EXAMPLES OF QUANTUM FIELD THEORIES WITHOUT DIVERGENCES

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Examples of non-polynomial Lagrangians are investigated. Particularly interesting is a Lagrangian similar to that in the non-linear electrodynamics of Born and Infeld but for the simple case of a scalar field. Quantisation is feasible but the principle of superposition is invalidated. Nevertheless, a probabilistic interpretation is still possible.

1. *Introduction*

The opinions about the origin of the convergence difficulties in quantum field theories are widely different. Some experts believe that the difficulties are of a mathematical origin and are connected with the procedure of quantisation, with the singular commutation relations, with the appearance of inequivalent representations and other ambiguities. They hope that a mathematical precisation of the objects of quantum field theory and of the prescriptions how to compute the quantities directly comparable with experiment will be necessary and sufficient to obtain a satisfactory theory. Some others represent an opposite view that no mathematical precisation can help to get rid of the fundamental convergence difficulties but a new physical idea is needed.

We are of the opinion that both are partly right: the mere precisation of the objects and rules of the game cannot remove the main sources of the difficulties in the conventional theory which are rather of a physical origin but, of course, a mathematical rigour will be also quite necessary.

The physical side of the problem concerns, above all, a proper choice of the model of the physical system. In particular, one has to decide whether the physical system is to be described in terms of a Lagrangian and, if so, what kind of a Lagrangian. In order to answer this question we have to advocate the History of Physics. The transition from the Newtonian to the Lagrangian formalism as well as from the Lagrangian to the Hamiltonian formalism constituted an admirable progress of theoretical physics. These formalisms could be taken over from the case of mechanics to the case of field theories and survived

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the revolutionary changes brought about by Relativity and Quantum Theory. Thus, there are good reasons to regard the Lagrangian and Hamiltonian formalisms as a durable gain and achievement of human thought. Therefore we would not like to abandon them unless it could be proved rigorously that no Lagrangian and Hamiltonian theory is able to account satisfactorily for the physical reality. Thus, in our opinion it is not advisable to abandon or modify any of the first principles constituting the foundations of the traditional field theory, *e. g.* to abandon the Lagrangian formulation altogether and assume the point of view of an autonomous S-matrix theory.

We assume that the field equations are to be derived from a variational principle with a local Lagrangian density $\mathscr{L}(x)$ dependent upon the arguments x *via* the field quantities and their first order derivatives taken at one and the same point of space-time. Mereover, the Lagrangian has to be of a form allowing for a canonical formulation. In situations where gravitation can be neglected the Lagrangian has to satisfy the requirements of Special Relativity, *i. e.* has to be a scalar with respect to the Poincaré group.

On the other hand, in order to avoid the peculiarities of the traditional theories, we have to look for some non-conventional forms of the Lagrangians. The Lagrangians considered hitherto were always polynomials in field quantities φ and their first order derivatives $\varphi_{,\mu}$ whereby the degree of the polynomials in the derivatives $\varphi_{,\mu}$ was not higher than two. However, the Lagrangian does not need to be as simple as that but may be a more complicated function, providing strongly non-linear field equations. In the traditional field theories the range of variability of the field quantities and their derivatives was assumed to be unlimited. The Lagrangians were real functions defined in the whole domain of their arguments φ, $\varphi_{,\mu}$. These requirements are sufficient but not necessary conditions for a canonical formulation and quantization.

In order to select a particular class of Lagrangians out of the infinity of various possibilities offered by arbitrarily complicated functions of φ and $\varphi_{,\mu}$ a new physical principle or, at least, a hint is necessary. A hint is provided by the following remark: at the beginning of this Century we witnessed twice revolutionary changes of utmost importance: the transition to relativistic and to quantum theories. Though different their basic assumptions are, they exhibit one feature in common: both incorporate a fundamental, dimensional constant c or h into the framework of the theory whereby the role of these constants was manifestly restrictive ($v \leqslant c$ or $\Delta q \Delta p \geqslant h$). Obviously, there is still place for an introduction of a third fundamental constant dimensionally independent of c and h whose role should be also manifestly restrictive.

The transition to non-linear Lagrangians which are no more polynomials offers a possibility of introducing such a constant restricting the range of variability of φ, or $\varphi^2_{,\mu}$, or of the whole Lagrangian.

2. *Precisation of the field concept*

For both physical as well as mathematical reasons it is not enough to assume the field φ to satisfy a certain hyperbolic field equation. One has to assume reasonable boundary conditions and to precise the mere concept of the field. Therefore we assume the following axiom: A solution of the field equation represents a physical field[1] if and only if it is of the following form

$$\varphi(\vec{x}, t) = \sum_{r=0}^{\infty} q_r(t)\varphi_r(\vec{x}) \tag{1}$$

where $\varphi_r(\vec{x})$ denotes a complete orthonormal set of functions with the following property: the integrals

$$I_{rs} = \int d^3x(\varphi_r(\vec{x})\vec{x}^s)^2, \qquad r, s = 0, 1, 2, \ldots \tag{1'}$$

exist for any finite r and s. This condition is satisfied *e. g.* by the eigenfunctions of the harmonic oscillator[2]. The coefficients q_r constitute a set of generalized coordinates describing the physical system called "the field". Field quantisation consists in quantising the system described by the enumerable set of coordinates q_r and their canonically conjugated momenta p_r, or equivalently by the complex coordinates a_r and a_r^+ where $a_r = 2^{-\frac{1}{2}}(q_r + ip_r)$.

The above axiom removes the difficulty connected with the appearance of inequivalent representations: these which are inequivalent to those based on (1) are not representations at all.

The quantum field theory is to be considered as a limit of the quantum theory of systems with a finite number of degrees of freedom

$$\varphi(\vec{x}, t) = \lim_{R\to\infty} \varphi_R(\vec{x}, t) \tag{2}$$

where

$$\varphi_R(\vec{x}, t) = \sum_{r=0}^{R} q_r(t)\varphi_r(\vec{x}). \tag{2'}$$

Inasmuch as φ_R decreases sufficiently rapidly for $|\vec{x}| \to \infty$ it is no more necessary to introduce two cutoff's (in the x-space and in the k-space separately) but only one cutoff $r \leqslant R$.

The limit transition $R \to \infty$ brings about with it singular commutation relations

$$[\pi(\vec{x}', t), \quad \varphi(\vec{x}'', t)] = -i\delta^{(3)}(\vec{x}' - \vec{x}'') \tag{3}$$

wherefrom it is seen that the operators φ and π are highly singular. Therefore the limit transition $R \to \infty$ is a very delicate problem and, most probably, is not feasible in the case of conventional interactions (polynomials in field quantities). In order to secure the existence of the limit transition we have to consider some examples of rather non-conventional Lagrangians.

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[1] For simplicity we consider a single, real scalar field.

[2] In this case r is an abbreviation for the set of three indices $n_1n_2n_3$ or nlm where l, m are the indices of spherical harmonics.

3. *An example of a bounded interaction*

Let us consider a Lagrangian

$$\tilde{\mathscr{L}} = \mathscr{L}_0 + \mathscr{L}' \tag{4}$$

where $\mathscr{L}_0$ denotes the usual Lagrangian for a free field whereas the interaction part of the Lagrangian is

$$\mathscr{L}' = gl^{-4}(1 - l^4\varphi^4)^{5/2} - gl^{-4}. \tag{5}$$

The domain of variability of φ is restricted by the form of the Lagrangian

$$l^2\varphi^2 - 1 \leqslant 0. \tag{6}$$

The Lagrangian $\mathscr{L}'$ as well as its derivatives $\frac{\partial \mathscr{L}'}{\partial \varphi}$ and $\frac{\partial^2 \mathscr{L}'}{\partial \varphi^2}$ are bounded in this domain and vanish for $l^2\varphi^2 \to 1$. This Lagrangian may be developed into a power series and yields terms proportional to φ^4, φ^8, ... describing an interaction of the field with itself by means of two-body, four-body, ... forces. In both limits $l^2\varphi^2 \to 0$ and $l^2\varphi^2 \to 1$ the field becomes a free field.

The fact that the derivatives $\frac{\partial \mathscr{L}'}{\partial \varphi}$ and $\frac{\partial^2 \mathscr{L}'}{\partial \varphi^2}$ are bounded ensures the existence of unique solutions which may be computed for any finite time interval by means of iteration starting with a free field satisfying initial conditions consistent with (6) but otherwise arbitrary. Thus, the classical field of this type is very regular.

Going over to quantisation we face the problem of attaching a well defined meaning to the interaction term (5). In order to precise this complicated operator we apply Wick's ordering prescription and secure the hermitian character of the interaction energy density by the following substitution

$$\mathscr{H}' \to \frac{1}{2} : (\mathscr{H}' + \mathscr{H}'^{+}):. \tag{7}$$

4. *An example of a temperate action*

Let us consider a field theory with the Lagrangian

$$\tilde{\mathscr{L}} = \frac{1}{a}\left(1 - (1 - 2a\mathscr{L})^{1/2}\right), \tag{8}$$

where a is a constant with dimension cm^4 and $\mathscr{L}$ is any conventional Lagrangian, for example

$$\mathscr{L} = -\frac{1}{2}\left(\varphi_{,\mu}^2 + m^2\varphi^2 + g\varphi^n\right). \tag{8'}$$

The Lagrangian $\tilde{\mathscr{L}}$ possesses an upper bond $1/a$. In the limit $a \to 0$ it goes over into the conventional Lagrangian (8′).

The relation

$$\pi = \frac{\partial \tilde{\mathscr{L}}}{\partial \dot{\varphi}} = \frac{\dot{\varphi}}{(1-2a\mathscr{L})^{1/2}} \tag{9}$$

is soluble uniquely with respect to $\dot{\varphi}$

$$\dot{\varphi} = \pi \left\{ \frac{1+a(\varphi_{,k}^2+m^2\varphi^2+g\varphi^n)}{1+a\pi^2} \right\}^{1/2} \tag{10}$$

whence this theory is canonical. The Hamiltonian density is

$$\mathscr{H} = \frac{1}{a} \{(1+a\pi^2)\,(1+a(\varphi_{,k}^2+m^2\varphi^2+g\varphi^n)\}^{1/2} - \frac{1}{a}\,. \tag{11}$$

It is positive definitive (at least for $g \geqslant 0$ and even n). The restriction

$$2a\mathscr{L} \leqslant 1 \tag{12}$$

is automatically satisfied if $\dot{\varphi}$ is replaced by the function (10).

Going over to quantum theory the expression appearing under the square root (11) has to be symmetrized. Assuming Wick's ordering prescription to the Hamiltonian (11) the positive definite character of the expression under the square root is lost and we have to secure the hermitian character of the energy density by a redefinition of the Hamiltonian

$$\mathscr{H} \to \frac{1}{2}\,(:\mathscr{H}:+:\mathscr{H}^{+}:). \tag{13}$$

This secures the hermitian character of the density but does not ensure its positive definitness. In order to guarantee a positive definite character of energy we have to impose a restriction upon the physically acceptable state functionals $|\Phi>$

$$\langle\Phi|:H:|\Phi\rangle \geqslant 0. \tag{14}$$

This restriction may be secured as an initial condition because H is a constant of the motion. Thus, it does not constitute an encroachment upon the dynamics but means a new situation in the theory of measurements.

The main novelty of such a theory consists in the fact that a superposition of physical states is not necessarily a physical state again. Fortunately, the principle of superposition is only a sufficient but not a necessary condition for a probabilistic interpretation. We may divide the eigenstates of an arbitrary observable A into physical $|A_n^{\mathrm{ph}}\rangle$ and unphysical $|A_k^{\mathrm{un}}\rangle$ and postulate that the probability of obtaining (in a measurement) an unphysical state is *a priori* zero whereas the coefficients

$$c_n^{\mathrm{ph}} = \langle A_n^{\mathrm{ph}}|\Phi\rangle \tag{15}$$

mean relative probability amplitudes for obtaining a physical state $|A_n^{\mathrm{ph}}\rangle$ in a measurement of A if the state just before the measurement was $|\Phi\rangle$. Thus, we have to normalize the coefficients (15) so as to have the sum over all physical states equal unity

$$\Sigma|c_n^{\mathrm{ph}}|^2 = 1. \tag{16}$$