time $s < t$ the particle was in the state specified by $\boldsymbol{y}$ and ϱ. Here, Ω is a Borel subset of the Euclidean space $\mathscr{R}^3$ and I is a subset of the set Z of all natural numbers. Assume that I is finite and consists of $2S+1$ elements:

$$|I| = 2S+1. \tag{2.1}$$

The number S is called the total spin of the particle and the discrete degree of freedom will also be called shortly "spin" for convenience.

* Address: Instytut Fizyki Teoretycznej, Wrocław, Cybulskiego 36, Poland.

We impose the following restrictions on the amplitudes (*cf.* [1]):

(*i*) $$(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda) = (t, \boldsymbol{x}, \lambda; s, \boldsymbol{y}, \varrho)^*$$

(*ii*) $$\lim_{t\to s} (s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda) = \delta_{\varrho\lambda}\delta(\boldsymbol{y}-\boldsymbol{x})$$

(*iii*) $$\sum_{\sigma\in I}\int_\Omega dz(s, \boldsymbol{y}, \varrho; \tau, \boldsymbol{z}, \sigma)(s, \boldsymbol{x}, \lambda; \tau, \boldsymbol{z}, \sigma)^* = \delta_{\varrho\lambda}\delta(\boldsymbol{y}-\boldsymbol{x})$$

(*iv*) $$\sum_{\sigma\in I}\int_\Omega dz(s, \boldsymbol{y}, \varrho; \tau, \boldsymbol{z}, \sigma)(\tau, \boldsymbol{z}, \sigma; t, \boldsymbol{x}, \lambda) = (s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda) \quad \text{for } s \leqslant \tau \leqslant t$$

(*v*) $$\lim_{\boldsymbol{z}'\to \boldsymbol{z}} (s, \boldsymbol{z}', \varrho; t, \boldsymbol{x}, \lambda) = (s, \boldsymbol{z}, \varrho; t, \boldsymbol{x}, \lambda).$$

Furthermore, we assume that the process is a diffusional one, what means that the following limits exist:

A. $$a^k_{\varrho\lambda}(s, \boldsymbol{y}) = \lim_{t\to s}(t-s)^{-1}\int_\Omega d\boldsymbol{x}(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda)(x_k-y_k)$$

B. $$b^{kj}_{\varrho\lambda}(s, \boldsymbol{y}) = \lim_{t\to s}(t-s)^{-1}\int_\Omega d\boldsymbol{x}(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda)(x_k-y_k)(x_j-y_j)$$

C. $$c_{\varrho\lambda}(s, \boldsymbol{y}) = \lim_{t\to s}(t-s)^{-1}\Big[\int_\Omega d\boldsymbol{x}(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda) - \delta_{\varrho\lambda}\Big]$$

D. $$\lim_{t\to s}(t-s)^{-1}\int_\Omega d\boldsymbol{x}(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda)(x_1-y_1)^{n_1}(x_2-y_2)^{n_2}(x_3-y_3)^{n_3} = 0$$

for all positive integers satisfying $n_1+n_2+n_3 \geqslant 3$.

It is now possible to derive, under the suitable differentiability assumptions [1], [5], the following differential equations (see Appendix):

$$-\partial_s[s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda) = \sum_{\sigma\in I} K_{\varrho\sigma}(s, \boldsymbol{y})(s, \boldsymbol{y}, \sigma; t, \boldsymbol{x}, \lambda) \tag{2.2}$$

$$\partial_t(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda) = \sum_{\sigma\in I} L_{\sigma\lambda}(t, \boldsymbol{x})(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \sigma) \tag{2.3}$$

where K and L are the differential operators:

$$K_{\varrho\lambda}(s, \boldsymbol{y}) = \frac{1}{2}\, b^{kj}_{\varrho\lambda}(s, \boldsymbol{y})\partial_k\partial_j + a^k_{\varrho\lambda}(s, \boldsymbol{y})\partial_k + c_{\varrho\lambda}(s, \boldsymbol{y}) \tag{2.4}$$

$$L_{\varrho\lambda}(t, \boldsymbol{x}) = \frac{1}{2}\, \partial_k\partial_j b^{kj}_{\varrho\lambda}(t, \boldsymbol{x}) - \partial_k a^k_{\varrho\lambda}(t, \boldsymbol{x}) + c_{\varrho\lambda}(t, \boldsymbol{x}). \tag{2.5}$$

Summation is, as usual, over repeated indices.

In order to simplify notation we introduce the matrices, *e.g.*,

$$[s, \boldsymbol{y}; t, \boldsymbol{x}] = \|(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda)\|_1^{2S+1}$$

$$K(s, \boldsymbol{y}) = \|K_{\varrho\lambda}(s, \boldsymbol{y})\|_1^{2S+1} = \frac{1}{2}\, b^{kj}(s, \boldsymbol{y})\partial_k\partial_j + a^k(s, \boldsymbol{y})\partial_k + c(s, \boldsymbol{y}) \tag{2.6}$$

$$L(t, \boldsymbol{x}) = \|L_{\varrho\lambda}(t, \boldsymbol{x})\|_1^{2S+1} = \frac{1}{2}\, \partial_k\partial_j b^{kj}(t, \boldsymbol{x}) - \partial_k a^k(t, \boldsymbol{x}) + c(t, \boldsymbol{x}).$$

The basic equations (2.2) and (2.3) take now the form

$$\{\mathbf{1}\partial_s + K(s, \boldsymbol{y})\}\,[s, \boldsymbol{y}; t, \boldsymbol{x}] = 0 \tag{2.7}$$

$$[s, \boldsymbol{y}; t, \boldsymbol{x}]\,\{-\mathbf{1}\partial_t + L(t, \boldsymbol{x})\} = 0. \tag{2.8}$$

It follows from the first two postulates, together with the equations (2.7) and (2.8), that

$$\{K(s, \boldsymbol{y}) + L^+(s, \boldsymbol{y})\}\, f(\boldsymbol{y}) = 0 \tag{2.9}$$

(the sign $^+$ means complex conjugation and transposition of matrix indices) for any twice differentiable function f. Putting for f a constant, a linear function and a quadratic function in $\boldsymbol{y}$ and using (2.6), we get the respective following identities

$$\begin{aligned} c(s, \boldsymbol{y}) + c(s, \boldsymbol{y})^+ &= \partial_k a^k(s, \boldsymbol{y}) - \frac{1}{2}\,\partial_k\partial_j b^{kj}(s, \boldsymbol{y}) \\ a^k(s, \boldsymbol{y}) - a^k(s, \boldsymbol{y})^+ &= \partial_j b^{kj}(s, \boldsymbol{y}) \\ b^{kj}(s, \boldsymbol{y}) + b^{kj}(s, \boldsymbol{y})^+ &= 0. \end{aligned} \tag{2.10}$$

We should stress here that the sign $^+$ concerns only the discrete indices and does not mean the hermitian adjoint of the differential operator.

These identities show that

$$\begin{aligned} b^{kj} &= i\beta^{kj} \\ a^k &= \xi^k + \frac{1}{2}\,\partial_j b^{kj} \\ c &= \frac{1}{2}\,\partial_k \xi^k + i\eta \end{aligned} \tag{2.11}$$

where β^{kj}, ξ^k and η are arbitrary Hermitean matrices.

3. *Wave functions*

Let $\{u_\varrho(\boldsymbol{x}), \boldsymbol{x} \in \Omega, \varrho = 1, 2, \ldots, 2S+1\}$ be a set of square integrable functions. We consider them as the initial wave functions of a particle. The future and past wave functions are then defined as follows (*cf.* [1])

$$\psi_\lambda(t, \boldsymbol{x}; s, u) = \sum_{\varrho \in I} \int_\Omega d\boldsymbol{y}\, u_\varrho(\boldsymbol{y})(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda) \tag{3.1}$$

$$\varphi_\varrho(s, \boldsymbol{y}; t, u^*) = \sum_{\lambda \in I} \int_\Omega d\boldsymbol{x}\,(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda) u_\lambda^*(\boldsymbol{x}). \tag{3.2}$$

Both of these functions are square integrable with the same norms as u. This follows from the unitarity postulate (*iii*). Here $\psi_\lambda(t, \boldsymbol{x}; s, u)$ is the probability density amplitude of finding the particle with the degree of freedom λ at the point $\boldsymbol{x}$ at the time t if it is known that at time $s < t$ the particle was distributed in space according to the wave function $\{u_\varrho(\boldsymbol{y}); \varrho \in I\}$. Accordingly, $\varphi_\varrho(s, \boldsymbol{y}; t, u^*)$ is the probability density amplitude of finding the particle with the degree of freedom ϱ at the point $\boldsymbol{y}$ at the time s if it is known that at the $t > s$ the particle will be distributed in space with the wave function $\{u_\lambda^*(\boldsymbol{x}); \lambda \in I\}$.

Owing to the second postulate (*ii*) we have

$$\psi_\lambda(s, \boldsymbol{x}; s, u) = u_\lambda(\boldsymbol{x}), \quad \varphi_\varrho(s, \boldsymbol{y}; s, u^*) = u_\varrho^*(\boldsymbol{x}). \tag{3.3}$$

From the first postulate (*i*) we get the relation

$$\psi_\lambda^*(t, \boldsymbol{x}; s, u) = \varphi_\lambda(t, \boldsymbol{x}; s, u^*). \tag{3.4}$$

It will be convenient to use matrix notation in what follows. Therefore, we define a row

$$\psi(t, \boldsymbol{x}; s, u) = [\psi_1(t. \boldsymbol{x}; s, u), \ldots, \psi_{2S+1}(t, \boldsymbol{x}; s, u)] \tag{3.5}$$

and a column

$$\varphi(s, \boldsymbol{y}; t, u^*) = \begin{Bmatrix} \varphi_1(s, \boldsymbol{y}; t, u^*) \\ \vdots \\ \varphi_{2S+1}(s, \boldsymbol{y}; t, u^*) \end{Bmatrix}. \tag{3.6}$$

Similarly, let $u(\boldsymbol{y})$ be a row

$$u(\boldsymbol{y}) = [u_1(\boldsymbol{y}), \ldots, u_{2S+1}(\boldsymbol{y})]. \tag{3.7}$$

Then we have with this notation

$$\begin{aligned} \psi(t, \boldsymbol{x}; s, u) &= \int_\Omega d\boldsymbol{y} u(\boldsymbol{y})[s, \boldsymbol{y}; t, \boldsymbol{x}] \\ \varphi(s, \boldsymbol{y}; t, u^*) &= \int_\Omega d\boldsymbol{x}[s, \boldsymbol{y}; t, \boldsymbol{x}]u^+(\boldsymbol{x}) \\ \psi^+(t, \boldsymbol{x}; s, u) &= \varphi(t, \boldsymbol{x}; s, u^*). \end{aligned} \tag{3.8}$$

Using the equations (2.7) and (2.8) together with the last formulae we get the Schrödinger equations

$$\{\partial_s + K(s, \boldsymbol{y})\}\varphi(s, \boldsymbol{y}; t, u^*) = 0 \tag{3.9}$$

$$\psi(t, \boldsymbol{x}; s, u)\{-\partial_t + L(t, \boldsymbol{x})\} = 0. \tag{3.10}$$

The operation ihL plays the role of the Hamiltonian of the system. Note that in standard quantum-mechanical notation these equations should be transposed. The ψ wave function is given by our ψ^T column function.

It is possible to restore the amplitude if a complete family of initial wave functions is known. Namely, if $\{u^{(n)}(\boldsymbol{x}); n = 1, 2, \ldots\}$ is a set of rows such that

$$\sum_{n=1}^{\infty} u_\varrho^{(n)*}(\boldsymbol{y})u_\lambda^{(n)}(\boldsymbol{x}) = \delta_{\varrho\lambda}\delta(\boldsymbol{y}-\boldsymbol{x}) \tag{3.11}$$

then it may be easily verified that

$$[s, \boldsymbol{y}; t, \boldsymbol{x}] = \sum_{n=1}^{\infty} \varphi(s, \boldsymbol{y}; t, u^{(n)*})u^{(n)}(\boldsymbol{x}) \tag{3.12}$$

$$[s, \boldsymbol{y}; t, \boldsymbol{x}] = \sum_{n=1}^{\infty} u^{(n)}(\boldsymbol{y})^+\psi(t, \boldsymbol{x}; s, u^{(n)}). \tag{3.13}$$

These are useful relations since they enable us to find transformation properties of the amplitude when they are known for the wave functions. Moreover, due to the postulate (*iv*) there is satisfied the relation

$$\sum_{n=1}^{\infty} \varphi(s, \boldsymbol{y}; \tau, u^{(n)*})\psi(t, \boldsymbol{x}; \tau, u^{(n)}) = [s, \boldsymbol{y}; t, \boldsymbol{x}]. \tag{3.14}$$

We end this section with a derivation of the continuity equation which follows from the Schrödinger equation. Namely, we have

$$\partial_t \sum_{\lambda\in I} |\psi_\lambda(t, \boldsymbol{x})|^2 = \partial_t(\psi(t, \boldsymbol{x})\psi^+(t, \boldsymbol{x}))$$

by virtue of (3.8), (3.9) and (3.10)

$$= \psi(t, \boldsymbol{x})\overleftarrow{L}(t, \boldsymbol{x})\varphi(t, \boldsymbol{x}) - \psi(t, \boldsymbol{x})\vec{K}(t, \boldsymbol{x})\varphi(t, \boldsymbol{x})$$

due to (2.6)

$$= \frac{1}{2}\{\partial_k\partial_j(\psi b^{kj})\varphi - \psi b^{kj}\partial_k\partial_j\varphi\} - \partial_k(\psi a^k\varphi) = -\partial_k J^k(t, \boldsymbol{x}), \tag{3.15}$$

where the current J^k is

$$J^k(t, \boldsymbol{x}) = \psi(t, \boldsymbol{x})\left\{a^k(t, \boldsymbol{x}) - \frac{1}{2}\partial_j b^{kj}(t, \boldsymbol{x}) + \frac{1}{2} b^{kj}(t, \boldsymbol{x})\vec{\partial}_j - \frac{1}{2}\overleftarrow{\partial}_j b^{kj}(t, \boldsymbol{x})\right\}\psi^+(t, \boldsymbol{x}). \tag{3.16}$$

An analogous equation is valid for φ and may be obtained from the latter by substitution $\psi \to \varphi^+$.

4. *Electromagnetic structure of the Schrödinger equation*

In this section we shall explain a possible dependence of the quantum diffusion coefficients a^k, b^{kj} and c on the potentials $\boldsymbol{A}$ and φ of a possible external electromagnetic field. This investigation is in principle similar to that given for the case of a spinless particle [4], but it contains one specific new feature.

As is well known, the vectors describing an electromagnetic field are

$$\vec{\mathscr{E}} = -\frac{1}{c}\boldsymbol{A} - \nabla\Phi$$

$$\vec{\mathscr{H}} = \nabla\times\boldsymbol{A}$$

where c is the velocity of light. The potentials $\boldsymbol{A}$ and Φ are specified up to the gauge transformation

$$\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \nabla\Lambda$$

$$\Phi \to \Phi' = \Phi - \frac{1}{c}\dot{\Lambda} \tag{4.2}$$

where the gauge function Λ is arbitrary. This leads to the conclusion that both equations

$$-\partial_s\varphi(s, \boldsymbol{y}) = K(s, \boldsymbol{y}, \boldsymbol{A}, \Phi)\varphi(s, \boldsymbol{y}) \tag{4.3}$$

$$-\partial_s\varphi'(s, \boldsymbol{y}) = K(s, \boldsymbol{y}, \boldsymbol{A}', \Phi')\varphi'(s, \boldsymbol{y}) \tag{4.4}$$

should be physically equivalent, *i.e.* their solutions may differ only by a phase factor

$$\varphi'_\varrho(s, \boldsymbol{y}) = \varphi_\varrho(s, \boldsymbol{y})e^{iF_\varrho(s,\boldsymbol{y})}. \tag{4.5}$$

Here F is a real function of s, $\boldsymbol{y}$, ϱ and a functional of Λ. This functional dependence should be homogeneous and linear since at $\Lambda = 0$ both equations coincide and the gauge transformations form a group. Using this connection between φ and φ' we easily obtain from (4.3) and (4.4) the conditions

$$b^{kj}_{\varrho\lambda}(\boldsymbol{A}', \Phi') = b^{kj}_{\varrho\lambda}(\boldsymbol{A}, \Phi) \tag{4.6}$$

$$a^{k}_{\varrho\lambda}(\boldsymbol{A}', \Phi') = a^{k}_{\varrho\lambda}(\boldsymbol{A}, \Phi) - ib^{kj}_{\varrho\lambda}(\boldsymbol{A}, \Phi)\partial_j F_\lambda(\Lambda) \tag{4.7}$$

$$\begin{aligned} c_{\varrho\lambda}(\boldsymbol{A}', \Phi') = {} & c_{\varrho\lambda}(\boldsymbol{A}, \Phi) - i\dot{F}_\lambda(\Lambda)\delta_{\varrho\lambda} - ia^{k}_{\varrho\lambda}(\boldsymbol{A}, \Phi)\partial_k F_\lambda(\Lambda) + \\ & - \frac{i}{2} b^{kj}_{\varrho\lambda}(\boldsymbol{A}, \Phi)\partial_k\partial_j F_\lambda(\Lambda) - \frac{1}{2} b^{kj}_{\varrho\lambda}(\boldsymbol{A}, \Phi)\partial_k F_\lambda(\Lambda)\partial_j F_\lambda(\Lambda). \end{aligned} \tag{4.8}$$

The dependence on s and $\boldsymbol{y}$ is suppressed since we are mainly interested here in the functional dependence on the potentials.

Solving these equations we get that

$$F_\lambda(\Lambda) = \alpha(\lambda)\Lambda \tag{4.9}$$

where $\alpha(\lambda)$ is a real number depending only on spin. The following form of dependence of a^k, b^{kj} and c on the potentials $\boldsymbol{A}$, φ is obtained:

$$b^{kj}_{\varrho\lambda}(\boldsymbol{A}, \Phi) = b^{kj}_{\varrho\lambda}(\vec{\mathscr{E}}, \vec{\mathscr{H}}) \tag{4.10}$$

$$a^{k}_{\varrho\lambda}(\boldsymbol{A}, \Phi) = a^{k}_{\varrho\lambda}(\vec{\mathscr{E}}, \vec{\mathscr{H}}) - ib^{kj}_{\varrho\lambda}(\vec{\mathscr{E}}, \vec{\mathscr{H}})\alpha(\lambda)A_j \tag{4.11}$$

$$\begin{aligned} c_{\varrho\lambda}(\boldsymbol{A}, \Phi) = {} & c_{\varrho\lambda}(\vec{\mathscr{E}}, \vec{\mathscr{H}}) + ic\alpha(\lambda)\,\Phi\delta_{\varrho\lambda} - i\alpha(\lambda)\,a^{k}_{\varrho\lambda}(\vec{\mathscr{E}}, \vec{\mathscr{H}})\,A_k - \\ & - \frac{i}{2}\alpha(\lambda)b^{kj}_{\varrho\lambda}(\vec{\mathscr{E}}, \vec{\mathscr{H}})\partial_k A_j - \frac{\alpha^2(\lambda)}{2} b^{kj}_{\varrho\lambda}(\vec{\mathscr{E}}, \vec{\mathscr{H}})A_k A_j. \end{aligned} \tag{4.12}$$

Here, $a^{k}_{\varrho\lambda}(\vec{\mathscr{E}}, \vec{\mathscr{H}})$, $b^{kj}_{\varrho\lambda}(\vec{\mathscr{E}}, \vec{\mathscr{H}})$, $c_{\varrho\lambda}(\vec{\mathscr{E}}, \vec{\mathscr{H}})$ are almost arbitrary functionals of $\vec{\mathscr{E}}, \vec{\mathscr{H}}$. The only restriction of an algebraic nature follows from the general structure of the coefficients a^k, b^{kj} and c (see (2.11)). Namely, the general formula

$$c(\boldsymbol{A}, \Phi) + c^+(\boldsymbol{A}, \Phi) = \frac{1}{2}\partial_k\{a^k(\boldsymbol{A}, \Phi) + a^{k+}(\boldsymbol{A}, \Phi)\}$$

together with the above expressions for a^k and c yields the restriction

$$c(\vec{\mathscr{E}}, \vec{\mathscr{H}}) + c^+(\vec{\mathscr{E}}, \vec{\mathscr{H}}) = \frac{1}{2}\partial_k\{a^k(\vec{\mathscr{E}}, \vec{\mathscr{H}}) + a^{k+}(\vec{\mathscr{E}}, \vec{\mathscr{H}})\}. \tag{4.13}$$

Thus, we have arrived at the most general Schrödinger equations describing a particle with spin moving in an external electromagnetic field:

$$-[\partial_s+ic\alpha(\varrho)\Phi(s,\boldsymbol{y})]\varphi_\varrho(s,\boldsymbol{y})$$

$$=\sum_{\sigma\in I}\left\{\frac{1}{2}\,b^{kj}_{\varrho\sigma}(s,\boldsymbol{y},\vec{\mathscr{E}},\vec{\mathscr{H}})\,[\partial_k-i\alpha(\sigma)A_k(s,\boldsymbol{y})]\,[\partial_j-i\alpha(\sigma)A_j(s,\boldsymbol{y})]+\right. \tag{4.14}$$

$$\left.+a^{k}_{\varrho\sigma}(s,\boldsymbol{y},\vec{\mathscr{E}},\vec{\mathscr{H}})\,[\partial_k-i\alpha(\sigma)A_k(s,\boldsymbol{y})]+c_{\varrho\sigma}(s,\boldsymbol{y},\vec{\mathscr{E}},\vec{\mathscr{H}})\right\}\varphi_\sigma(s,\boldsymbol{y})$$

$$[\partial_t-ic\alpha(\lambda)\Phi(t,\boldsymbol{x})]\psi_\lambda(t,\boldsymbol{x})$$

$$=\sum_{\sigma\in I}\left\{\frac{1}{2}\,[\partial_k+i\alpha(\sigma)A_k(t,\boldsymbol{x})]\,[\partial_j+i\alpha(\sigma)A_j(t,\boldsymbol{x})]b^{kj}_{\sigma\lambda}(t,\boldsymbol{x},\vec{\mathscr{E}},\vec{\mathscr{H}})+\right. \tag{4.15}$$

$$\left.-[\partial_k+i\alpha(\sigma)A_k(t,\boldsymbol{x})]a^{k}_{\sigma\lambda}(t,\boldsymbol{x},\vec{\mathscr{E}},\vec{\mathscr{H}})+c_{\sigma\lambda}(t,\boldsymbol{x},\vec{\mathscr{E}},\vec{\mathscr{H}})\right\}\psi_\sigma(t,\boldsymbol{x}).$$

The same equations are valid for the transition amplitudes. We shall see later that when the additional degree of freedom becomes an ordinary spin of value 1/2 these equations contain the Pauli equation as a special case.

5. *An ordinary spin as the additional degree of freedom*

We say that the discrete indices $\varrho,\lambda,\sigma\,\ldots$ correspond to different orientations of an ordinary spin of value S (see (2.1)) if the wave functions transform under the $\mathscr{R}^3$ — space rotations in the following manner:

$$\varphi'_\varrho(s,\boldsymbol{y})=\sum_{\sigma=1}^{2S+1}D_{\varrho\sigma}(R)\varphi_\varrho(s,R^{-1}\boldsymbol{y}). \tag{5.1}$$

Here, $D(R)$ is the matrix of the irreducible $(2S+1)$-dimensional unitary representation of the SU_2 group, R is a rotation. In the matrix notation this will be

$$\varphi'(s,\boldsymbol{y})=D(R)\varphi(s,R^{-1}\boldsymbol{y}) \tag{5.2}$$

$$\psi'(t,\boldsymbol{x})=\psi(t,R^{-1}\boldsymbol{x})D^{+}(R). \tag{5.3}$$

Using the formula (3.13) we may get a transformation law for the amplitude,

$$[s,\boldsymbol{y};t,\boldsymbol{x}]'=\sum_n u^{(n)\prime}(\boldsymbol{y})^{+}\psi'(t,\boldsymbol{x};s,u^{(n)})$$

$$=D(R)\sum_n u^{(n)}(R^{-1}\boldsymbol{y})\psi(t,R^{-1}\boldsymbol{x};s,u^{(n)})D^{+}(R)$$

$$=D(R)[s,R^{-1}\boldsymbol{y};t,R^{-1}\boldsymbol{x}]D^{+}(R). \tag{5.4}$$

The Schrödinger equations for the amplitude should be covariant under rotations, *i.e.*,

$$-\partial_s[s,\boldsymbol{y};t,\boldsymbol{x}]'=K'(s,\boldsymbol{y})[s,\boldsymbol{y};t,\boldsymbol{x}]' \tag{5.5}$$

$$\partial_t[s,\boldsymbol{y};t,\boldsymbol{x}]'=[s,\boldsymbol{y};t,\boldsymbol{x}]'L'(t,\boldsymbol{x}) \tag{5.6}$$

where $K'(s, \boldsymbol{y})$ and $L'(t, \boldsymbol{x})$ are differential operators of the same type as before with transformed coefficients. Using (5.4) we easily conclude from (3.9) and (3.10) that

$$K'(s, \boldsymbol{y}) = D(R)K(s, R^{-1}\boldsymbol{y})D^{+}(R) \tag{5.7}$$

$$L'(t, \boldsymbol{x}) = D(R)L(t, R^{-1}\boldsymbol{x})D^{+}(R) \tag{5.8}$$

where

$$D(R)K(s, R^{-1}\boldsymbol{y})D^{+}(R) = \frac{1}{2}\, D(R)b^{kj}(s, R^{-1}\boldsymbol{y})D^{+}(R)\, \frac{\partial^2}{\partial(R^{-1}\boldsymbol{y})_k \partial(R^{-1}\boldsymbol{y})_j}$$

$$+ D(R)a^k(s, R^{-1}\boldsymbol{y})D^{+}(R)\, \frac{\partial}{\partial(R^{-1}\boldsymbol{y})_k} + D(R)c(s, R^{-1}\boldsymbol{y})D^{+}(R). \tag{5.9}$$

We see here that the following transformation properties will be appropriate for the covariance of the Schrödinger equations:

$$R_{mk}R_{nj}D(R)b^{kj}(s, R^{-1}\boldsymbol{y})D^{+}(R) = b^{mn}(s, \boldsymbol{y})' \tag{5.10}$$

$$R_{lk}D(R)a^{k}(s, R^{-1}\boldsymbol{y})D^{+}(R) = a^{l}(s, \boldsymbol{y})' \tag{5.11}$$

$$D(R)c(s, R^{-1}\boldsymbol{y})D^{+}(R) = c(s, \boldsymbol{y})'. \tag{5.12}$$

We made use here of the obvious fact that $\partial/\partial(R^{-1}\boldsymbol{y})_k = R_{jk}\partial/\partial y_j$. Thus, b^{kj}, a^k and c should have tensorial transformation properties of second, first and zero rank, respectively.

In order to see any possible internal structure of these tensors, let us discuss the equation (5.12). We can rewrite it in the following way:

$$c'_{\lambda\varrho} = (DcD^{+})_{\lambda\varrho} = D_{\lambda\alpha}c_{\alpha\beta}D^{+}_{\beta\varrho} = D_{\lambda\alpha}D^{*}_{\varrho\beta}c_{\alpha\beta}.$$

Hence, the transformation $c \to c'$ is a tensor product of transformations D and D^*. As is known, the complex conjugated representation D^* of the SU_2 group is unitarily equivalent to the representation D (*e.g.*, see [6]). So the above transformation gives a representation of the SU_2 group which is equivalent to the tensor product $D \otimes D$ of representations of $2S+1$ dimensions each. It then splits into a direct sum of representations labeled by spin L ranging from 0 up to $2S$:

$$D^S \otimes D^S = D^0 \oplus D^1 \oplus \ldots \oplus D^{2S}. \tag{5.13}$$

It can be understood that the matrix c is a linear combination of $(2S+1)^2$ matrices collected into sets which are composed, respectively, of one matrix ($L = 0$), three matrices ($L = 1$), five matrices ($L = 2$), and so on, and each set forms an irreducible representation of the SU_2 group.

The same argument goes for each of the matrices a^k and b^{kj}. Let us consider the simplest example, *i.e.*, the case of spin 1/2.

6. *Particle with spin 1/2*

In this case all of the coefficients b^{kj}, a^k and c are matrices of size 2×2. It is well known that one may represent them uniquely as linear combinations of three Pauli matrices ($L = 1$) supplemented by the unit matrix ($L = 0$):

$$\begin{aligned} c &= c_0\mathbf{1}+c_l\sigma^l \\ a^k &= a_0^k\mathbf{1}+a_l^k\sigma^l \\ b^{kj} &= b_0^{kj}\mathbf{1}+b_l^{kj}\sigma^l \end{aligned} \tag{6.1}$$

where the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

have the following transformation properties

$$D(R)\sigma^l D^+(R) = R_{kl}\sigma^k. \tag{6.2}$$

We put this into the equation (5.12):

$$\begin{aligned} D(R)c(s, R^{-1}\boldsymbol{y})D^+(R) &= c_0(s, R^{-1}\boldsymbol{y})D(R)\mathbf{1}D^+(R)+c_l(s, R^{-1}\boldsymbol{y})D(R)\sigma^l D^+(R) \\ &= c_0(s, R^{-1}\boldsymbol{y})\mathbf{1}+R_{kl}c_l(s, R^{-1}\boldsymbol{y})\sigma^k \\ &= c_0(s, \boldsymbol{y})'\mathbf{1}+c_k(s, \boldsymbol{y})'\sigma^k = c(s, \boldsymbol{y})'. \end{aligned}$$

We see thus that the required covariance is fulfilled if c_0 is a scalar function and the three functions c_l form a vector.

Similarly, equation (5.11) gives

$$\begin{aligned} R_{lk}D(R)a^k(s, R^{-1}\boldsymbol{y})D^+(R) &= R_{lk}a_0^k(s, R^{-1}\boldsymbol{y})\mathbf{1}+R_{lk}R_{mn}a_n^k(s, R^{-1}\boldsymbol{y})\sigma^m \\ &= a_0^l(s, \boldsymbol{y})'\mathbf{1}+a_m^l(s, \boldsymbol{y})'\sigma^m = a^l(s, \boldsymbol{y})'. \end{aligned}$$

Now we see that the functions a_0^k should form a vector and a_n^k a tensor of second rank. In the same manner we conclude from Eq. (5.10) that b_0^{kj} should be a tensor of second rank and b_l^{kj} a tensor of third rank.

We ought to notice that the transformation properties of coefficients a^k given by (5.11) exclude any possible dependence of the parameter α on spin index. Indeed, from the formula (4.11) one sees that $a^k(\boldsymbol{A}, \varphi)$ transforms properly only if

$$D(R)\alpha D^+(R) = \alpha \quad \text{for each } R \tag{6.3}$$

where α is a diagonal matrix with nonzero elements $\alpha(\varrho)$, $\varrho = 1, 2, \ldots, 2S+1$. Equation (6.3) says that α commutes with all with $D(R)$ which form an irreducible representation of the SU_2 group, hence α is proportional to the unit matrix and dependence of charge on ordinary spin is excluded. We say charge because the coefficient α is usually related to electric charge:

$$-\alpha\hbar c = q. \tag{6.4}$$

As we have said before, the equations (4.14) and (4.15) contain the Pauli equation as a special case. In order to derive it we assume that

$$b^{kj}_{\varrho\lambda}(s, \boldsymbol{y}) = i\,\frac{\hbar}{m}\,\delta_{jk}\delta_{\varrho\lambda} \tag{6.5}$$

$$a^{k}_{\varrho\lambda}(s, \boldsymbol{y}, \boldsymbol{A}, \Phi) = -\frac{q}{mc}\,A_k(s, \boldsymbol{y})\,\delta_{\varrho\lambda} \tag{6.6}$$

$$c_{\varrho\lambda}(s, \boldsymbol{y}, \boldsymbol{A}, \Phi) = c_{\varrho\lambda}(s, \boldsymbol{y}, \vec{\mathscr{E}}, \vec{\mathscr{H}}) - i\,\frac{q}{\hbar}\,\Phi(s, \boldsymbol{y})\,\delta_{\varrho\lambda} - \frac{q}{2\,mc}\,\operatorname{div}\boldsymbol{A}\,\delta_{\varrho\lambda} - \frac{iq^2}{2\hbar mc}\,\boldsymbol{A}^2\delta_{\varrho\lambda} \tag{6.7}$$

and

$$c^{T}(s, \boldsymbol{y}, \vec{\mathscr{E}}, \vec{\mathscr{H}}) = -\frac{i}{\hbar}\,V(s, \boldsymbol{y})\,\mathbf{1} - \frac{i}{\hbar}\,\beta(\vec{\mathscr{H}}\cdot\boldsymbol{\sigma}). \tag{6.8}$$

The phenomenological coefficient β has the value $\frac{q\hbar}{2\,mc}$, as may be derived from the relativistic theory of a particle with spin [7]. Putting the above expressions into (4.14) and (4.15) we end up with the following equations:

$$i\hbar\left(\partial_s - \frac{iq}{\hbar}\,\Phi\right)\varphi_\varrho(s, \boldsymbol{y}) = -\frac{\hbar^2}{2\,m}\left(\partial_k + \frac{iq}{\hbar c}\,A_k\right)\left(\partial_k + \frac{iq}{\hbar c}\,A_k\right)\varphi_\varrho(s, \boldsymbol{y}) +$$

$$+ V(s, \boldsymbol{y})\varphi_\varrho(s, \boldsymbol{y}) + \frac{q\hbar}{2\,mc}\sum_{\lambda=1}^{2}\mathscr{H}_k\sigma^{k}_{\lambda\varrho}\varphi_\lambda(s, \boldsymbol{y}), \qquad \varrho = 1, 2 \tag{6.9}$$

$$i\hbar\left(\partial_t + \frac{iq}{\hbar}\,\Phi\right)\psi_\lambda(t, \boldsymbol{x}) = -\frac{\hbar^2}{2\,m}\left(\partial_k - \frac{iq}{\hbar c}\,A_k\right)\left(\partial_k - \frac{iq}{\hbar c}\,A_k\right)\psi_\lambda(t, \boldsymbol{x}) +$$

$$+ V(t, \boldsymbol{x})\psi_\lambda(t, \boldsymbol{x}) + \frac{q\hbar}{2\,mc}\sum_{\varrho=1}^{2}\mathscr{H}_k\sigma^{k}_{\lambda\varrho}\psi_\varrho(t, \boldsymbol{x}), \qquad \lambda = 1, 2. \tag{6.10}$$

These equations coincide with the usual Pauli equations for a charged particle with spin 1/2. One should note that normally the ψ-function is to be ordered into a column, while its adjoint, φ, into a row.

We hope we have convinced the Reader that the theory of quantum stochastic processes is really adequate for the requirements of quantum mechanics. Clearly, a theory similar in principle may be set up along the above lines for a more complicated case of physical interest. Our quantum causality postulate (*iv*) expressed by the Smoluchowski-Chapman-Kolmogorov causality equation [8], [9] supplemented by other suitable assumptions is a basic postulate in our approach to the quantum thêory.

APPENDIX

The equation (2.2) may be derived in the following way. First we consider the amplitude

$$(s-\Delta s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda) = \sum_{\sigma\in I} \int_{\Omega} d\boldsymbol{z}(s-\Delta s, \boldsymbol{y}, \varrho; s, \boldsymbol{z}, \sigma)(s, \boldsymbol{z}, \sigma; t, \boldsymbol{x}, \lambda). \tag{A.1}$$

We expand the second factor in the integrand into the Taylor series around the point $\boldsymbol{z}=\boldsymbol{y}$

$$(s, \boldsymbol{z}, \sigma; t, \boldsymbol{x}, \lambda) = \left[1+(z_k-y_k)\partial/\partial y_k + \frac{1}{2}(z_k-y_k)(z_j-y_j)\frac{\partial^2}{\partial y_k \partial y_j} + \ldots\right](s, \boldsymbol{y}, \sigma; t, \boldsymbol{x}, \lambda) \tag{A.2}$$

and calculate the limit

$$\lim_{\Delta s\to 0} \frac{1}{\Delta s}[(s-\Delta s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda)-(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \delta)] = -\partial_s(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda).$$

Using the notation $A \div D$ of the second section we get from (A.1) and (A.2) the differential equations (2.2):

$$-\partial_s(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda) = \sum_{\sigma\in I}\left[\frac{1}{2} b^{kj}_{\varrho\sigma}(s, \boldsymbol{y})\,\partial_k\partial_j + a^{k}_{\varrho\sigma}(s, \boldsymbol{y})\,\partial_k + c_{\varrho\sigma}(s, \boldsymbol{y})\right](s, \boldsymbol{y}, \sigma; t, \boldsymbol{x}, \lambda).$$

A slightly different method has to be applied when deriving equations (2.3). Namely, we consider the quantity

$$\int_{\Omega} d\boldsymbol{x} f(\boldsymbol{x})\partial_t(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{\Omega} d\boldsymbol{x} f(\boldsymbol{x})\,[(s, \boldsymbol{y}\ \ \varrho; t+\Delta t, \boldsymbol{x}, \lambda)-(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda)] \tag{A.3}$$

where $f(\boldsymbol{x})$ is an auxiliary test function which we shall choose accordingly below. Using the quantum causality postulate (*iv*) and exchanging the variables of integration, $\boldsymbol{z}$ and $\boldsymbol{x}$, we obtain

$$\int_{\Omega} d\boldsymbol{x} f(\boldsymbol{x})(s, \boldsymbol{y}, \varrho; t+\Delta t, \boldsymbol{x}, \lambda)$$

$$= \sum_{\sigma\in I}\int_{\Omega} d\boldsymbol{x}(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \sigma)\int_{\Omega} d\boldsymbol{z}(t, \boldsymbol{x}, \sigma; t+\Delta t, \boldsymbol{z}, \lambda) f(\boldsymbol{z}).$$

Now we expand $f(\boldsymbol{z})$ around $\boldsymbol{z}=\boldsymbol{x}$

$$f(\boldsymbol{z}) = \left[1+(z_k-x_k)\partial_k + \frac{1}{2}(z_k-x_k)(z_j-x_j)\partial_k\partial_j + \ldots\right] f(\boldsymbol{x})$$

and calculate the limit (A.3), using again the notation $A\div D$ of section 2. We easily find that

$$\int_{\Omega} d\boldsymbol{x}\, f(\boldsymbol{x})\partial_t(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda)$$

$$= \sum_{\sigma\in I}\int_{\Omega} d\boldsymbol{x}(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \sigma)\left[c_{\sigma\lambda}(t, \boldsymbol{x}) + a^{k}_{\sigma\lambda}(t, \boldsymbol{x})\partial_k + \frac{1}{2} b^{kj}_{\sigma\lambda}(t, \boldsymbol{x})\partial_k\partial_j\right] f(\boldsymbol{x}).$$

Performing suitable integration by parts and assuming that $f(\boldsymbol{x})$ and $\partial_j f(\boldsymbol{x})$ vanish at the boundary of Ω we get

$$\int_\Omega d\boldsymbol{x}\, f(\boldsymbol{x}) \partial_t (s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda)$$

$$= \sum_{\sigma \in I} \int_\Omega d\boldsymbol{x} f(\boldsymbol{x}) \left[\frac{1}{2} \partial_k \partial_j b^{kj}_{\sigma\lambda}(t, \boldsymbol{x}) - \partial_k a^k_{\sigma\lambda}(t, \boldsymbol{x}) + c_{\sigma\lambda}(t, \boldsymbol{x})\right] (s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \sigma).$$

Since the function f is otherwise arbitrary, we obtain the differential equations (2.3) valid throughout Ω:

$$\partial_t(s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \lambda) = \sum_{\sigma \in I} \left[\frac{1}{2} \partial_k \partial_j b^{kj}_{\sigma\lambda}(t, \boldsymbol{x}) - \partial_k a^k_{\sigma\lambda}(t, \boldsymbol{x}) + c_{\sigma\lambda}(t, \boldsymbol{x})\right] (s, \boldsymbol{y}, \varrho; t, \boldsymbol{x}, \sigma).$$

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