BINDING ENERGY OF ^{3}H AND n-d DOUBLET SCATTERING LENGTH WITH RANK FOUR SEPARABLE POTENTIAL

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Separable charge independent nucleon-nucleon potentials containing an attraction and a hard shell repulsion in the $^1S_0$ state and the Yamaguchi tensor interaction in the $^3S_1+^3D_1$ state are used to calculate the ^{3}H ground state energy, E_T, and the n-d doublet scattering length, ^2a. For the potential which gives the best fit to the nucleon-nucleon data we get: $E_T = -8.81$ MeV, $^2a = 0.58$ fm. All our results fall on the Phillips line. The paper contains also a derivation of the system of coupled integral equations for the bound state and for the scattering problem for an arbitrary rank potential acting in the $^1S_0$ and $^3S_1+^3D_1$ states.

1. Introduction

The nuclear three-body problem[1] has attracted the attention of several physicists virtually since the beginning of nuclear physics. By investigating the properties of the three-nucleon system, in principle, we should be able to gain valuable information on the basic nucleon-nucleon interaction. The progress in gaining in this way reliable information on nuclear forces has been very slow because of the essential difficulties in solving the three-body problem. Thus for a long time the most detailed calculations — by Blatt and his collaborators (see, *e.g.*, [5]) — have been based on the variational principle. Since the work of Faddeev [6] an increasing number of calculations have been devoted to finding exact solutions of the three-body equations. In most of these calculations the partial wave projection of the two-body potential has been assumed to have the form:

$$V_l(r, r') = \sum_n \lambda_n v_n(r) v_n(r'), \tag{1.1}$$

known as the separable approximation. Alternatively, one may make the separable approximation on the level of the two-body t matrix. The two approximations are completely

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[1] For a recent review of the three-nucleon problem see Refs [1]—[4].

equivalent. This approach of solving exactly the three-body problem with a separable two-body potential was pioneered by Mitra [7], Kharchenko [8] and Sitenko and Kharchenko [9].

In the present paper we calculate the triton ground state energy, E_T, and the $n-d$ doublet scattering length, ${}^2a$, with the help of a charge independent separable two-body potential acting in the ${}^1S_0$ and ${}^3S_1+{}^3D_1$ states. In the ${}^1S_0$ state, in which the short range repulsion is best established, our potential contains both attraction and a hard shell repulsion. In the ${}^3S_1+{}^3D_1$ state, we apply the pure attractive Yamaguchi [10] potential which has a central and a tensor part. Thus our potential of rank four (*i.e.*, it contains four components) has probably all the essential features of the two-body interaction which are relevant in the low energy three-nucleon problem. (The interaction in other states seems to be much less important [11].)

The effect of the charge dependence of nuclear forces on E_T has been investigated in [12], where a reduction in the triton binding by 0.7–0.9 MeV due to the charge dependence was found. Although it is a sizeable effect, a big part of it certainly reflects the sensitivity of E_T to the value of the singlet scattering length which is known less precisely than the values of the other low energy nucleon-nucleon parameters. Let us mention also that the magnitude of the charge dependence of nuclear forces is not unambiguously determined [13]. Obviously, taking into account the charge dependence would increase essentially the size of the computations. Having all this in mind, in the present paper we assume that the two-body potential is charge independent. Consequently, the isotopic spin, $T = 1/2$, is a good quantum number of the triton[2], and we find it convenient to apply the isotopic spin formalism.

In the case of a separable interaction one may reduce the original three-body Schroedinger equation to a system of coupled one-dimensional integral equations. In the case of a rank three potential with a tensor force of the Yamaguchi type the derivation of the corresponding system of integral equations has been given by Bhakar [14] (for the bound state problem), by Bhasin [15] (for the scattering problem), by Mitra, Shrenk and Bhasin [16] and by Sitenko and Kharchenko [17] (for both the bound state and scattering problems), and by Fuda [18] (for the bound state problem). The generalization to the case of higher rank potentials for the bound state problem has been given by Stagat [19]. A simple and straightforward method of deriving the system of integral equations in the case of a central interaction for the bound state problem has been given in Ref. [12]. A generalization of this simple method to the case of an arbitrary rank potential with tensor forces acting in the ${}^1S_0$ and ${}^3S_1+{}^3D_1$ states is outlined in the present paper for both the bound state and the zero energy doublet scattering problem.

The two-body potential used in the present calculation is presented in Section 2. In Section 3 the kinematics of the three-nucleon system is described. In Section 4 we derive from the Schroedinger equation the system of coupled integral equations for both the triton and the zero energy $n-d$ doublet scattering in the case of an arbitrary rank potential of the type described in Section 2. The numerical method of solving our equations is described in Section 5. The results obtained are presented and discussed in Section 6. In the Appendix

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[2] The percentage of the $T = 3/2$ state in triton in the case of the charge dependent forces considered in [12] was found to be $1-2\times10^{-4}\%$.

we have collected the formulae used in the derivation of the system of equations given in Section 4.

The results of the present paper concerning the triton binding energy have been presented in [20] and those concerning the $n-d$ scattering length have been reported in [21].

2. *The nucleon-nucleon interaction*

The two-body nucleon-nucleon interaction is assumed to be charge independent, of a nonlocal separable type. In the relative momentum representation it has the form:

$$\langle \boldsymbol{k}'|\hat{V}|\boldsymbol{k}\rangle = \sum_{\nu=1}^{N_s} \lambda_\nu^s u_\nu(k')u_\nu(k)\hat{A}^s + \sum_{\nu=1}^{N_t} \lambda_\nu^t \hat{v}_\nu(\boldsymbol{k}')\hat{v}_\nu(\boldsymbol{k})\hat{A}^t, \tag{2.1}$$

where $\hat{A}^s(\hat{A}^t)$ is a projection operator onto the spin singlet and isospin triplet (spin triplet and isospin singlet) state of the two nucleons. Here and throughout this paper "hats" on a quantity denote operators in the spin and isospin space ($\hat{A}$, $\hat{v}$, $\hat{V}$). The momentum states are normalized according to:

$$\langle \boldsymbol{k}'|\boldsymbol{k}\rangle = (2\pi)^3\delta(\boldsymbol{k}'-\boldsymbol{k}), \tag{2.2}$$

where $\boldsymbol{k}$ is the momentum vector in units of $\hbar$.

The $\hat{v}$'s in Eq. (2.1) have the form:

$$\hat{v}_\nu(\boldsymbol{k}) = v_\nu(k) + \frac{1}{\sqrt{8}} w_\nu(k)\hat{S}(\boldsymbol{k}), \tag{2.3}$$

where $\hat{S}(\boldsymbol{k})$ is the tensor operator.

Our nucleon-nucleon interaction (2.1) is the so called Yamaguchi [10], [22] interaction. Actually, it is a generalization of the original Yamaguchi interaction, as it contains N_s spin singlet and N_t spin triplet terms. The interaction (2.1) acts only in the $^1S_0$ and $^3S_1+^3D_1$ states. The mixing of the $^3S_1$ and $^3D_1$ states is caused by the presence of the tensor operator, Eq. (2.3). Let us notice that whereas the original Yamaguchi potential in the triplet state ($N_t = 1$) acts just in the "eigen-S state", our potential in the case of $N_t > 1$ acts in both "eigen-S" and "eigen-D states".

Now, we are going to describe how the functions u_ν, v_ν, w_ν, which determine the shape of the interaction, and the strength parameters, λ_ν^s, λ_ν^t, are adjusted to the low and medium energy properties of the two-nucleon system.

A. The interaction in the $^1S_0$ state

To take care of the short range repulsion in the nucleon-nucleon interaction, best established in the $^1S_0$ state, we shall use two terms ($N_s = 2$) in our singlet-state interaction. The short range repulsion may be represented conveniently by a hard shell potential which is separable in each partial wave. Hence, by adding to the attractive Yamaguchi S state interaction [22] the S state part of the hard shell repulsion we obtain an interaction which contains both the long range attraction and the short range repulsion and is still separable.

This potential, first applied by Puff [23], has the deficiency of letting the two-body wave function leak into the region inside the hard shell. We remove this deficiency by cutting off the spatial representation of the attractive Yamaguchi part of the potential for distances smaller than the hard shell radius r_c. This procedure, explained in [24] in the case of the $\Lambda-\Lambda$ interaction, leads to the following form of the attractive part ($\nu = 1$) of the potential:

$$v_1(k) = \frac{1}{k(\beta_s^2+k^2)} [\beta_s \sin kr_c + k \cos kr_c] \exp(-\beta_s r_c), \tag{2.4}$$

whereas the repulsive hard shell part ($\nu = 2$) has the form:

$$v_2(k) = \sin kr_c/kr_c. \tag{2.5}$$

Since the $\nu = 2$ part of our interaction represents a hard shell repulsion, in our final expressions we go with the value of the strength parameter λ_2^s to the limit:

$$\lambda_2^s \to \infty. \tag{2.6}$$

The three free parameters of our $^1S_0$ interaction, λ_1^s, β_s, r_c, may be adjusted to the experimental values of the singlet scattering length, a_s, and of the singlet effective range, r_{0s}, and to the observed behaviour of the $^1S_0$ nucleon-nucleon phase shift, $\delta(^1S_0)$, at higher energies. The calculation of a_s, r_{0s} and ctg $\delta(^1S_0)$, with the help of our potential, Eqs (2.4–2.6), is a standard one (see, *e.g.*, Ref. [23]), and we simply write the resulting equations for a_s, r_{0s}:

$$[1-(r_c/3a_s)-(r_{0s}/r_c)]/[1-(r_c/a_s)]^2$$
$$= [\tfrac{1}{3}-(2/\beta_s r_c)^2]/[1-(a_s/r_c)]-(3/\beta_s r_c)-1, \tag{2.7β}$$

$$8\pi\beta_s^3/\lambda_1^s m = -\{1+2/[\beta_s(r_c-a_s)]\} \exp(-2\beta_s r_c), \tag{2.7λ}$$

and the expression for ctg $\delta(^1S_0)$:

$$\operatorname{ctg} \delta(^1S_0) = -\operatorname{ctg} kr_c + \left\{\sin kr_c e^{\beta_s r_c}\left[\frac{\sin kr_c}{2k} \times \right.\right.$$
$$\left.\left. \times \left(\frac{8\pi}{\lambda_1^s m}(\beta_s^2+k^2)^2 + e^{-\beta_s r_c}(\beta_s - k^2/\beta_s)\right) + \cos kr_c\, e^{-\beta_s r_c}\right]\right\}^{-1}, \tag{2.8}$$

where $m = 0.02412\ \mathrm{MeV^{-1}fm^{-2}}$ is the nucleon mass divided by $\hbar^2$.

The values

$$a_s = -23.7\ \mathrm{fm},\quad r_{0s} = 2.5\ \mathrm{fm}, \tag{2.9}$$

have been used in calculating, with the help of Eqs (2.7β–2.7λ), the values of the parameters β_s, λ_1^s for each value of the hard shell radius, r_c, considered. The results are shown in Table I. To choose the best value of r_c we have compared the phase shift $\delta(^1S_0)$, calculated with the help of Eq. (2.8), as a function of the laboratory energy, $E_{\mathrm{LAB}} = 2k^2/m$, with the corresponding YLAM phase shift [25]. The best overall agreement with YLAM, shown in Fig. 1, has been achieved for $r_c = 0.25$ fm.

Notice, however, that our potential gives a rather poor agreement with the YLAM phase shifts in the range of laboratory energies around 10 MeV, which might be more important

TABLE I

The parameters of the two-body interaction in the $^1S_0$ state

r_c (fm)	β_s (fm^{-1})	λ_1^s (MeV fm^{-1})
0.00	1.254	-1.925×10^3
0.10	1.370	-3.318×10^3
0.15	1.436	-4.485×10^3
0.25	1.590	-8.807×10^3
0.40	1.893	-30.808×10^3

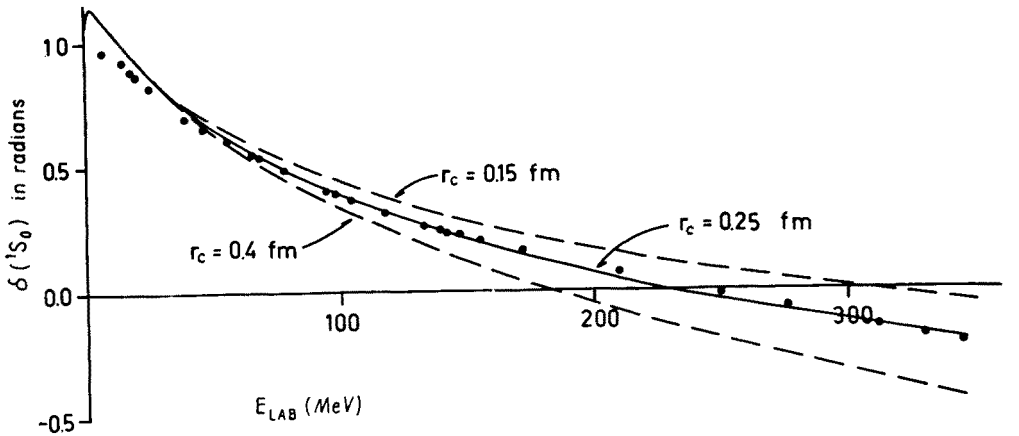

Fig. 1. The $^1S_0$ phase shift calculated for three values of the hard shell radius, r_c. Dots denote the YLAM $^1S_0$ phase shift

in the triton problem than the higher energy range, where the agreement is satisfactory. Unfortunately, we cannot change this situation by changing the value of r_c — at least with the shape (2.4) of v_1 (see Fig. 1).

B. The interaction in the $^3S_1+^3D_1$ state

For computational reasons, in the present three-body calculations we have restricted ourselves to a one term triplet interaction ($N_t = 1$). For the functions $v_1 = v$ and $w_1 = w$, Eq. (2.3), we use the Yamaguchi form [10]:

$$v(k) = 1/(\beta_t^2+k^2), \qquad w(k) = -t_Y k^2/(\gamma^2+k^2)^2. \tag{2.10}$$

The values of the parameters of this triplet interaction, which we apply in the present work, were found in [10] to be:

$$\lambda_1^t = \lambda^t = -2.561\times10^3 \text{ MeV fm}^{-1}, \quad t_Y = 1.784,$$
$$\beta_t = 1.334 \text{ fm}^{-1}, \quad \gamma = 1.568 \text{ fm}^{-1}. \tag{2.11}$$

These values reproduce the binding energy of the deuteron, $\varepsilon_D = 2.225$ MeV, the triplet

scattering length, $a_t = 5.38$ fm, the deuteron quadrupole moment, $Q = 0.274$ fm^2, and give for the percentage of the D state in the deuteron, $P_D = 4\%$.

We have considered also the pure central force case in which we have dropped the tensor part of the interaction and have adjusted the parameters of the central part, v, to $\varepsilon_D = 2.225$ MeV and $a_t = 5.38$ fm. In this case we have:

$$\lambda^t = -4.282\times 10^3 \text{ MeV fm}^{-1}, \quad \beta_t = 1.449 \text{ fm}^{-1}, \quad t_Y = 0. \tag{2.12}$$

In deriving the equations for the neutron-deutron scattering, we shall need the form of the normalized deuteron wave function, φ^D, produced by the interaction (2.1). In momentum representation, we have:

$$\langle \boldsymbol{k}|\varphi^D\rangle = \hat{\varphi}^D(\boldsymbol{k})\chi_{1M}\zeta_0, \tag{2.13}$$

where χ_{1M} is the triplet spin state with the z-projection of the total spin equal M, and ζ_0 the singlet isospin state of two nucleons. For $\hat{\varphi}^D(\boldsymbol{k})$ we have:

$$\hat{\varphi}^D(\boldsymbol{k}) = -\frac{N}{\varepsilon_D + k^2/m}\sum_{\nu=1}^{N_t} c_\nu \hat{v}_\nu(\boldsymbol{k}), \tag{2.14}$$

where the coefficients c_ν satisfy the system of linear homogeneous equations:

$$(1/\lambda_\nu^t)c_\nu - \sum_{\mu=1}^{N_t} E_{\nu\mu}c_\mu = 0, \qquad \nu = 1, 2, \ldots, N_t, \tag{2.15}$$

with

$$E_{\nu\mu} = -\int \frac{d\boldsymbol{k}}{(2\pi)^3}\,[v_\nu(k)\, v_\mu(k) + w_\nu(k)w_\mu(k)]/(\varepsilon_D + k^2/m). \tag{2.16}$$

By requiring the determinant of the system of equations (2.15) to vanish, we get the eigenvalue equation:

$$\text{Det}\,\{(1/\lambda_\nu^t)\delta_{\mu\nu} - E_{\mu\nu}\} = 0. \tag{2.17}$$

With Eq. (2.17) being safisfied, we may make solution of Eq. (2.15) unique by imposing the condition:

$$\sum_{\nu=1}^{N_t} c_\nu^2 = 1. \tag{2.18}$$

For the normalization constant N we have:

$$N^2 = 1/\sum_{\mu\nu} \mathcal{N}_{\mu\nu}c_\mu c_\nu, \tag{2.19}$$

where

$$\mathcal{N}_{\mu\nu} = \frac{\partial}{\partial \varepsilon_D} E_{\mu\nu}. \tag{2.20}$$

We have introduced a minus sign in Eq. (2.14) for reasons of convenience. Otherwise, this sign is irrelevant.

The equations (2.14–2.20) are an obvious generalization of the equations of Ref. [10].

3. *The kinematics of the three-nucleon system*

By $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_3$ we denote the momenta (in units of $\hbar$) of the three nucleons. Throughout this paper we shall work in the bary-centric subspace of our three-nucleon system:

$$\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3=0. \tag{3.1}$$

Consequently, we have two independent vectors to label the states of the system. Following Faddeev [6] (see also [26]), we introduce three such pairs of vectors (see Fig. 2):

$$\boldsymbol{q}_i=\boldsymbol{p}_j+\boldsymbol{p}_k, \qquad \boldsymbol{k}_i=\tfrac{1}{2}(\boldsymbol{p}_k-\boldsymbol{p}_i), \tag{3.2}$$

where the indices i, j, k form one of the three cyclic permutations of 1, 2, 3. The momentum $\boldsymbol{q}_i$ is the total momentum of the particles j and k (which we shall call the i-th pair), and $\boldsymbol{k}_i$ is the relative momentum of the i-th pair. Each pair of the vectors $\boldsymbol{q}_i$, $\boldsymbol{p}_i$, according to the equations:

$$\boldsymbol{q}_i|\boldsymbol{q}'\boldsymbol{k}'\rangle_i=\boldsymbol{q}'|\boldsymbol{q}'\boldsymbol{k}'\rangle_i, \qquad \boldsymbol{k}_i|\boldsymbol{q}'\boldsymbol{k}'\rangle_i=\boldsymbol{k}'|\boldsymbol{q}'\boldsymbol{k}'\rangle_i, \tag{3.3}$$

labels the momentum states of three different bases, $|\boldsymbol{q}\boldsymbol{k}\rangle_i$ $(i=1, 2, 3)$, normalized according to:

$${}_i\langle\boldsymbol{q}\boldsymbol{k}|\boldsymbol{q}'\boldsymbol{k}'\rangle_i=(2\pi)^6\delta(\boldsymbol{q}-\boldsymbol{q}')\delta(\boldsymbol{k}-\boldsymbol{k}'). \tag{3.4}$$

Let us notice that Eqs (3.2) imply:

$$\boldsymbol{q}_2=-\frac{1}{2}\boldsymbol{q}_1+\boldsymbol{k}_1, \qquad \boldsymbol{q}_1=-\frac{1}{2}\boldsymbol{q}_2-\boldsymbol{k}_2,$$

$$\boldsymbol{k}_2=-\frac{3}{4}\boldsymbol{q}_1-\frac{1}{2}\boldsymbol{k}_1, \qquad \boldsymbol{k}_1=\frac{3}{4}\boldsymbol{q}_2-\frac{1}{2}\boldsymbol{k}_2. \tag{3.5}$$

These equations enable us to interchange the bases $|\ \rangle_1$ and $|\ \rangle_2$. Namely, the corresponding transformation coefficients are:

$$\begin{aligned}{}_1\langle\boldsymbol{q}\boldsymbol{k}|\boldsymbol{q}'\boldsymbol{k}'\rangle_2&={}_2\langle\boldsymbol{q}'\boldsymbol{k}'|\boldsymbol{q}\boldsymbol{k}\rangle_1\\&=(2\pi)^6\,\delta\left(\boldsymbol{q}'+\frac{1}{2}\boldsymbol{q}-\boldsymbol{k}\right)\delta\left(\boldsymbol{k}'+\frac{3}{4}\boldsymbol{q}+\frac{1}{2}\boldsymbol{k}\right)\\&=(2\pi)^6\,\delta\left(\boldsymbol{q}+\frac{1}{2}\boldsymbol{q}'+\boldsymbol{k}'\right)\delta\left(\boldsymbol{k}-\frac{3}{4}\boldsymbol{q}'+\frac{1}{2}\boldsymbol{k}'\right).\end{aligned} \tag{3.6}$$

By replacing the indices: $1\to 2$, $2\to 3$, and $1\to 3$, $2\to 1$, we may obtain from Eqs (3.5–3.6) all the remaining transformation equations.

As far as the total kinetic energy operator, K, is concerned, the choice of any of the three bases is irrelevant in the sense that we have:

$$K|\boldsymbol{q}\boldsymbol{k}\rangle_i=\left(\frac{3q^2}{4m}+\frac{k^2}{m}\right)|\boldsymbol{q}\boldsymbol{k}\rangle_i, \qquad \text{for } i=1, 2, 3. \tag{3.7}$$

To classify the spin states of the system, again, we introduce three different bases with the help of the operators:

$$\boldsymbol{I}_1=\boldsymbol{s}_2+\boldsymbol{s}_3, \qquad \boldsymbol{I}_2=\boldsymbol{s}_1+\boldsymbol{s}_3, \qquad \boldsymbol{I}_3=\boldsymbol{s}_1+\boldsymbol{s}_2, \tag{3.8}$$

where $\boldsymbol{s}_j = \frac{1}{2}\boldsymbol{\sigma}_j$ is the spin operator of the j-th nucleon, $\boldsymbol{I}_i$ is the total spin of the i-th pair of nucleons. To get the total spin of the three-nucleon system, $\boldsymbol{I}^{\mathrm{tot}}$, we may use any of the three equations:

$$\boldsymbol{I}^{\mathrm{tot}} = \boldsymbol{s}_1 + \boldsymbol{I}_1, \quad \boldsymbol{I}^{\mathrm{tot}} = \boldsymbol{s}_2 + \boldsymbol{I}_2, \quad \boldsymbol{I}^{\mathrm{tot}} = \boldsymbol{I}_3 + \boldsymbol{s}_3. \tag{3.9}$$

Each operator $\boldsymbol{I}_j^2$, together with the operator $(\boldsymbol{I}^{\mathrm{tot}})^2$ and with the operator of the z-component of the total spin, I_z^{tot}, according to the equations:

$$\left.\begin{aligned} \boldsymbol{I}_j^2 |I\, I^{\mathrm{tot}} M\rangle_j^\sigma &= I(I+1)|I\, I^{\mathrm{tot}} M\rangle_j^\sigma, \\ (\boldsymbol{I}^{\mathrm{tot}})^2 |I\, I^{\mathrm{tot}} M\rangle_j^\sigma &= I^{\mathrm{tot}}(I^{\mathrm{tot}}+1)|I\, I^{\mathrm{tot}} M\rangle_j^\sigma, \\ I_z^{\mathrm{tot}} |I\, I^{\mathrm{tot}} M\rangle_j^\sigma &= M|I\, I^{\mathrm{tot}} M\rangle_j^\sigma, \end{aligned}\right\} \tag{3.10}$$

labels the normalized spin states of three different bases, $|I\, I^{\mathrm{tot}} M\rangle_j^\sigma$ $(j = 1, 2, 3)$.

Actually, we shall need only the spin doublet states with $I^{\mathrm{tot}} = \frac{1}{2}$, for which we introduce the simplified notation:

$$|I, I^{\mathrm{tot}} = \frac{1}{2}, M\rangle_j^\sigma = |I\rangle_j^\sigma. \tag{3.11}$$

In the presence of the tensor forces I^{tot} is not a good quantum number, and one expects the presence of the spin quartet states $(I^{\mathrm{tot}} = \frac{3}{2})$ which, however, will be generated automatically by the action of tensor operators on the doublet states.

In constructing the states $|I\rangle_j^\sigma$, one has to fix the order of adding spins of the three nucleons. Here, we accept the order of coupling indicated in Eqs (3.8) and (3.9), which is the order applied in [27].

To be able to change from one base to another it is necessary to know the transformation coefficients (diagonal in M):

$$U^{ik}(I, K) = U^{ki}(K, I) = {}_k^\sigma\langle K|I\rangle_i^\sigma = {}_i^\sigma\langle I|K\rangle_k^\sigma, \quad (i \neq k). \tag{3.12}$$

The U^{ik} coefficients may be calculated easily (see [27]), and with our order of coupling one gets for them the following values:

$$\begin{aligned} U^{13}(1,1) = -U^{13}(0,0) = U^{12}(0,0) = -U^{12}(1,1) = U^{23}(0,0) \\ = U^{23}(1,1) = 1/2, \end{aligned} \tag{3.13}$$

$$U^{13}(1,0) = U^{13}(0,1) = U^{12}(1,0) = U^{12}(0,1) = U^{23}(0,1) = -U^{23}(1,0) = \sqrt{3}/2.$$

Notice that we have:

$$U^{23}(I, J) = (-)^I\, U^{21}(I, J). \tag{3.14}$$

For the sake of completeness let us mention that in the case of the quartet state $(I^{\mathrm{tot}} = 3/2)$ the spin of each nucleon pair is equal one, $I_i = 1$ $(i = 1, 2, 3)$, and for the transformation coefficients we have:

$${}_i^\sigma\langle 1, \frac{3}{2}, M|1, \frac{3}{2}, M'\rangle_j^\sigma = \delta_{MM'}. \tag{3.15}$$

With the isospin states, we proceed in exactly the same way as with the spin states. We apply the same coupling order, and introduce the notation:

$$|T, T^{\mathrm{tot}} = 1/2, T_3^{\mathrm{tot}}\rangle_j^{\tau} = |T\rangle_j^{\tau}, \tag{3.16}$$

where T is the isospin of the j-th pair of nucleons, T^{tot} is the total isospin of the three-nucleon system, and T_3^{tot} is its third component. In the case of charge independent forces T^{tot} is a good quantum number. The transformation coefficients for the isospin states are the same as for the spin states,

$${}_i^{\tau}\langle T|T'\rangle_k^{\tau} = U^{ik}(T, T'). \tag{3.17}$$

Collecting the spatial, spin and isospin states, we may write the complete set of states of the three-nucleon system in the three different bases:

$$|\boldsymbol{q}\boldsymbol{k}; I I^{\mathrm{tot}} M; T T^{\mathrm{tot}} T_3^{\mathrm{tot}}\rangle_i = |\boldsymbol{q}\boldsymbol{k}\rangle_i |I I^{\mathrm{tot}} M\rangle_i^{\sigma} |T T^{\mathrm{tot}} T_3^{\mathrm{tot}}\rangle_i^{\tau}. \tag{3.18}$$

Obviously, because of the Pauli principle, only certain combinations of these states are realized in nature.

The advantage of the i-th base is that it forms the most convenient base for the description of the interaction between the two nucleons of the i-th pair. Often the name "i-th natural base" is used. Since in the three-nucleon system there is an interaction between all three nucleons, we need all three natural bases for the description of the system.

In constructing the three-nucleon wave function we shall find it convenient to have the following spin and isospin states defined:

$$|1\rangle\rangle_i = |1\rangle_i^{\sigma}|0\rangle_i^{\tau}, \quad |0\rangle\rangle_i = |0\rangle_i^{\sigma}|1\rangle_i^{\tau}. \tag{3.19}$$

Obviously, the states $|I\rangle\rangle_i$ are antisymmetric in spin and isospin coordinates of the i-th pair of nucleons. The total spin and isospin of the states $|I\rangle\rangle_i$ is equal 1/2, whereas the spin of the i-th pair of nucleons is I.

In the next section we shall impose on the wave function of the three-nucleon system the condition of complete antisymmetry. For this reason we should know the behaviour of the momentum, spin and isospin states introduced above under the exchange of the coordinates of two nucleons. Let us introduce the exchange operator of all the coordinates of the nucleons i and j:

$$P_{ij} = P_{ij}^{r} P_{ij}^{\sigma} P_{ij}^{\tau} = P_{ij}^{r} P_{ij}^{\sigma\tau}, \tag{3.20}$$

where the superscripts r, σ, τ denote exchange operators of spatial, spin and isospin coordinates, respectively.

Now, the following relations hold (for $i \neq j \neq k \neq i$):

$$P_{ij}^{r}|\boldsymbol{q}\boldsymbol{k}\rangle_k = |\boldsymbol{q}-\boldsymbol{k}\rangle_k, \quad P_{ij}^{r}|\boldsymbol{q}\boldsymbol{k}\rangle_i = |\boldsymbol{q}-\boldsymbol{k}\rangle_j, \tag{3.21}$$

$$P_{ij}^{\sigma\tau}|I\rangle\rangle_k = -|I\rangle\rangle_k, \quad P_{ij}^{\sigma\tau}|I\rangle\rangle_i = -\varepsilon_k|I\rangle\rangle_j, \tag{3.22}$$

where

$$\varepsilon_k = \begin{cases} -1 & \text{for } k = 2, 3, \\ +1 & \text{for } k = 1. \end{cases} \tag{3.23}$$

Some of these relations are obvious (the first relation of (3.21) and the first relation of (3.22)), the others may be checked easily. For the second relation of (3.22) our order of coupling spins and isospins is relevant.

4. The Schroedinger equation of the three-nucleon system

The Schroedinger equation satisfied by the state vector $|\Psi\rangle$ of the three-nucleon system whose energy is E has the form:

$$D|\Psi\rangle = \hat{V}|\Psi\rangle, \tag{4.1}$$

where

$$D = E - K, \tag{4.2}$$

and

$$\hat{V} = \hat{V}_1 + \hat{V}_2 + \hat{V}_3,$$

where $\hat{V}_i$ is the interaction between the two nucleons of the i-th pair (see Fig. 2). If we introduce the notation:

$$|\Phi_i\rangle = \hat{V}_i|\Psi\rangle, \tag{4.4}$$

we may write the Schroedinger equation (4.1) in the form:

$$|\Phi_i\rangle = \hat{V}_i D^{-1} \sum_{j=1}^{3} |\Phi_j\rangle, \tag{4.5}$$

where D^{-1} is regular and uniquely defined for negative values of E, to be considered throughout this paper.

Each state vector $|\Phi_i\rangle$ shall be written in its "natural" momentum representation, for which we introduce the following notation:

$${}_i\langle \boldsymbol{q}\boldsymbol{k}|\Phi_i\rangle = \boldsymbol{\Phi}_i(\boldsymbol{q}\boldsymbol{k}), \tag{4.6}$$

where here and further on we use bold face charcters, like $\boldsymbol{\Phi}_i$ to denote wave functions in momentum space and state vectors in spin and isospin space.

By using for each $|\Phi_j\rangle$ its natural momentum representation, we get from Eq. (4.5):

$$\boldsymbol{\Phi}_i(\boldsymbol{q}\boldsymbol{k}) = \sum_{j=1}^{3} (2\pi)^{-6} \int d\boldsymbol{q}'\, d\boldsymbol{k}'\, {}_i\langle \boldsymbol{q}\boldsymbol{k}|\hat{V}_i D^{-1}|\boldsymbol{q}'\boldsymbol{k}'\rangle_j \boldsymbol{\Phi}_j(\boldsymbol{q}'\boldsymbol{k}'). \tag{4.7}$$

Now, we have:

$${}_i\langle \boldsymbol{q}\boldsymbol{k}|\hat{V}_i|\boldsymbol{q}'\boldsymbol{k}'\rangle_i = (2\pi)^3 \delta(\boldsymbol{q}-\boldsymbol{q}')\, \langle \boldsymbol{k}|\hat{V}_i|\boldsymbol{k}'\rangle, \tag{4.8}$$

where $\langle \boldsymbol{k}|\hat{V}_i|\boldsymbol{k}'\rangle$ is the two-body interaction in the relative momentum representation (it is an operator acting on the spin and isospin coordinates of the i-th pair of nucleons). When we exploit Eqs (3.6) and (3.7) we may transform Eq. (4.7) easily into:

$$\boldsymbol{\Phi}_i(\boldsymbol{q}\boldsymbol{k}) = \int \frac{d\boldsymbol{k}'}{(2\pi)^3} \langle \boldsymbol{k}|\hat{V}_i|\boldsymbol{k}'\rangle\, \mathscr{D}^{-1}(qk')\, \boldsymbol{\Phi}_i(\boldsymbol{q}\boldsymbol{k}') +$$

$$+ \sum_{k \neq i} \int \frac{d\boldsymbol{q}'}{(2\pi)^3} \left\langle \boldsymbol{k}\, |\hat{V}_i| \mp \left(\boldsymbol{q}' + \frac{1}{2}\boldsymbol{q}\right) \right\rangle D(\boldsymbol{q}\boldsymbol{q}')^{-1}\, \boldsymbol{\Phi}_k \left(\boldsymbol{q}', \pm \left(\boldsymbol{q} \pm \frac{1}{2}\boldsymbol{q}'\right)\right), \tag{4.9}$$

where

$$\mathscr{D}(qk) = E - \frac{3q^2}{4m} - \frac{k^2}{m}, \tag{4.10}$$

$$D(\boldsymbol{q}\boldsymbol{q}') = E - \frac{1}{2m}[q^2 + (\boldsymbol{q}+\boldsymbol{q}')^2 + \boldsymbol{q}'^2] = \mathscr{D}\left(q', \left|\boldsymbol{q} + \frac{1}{2}\boldsymbol{q}'\right|\right). \tag{4.11}$$

The lower signs in Eq. (4.9) apply when i and k are a cyclic pair of numbers 1, 2, 3 (*e.g.*, $i = 1, k = 2$), the lower signs in the reversed case (*e.g.*, $i = 1, k = 3$).

In the case of a general two-body interaction, $\hat{V}_i$, the system of integral equations (4.9) seems to be, at present, numerically unmanageable. If, however, we assume for $\hat{V}_i$ the nonlocal separable form, Eq. (2.1), the system of equations (4.9) is simplified tremendously, and may be reduced to a system of one-dimensional integral equations. From now on we assume for $\hat{V}_i$ the form (2.1). Since we have three different pairs of nucleons, we have to indicate in Eq. (2.1) the particular i-th pair of interacting nucleons by making the substitutions:

$$\hat{A}^s \to \hat{A}^s_i, \quad \hat{A}^t \to \hat{A}^t_i, \quad \hat{v}_\nu \to \hat{v}_{i\nu},$$

where $\hat{v}_{i\nu}$ is given by Eq. (2.3) with $\hat{S} \to \hat{S}_i$. When this separable form of $\hat{V}_i$ is introduced into Eq. (4.9) one is led to the following Ansatz for $\boldsymbol{\Phi}_i$:

$$\boldsymbol{\Phi}_i(\boldsymbol{q}\boldsymbol{k}) = \sum_\nu \{u_\nu(k)\boldsymbol{\chi}^s_{i\nu}(q) + \hat{v}_{i\nu}(\boldsymbol{k})\boldsymbol{\chi}^t_{i\nu}(\boldsymbol{q})\}, \tag{4.12}$$

where

$$\boldsymbol{\chi}^s_{i\nu}(q) = s_{i\nu}(q)|0\rangle_i, \tag{4.13}$$

$$\boldsymbol{\chi}^t_{i\nu}(\boldsymbol{q}) = \left\{t_{i\nu}(q) + \frac{1}{\sqrt{8}}\, T_{i\nu}(q)\, \hat{S}_i(\boldsymbol{q})\right\}|\, 1 \gg_i . \tag{4.14}$$

The sum over ν in Eq. (4.12), and in similar equations to follow, runs from 1 to N_s in the first term and from 1 to N_t in the second term of this equation. The unknown functions in the Ansatz are $s_{i\nu}$, $t_{i\nu}$, $T_{i\nu}$. We shall prove the correctness of our Ansatz by simply showing that with this Ansatz we are able to solve Eq. (4.9).

First, let us derive the conditions imposed on $s_{i\nu}$, $t_{i\nu}$, $T_{i\nu}$ by the Pauli principle:

$$P_{ij}|\Psi\rangle = -|\Psi\rangle \tag{4.15}$$

In the case: $k \neq j \neq i \neq k$, we have:

$$P_{kj}|\Phi_i\rangle = P_{kj}\hat{V}_i|\Psi\rangle = \hat{V}_i P_{kj}|\Psi\rangle = -|\Phi_i\rangle \tag{4.16}$$

On the other hand,

$${}_i\langle \boldsymbol{q}\boldsymbol{k}|P^\tau_{kj}P^{\sigma\tau}_{kj}|\Phi_i\rangle = {}_i\langle \boldsymbol{q}-\boldsymbol{k}|P^{\sigma\tau}_{kj}|\Phi_i\rangle = P^{\sigma\tau}_{kj}\boldsymbol{\Phi}_i(\boldsymbol{q}-\boldsymbol{k}), \tag{4.17}$$

where we have used the property (3.21). Since, however, according to our Ansatz

$$\boldsymbol{\Phi}_i(\boldsymbol{q}-\boldsymbol{k}) = \boldsymbol{\Phi}_i(\boldsymbol{q}\boldsymbol{k}), \tag{4.18}$$

Eqs (4.17) and (4.16) imply:

$$P^{\sigma\tau}_{kj}\boldsymbol{\Phi}_i(\boldsymbol{q}\boldsymbol{k}) = -\boldsymbol{\Phi}_i(\boldsymbol{q}\boldsymbol{k}) \tag{4.19}$$

This requirement of the Pauli principle, however, is automatically fulfilled by our Ansatz, as one may see immediately with the help of the first of Eqs (3.22).

In the case: $k = i$, we have:

$$P_{ij}|\Phi_i\rangle = P_{ij}\hat{V}_i|\Psi\rangle = \hat{V}_i P_{ij}|\Psi\rangle = -|\Phi_j\rangle. \tag{4.20}$$

On the other hand, with the help of the property (3.21),

$${}_j\langle \boldsymbol{qk}|P_{ij}^{\tau}P_{ij}^{\sigma\tau}|\Phi_i\rangle = {}_i\langle \boldsymbol{q}-\boldsymbol{k}|P_{ij}^{\sigma\tau}|\Phi_i\rangle = P_{ij}^{\sigma\tau}\boldsymbol{\Phi}_i(\boldsymbol{q}-\boldsymbol{k}). \tag{4.21}$$

Equations (4.20) and (4.21), together with Eq. (4.18) imply:

$$P_{ij}^{\sigma\tau}\boldsymbol{\Phi}_i(\boldsymbol{q}, \boldsymbol{k}) = -\boldsymbol{\Phi}_j(\boldsymbol{qk}). \tag{4.22}$$

This requirement of the Pauli principle has definite consequences for our functions $s_{i\nu}$, $t_{i\nu}$, $T_{i\nu}$. Namely, with our Ansatz, Eq. (4.12), we have:

$$P_{ij}^{\sigma\tau}\boldsymbol{\Phi}_i(\boldsymbol{qk}) = \sum_{\nu}\{u_\nu(k)P_{ij}^{\sigma\tau}\boldsymbol{\chi}_{i\nu}^{s}(q) + \hat{v}_{j\nu}(\boldsymbol{k})P_{ij}^{\sigma\tau}\boldsymbol{\chi}_{i\nu}^{t}(\boldsymbol{q})\}. \tag{4.23}$$

Now, from Eqs (4.13–4.14), we get with the help of the second of Eqs (3.22):

$$\left.\begin{aligned} P_{ij}^{\sigma\tau}\boldsymbol{\chi}_{i\nu}^{s}(q) &= S_{i\nu}(q)\, P_{ij}^{\sigma\tau}|\,0\gg_i = -\varepsilon_k S_{i\nu}(q)|\,0\gg_j, \\ P_{ij}^{\sigma\tau}\boldsymbol{\chi}_{i\nu}^{t}(\boldsymbol{q}) &= -\varepsilon_k\left\{t_{i\nu}(q) + \frac{1}{\sqrt{8}}\, T_{i\nu}(q)\, \hat{S}_j(\boldsymbol{q})\right\}|\,1\gg_j. \end{aligned}\right\} \tag{4.24}$$

Eqs (4.23–4.24), when compared with the requirement (4.22), imply the following conditions:

$$s_{j\nu}(q) = \varepsilon_k s_{i\nu}(q),\ t_{j\nu}(q) = \varepsilon_k t_{i\nu}(q),\ T_{j\nu}(q) = \varepsilon_k T_{i\nu}(q), \tag{4.25}$$

where $j \neq i \neq k \neq j$. Since $\varepsilon_j = \varepsilon_k\varepsilon_i$, Eq. (3.23), we satisfy the conditions (4.25) by putting:

$$s_{i\nu}(q) = \varepsilon_i s_\nu(q),\ t_{i\nu}(q) = \varepsilon_i t_\nu(q),\ T_{i\nu}(q) = \varepsilon_i T_\nu(q). \tag{4.26}$$

Thus, our final Ansatz for $\boldsymbol{\Phi}_i$ which obeys the Pauli principle is given by Eq. (4.12) with

$$\boldsymbol{\chi}_{i\nu}^{s}(q) = \varepsilon_i s_\nu(q)|0\gg_i, \tag{4.27}$$

$$\boldsymbol{\chi}_{i\nu}^{t}(\boldsymbol{q}) = \varepsilon_i\left[t_\nu(q) + \frac{1}{\sqrt{8}}\, T_\nu(q)\, \hat{S}_i(\boldsymbol{q})\right]|\,1\gg_i. \tag{4.28}$$

To get the equations for the N_s+2N_t unkown functions $s_\nu(q)$, $t_\nu(q)$, $T_\nu(q)$, we insert our Ansatz into Eq. (4.9). By comparing the coefficients at $u_\nu(k)$ and $\hat{v}_{i\nu}(\boldsymbol{k})$ on both sides of the equation, we easily get:

$$\boldsymbol{\chi}_{i\nu}^{s}(q)/\lambda_\nu^{s} = \sum_{\mu}\left\{\mathscr{S}_{\nu\mu}^{s}(q)\ \boldsymbol{\chi}_{i\mu}^{s}(q) + \sum_{k\neq i}\int \frac{d\boldsymbol{q}'}{(2\pi)^3}\, D(\boldsymbol{qq}')^{-1}\, u_\nu(c)\hat{A}_i^{s}\times\right.$$

$$\left.\times[u_\mu(a)\boldsymbol{\chi}_{k\mu}^{s}(q') + \hat{v}_{k\mu}(\boldsymbol{a})\boldsymbol{\chi}_{k\mu}^{t}(\boldsymbol{q}')]\right\}, \tag{4.29}$$

$$\chi^t_{i\nu}(\boldsymbol{q})/\lambda^t_\nu = \sum_\mu \Big\{ \mathscr{I}^t_{\nu\mu}(q)\chi^t_{i\mu}(\boldsymbol{q}) + \sum_{k\neq i} \int \frac{d\boldsymbol{q}'}{(2\pi)^3} D(\boldsymbol{q}\boldsymbol{q}')^{-1}\, \hat{v}_{i\nu}(\boldsymbol{c})\hat{\Lambda}^t_i \times$$

$$\times [u_\mu(a)\chi^s_{k\mu}(q') + \hat{v}_{k\mu}(\boldsymbol{a})\chi^t_{k\mu}(\boldsymbol{q}')] \Big\}, \tag{4.30}$$

where

$$\left.\begin{aligned} \mathscr{I}^s_{\nu\mu}(q) &= \int \frac{d\boldsymbol{k}}{(2\pi)^3}\, \mathscr{D}^{-1}(qk) u_\nu(k) u_\mu(k), \\ \mathscr{I}^t_{\nu\mu}(q) &= \int \frac{d\boldsymbol{k}}{(2\pi)^3}\, \mathscr{D}^{-1}(qk)[v_\nu(k)v_\mu(k) + w_\nu(k)w_\mu(k)], \end{aligned}\right\} \tag{4.31}$$

$$\boldsymbol{a} = \boldsymbol{q} + \tfrac{1}{2}\boldsymbol{q}', \quad \boldsymbol{c} = \tfrac{1}{2}\boldsymbol{q} + \boldsymbol{q}'. \tag{4.32}$$

Now, the right hand sides of Eqs (4.29) and (4.30) have to be calculated. In particular, it is important to perform the integration over the azimuthal dependence of $\boldsymbol{q}'$. This calculation is perfectly straightforward. Notice, however, that, *e. g.*, the term $\hat{v}_{i\nu}(\boldsymbol{c})\hat{v}_{k\mu}(\boldsymbol{a})\chi^t_{k\mu}(\boldsymbol{q}')$ gives rise to a product: $\hat{S}_i(\boldsymbol{c})\, \hat{S}_k(\boldsymbol{a})\, \hat{S}_k(\boldsymbol{q}')$, which has to be integrated over the azimuthal angle of $\boldsymbol{q}'$. This type of terms makes the calculation lengthy, and we feel that it would not be reasonable to present the whole calculation in detail. Instead, we have restricted ourselves to collect, in the Appendix, the identities and relations which we have used in the calculation. The result of this straightforward calculation is that the right hand side of Eq. (4.29) has the form $\varepsilon_i \mathscr{A}(q)|0\rangle\!\rangle_i$, and the right hand side of Eq. (4.30) has the form $\varepsilon_i[\mathscr{B}(q) + \mathscr{C}(q)\, \hat{S}_i(\boldsymbol{q})]|1\rangle\!\rangle_i$. This means that the right hand sides of these equations have the same form as their left hand sides which proves the correctness of our Ansatz for $\boldsymbol{\Phi}_i$. By comparing the coefficients at $\varepsilon_i|0\rangle\!\rangle_i$ on both sides of Eq. (4.29), and those at $\varepsilon_i|1\rangle\!\rangle_i$ and at $\varepsilon_i\hat{S}_i(\boldsymbol{q})|1\rangle\!\rangle_i$ on both sides of Eq. (4.30), we get the following system of coupled one-dimensional integral equations for the functions $s_\nu(q)$, $t_\nu(q)$, $T_\nu(q)$:

$$s_\nu(q)/\lambda^s_\nu - \sum_\mu \mathscr{I}^s_{\nu\mu}(q)s_\mu(q) = \sum_\mu \int_0^\infty dq' q'^2\, [K^{ss}_{\nu\mu}(qq')\, s_\mu(q') +$$

$$+ K^{st}_{\nu\mu}(qq')t_\mu(q') + K^{sT}_{\nu\mu}(qq')\, T_\mu(q')], \tag{4.33s}$$

$$t_\nu(q)/\lambda^t_\nu - \sum_\mu \mathscr{I}^t_{\nu\mu}(q)t_\mu(q) = \sum_\mu \int_0^\infty dq' q'^2\, [K^{ts}_{\nu\mu}(qq')\, s_\mu(q') +$$

$$+ K^{tt}_{\nu\mu}(qq')t_\mu(q') + K^{tT}_{\nu\mu}(qq')T_\mu(q')], \tag{4.33t}$$

$$T_\nu(q)/\lambda^t_\nu - \sum_\mu \mathscr{I}^t_{\nu\mu}(q)T_\mu(q) = \sum_\mu \int_0^\infty dq' q'^2\, [K^{Ts}_{\nu\mu}(qq')\, s_\mu(q') +$$

$$+ K^{Tt}_{\nu\mu}(qq')\, t_\mu(q') + K^{TT}_{\nu\mu}(qq')T_\mu(q')]. \tag{4.33T}$$

Here

$$K^{xy}_{\nu\mu}(qq') = \left(\frac{1}{2\pi}\right)^3 \int d\hat{q}'\, D(\boldsymbol{q}\boldsymbol{q}')^{-1}\, A^{xy}_{\nu\mu}(\boldsymbol{q}\boldsymbol{q}'), \tag{4.34}$$

where

$$A^{ss}_{\nu\mu}(\boldsymbol{qq}') = \frac{1}{2}\, u_\nu(c)u_\mu(a),$$

$$A^{st}_{\nu\mu}(\boldsymbol{qq}') = -\frac{3}{2}\, u_\nu(c)v_\mu(a),$$

$$A^{sT}_{\nu\mu}(\boldsymbol{qq}') = -\frac{3}{2}\, Q(\boldsymbol{aq}')u_\nu(c)w_\mu(a),$$

$$A^{ts}_{\nu\mu}(\boldsymbol{qq}') = -\frac{3}{2}\, v_\nu(c)u_\mu(a),$$

$$A^{tt}_{\nu\mu}(\boldsymbol{qq}') = \frac{1}{2}\, v_\nu(c)v_\mu(a) - Q(\boldsymbol{ac})w_\nu(c)w_\mu(a),$$

$$A^{tT}_{\nu\mu}(\boldsymbol{qq}') = \frac{1}{2}\, Q(\boldsymbol{aq}')v_\nu(c)w_\mu(a) - Q(\boldsymbol{q}'\boldsymbol{c})w_\nu(c)v_\mu(a) +$$

$$+ \frac{1}{\sqrt{8}}\,[Q(\boldsymbol{aq}') + Q(\boldsymbol{ca}) + Q(\boldsymbol{cq}') - 1]w_\nu(c)w_\mu(a),$$

$$A^{Ts}_{\nu\mu}(\boldsymbol{qq}') = -\frac{3}{2}\, Q(\boldsymbol{cq})w_\nu(c)u_\mu(a),$$

$$A^{Tt}_{\nu\mu}(\boldsymbol{qq}') = \frac{1}{2}\, Q(\boldsymbol{cq})w_\nu(c)v_\mu(a) - Q(\boldsymbol{aq})v_\nu(c)w_\mu(a) +$$

$$+ \frac{1}{\sqrt{8}}\,[Q(\boldsymbol{cq}) + Q(\boldsymbol{aq}) + Q(\boldsymbol{ca}) - 1]w_\nu(c)w_\mu(a),$$

$$A^{TT}_{\nu\mu}(\boldsymbol{qq}') = -Q(\boldsymbol{q}'\boldsymbol{q})v_\nu(c)v_\mu(a) + \frac{1}{\sqrt{8}}\,[Q(\boldsymbol{aq}) + Q(\boldsymbol{q}'\boldsymbol{q}) +$$

$$+ Q(\boldsymbol{aq}') - 1]v_\nu(c)w_\mu(a) + \frac{1}{\sqrt{8}}\,[Q(\boldsymbol{cq}) + Q(\boldsymbol{q}'\boldsymbol{q}) +$$

$$+ Q(\boldsymbol{cq}') - 1]w_\nu(c)v_\mu(a) + \frac{1}{8}\,[2Q(\boldsymbol{cq}) +$$

$$+ 2Q(\boldsymbol{aq}') - Q(\boldsymbol{aq}) - Q(\boldsymbol{q}'\boldsymbol{q}) - Q(\boldsymbol{ca}) - Q(\boldsymbol{cq}')]w_\nu(c)w_\mu(a), \tag{4.35}$$

with

$$Q(\boldsymbol{ab}) = \frac{1}{2}\,[3(\boldsymbol{ab}/ab) - 1] = P_2(\hat{\boldsymbol{a}}\hat{b}). \tag{4.36}$$

Let us notice that the change of variables: $\boldsymbol{q} \leftrightarrow \boldsymbol{q}'$, implies the change $\boldsymbol{a} \leftrightarrow \boldsymbol{c}$ (see Eq. (4.32)). Consequently, we have $A^{xy}_{\nu\mu}(\boldsymbol{qq}') = A^{xy}_{\mu\nu}(\boldsymbol{q}'\boldsymbol{q})$. Furthermore, since $D(\boldsymbol{qq}') = D(\boldsymbol{q}'\boldsymbol{q})$, Eq. (4.11), we conclude that

$$K^{yx}_{\nu\mu}(\boldsymbol{qq}') = K^{yx}_{\mu\nu}(\boldsymbol{q}'\boldsymbol{q}). \tag{4.37}$$

If we make the substitutions:

$$s_\nu(q) = \tilde{s}_\nu(q)/q,\ t_\nu(q) = \tilde{t}_\nu(q)/q,\ T_\nu(q) = \tilde{T}_\nu(q)/q, \tag{4.38}$$

we may write Eq. (4.33s) in the form:

$$\tilde{s}_\nu(q)/\lambda^s_\nu = \sum_\mu \{\mathscr{I}^s_{\nu\mu}(q) + \int_0^\infty dq'\ [\tilde{K}^{ss}_{\nu\mu}(qq')\tilde{s}_\mu(q') + \\ + \tilde{K}^{st}_{\nu\mu}(qq')\tilde{t}_\mu(q') + \tilde{K}^{sT}_{\nu\mu}(qq')\tilde{T}_\mu(q')]\}, \tag{4.39s}$$

where

$$\tilde{K}^{xy}_{\nu\mu}(qq') = qK^{xy}_{\nu\mu}(qq')q'. \tag{4.40}$$

Similarly, we may write analogical equations (4.39t), (4.39T).

Since Eq. (4.37) implies:

$$\tilde{K}^{xy}_{\nu\mu}(qq') = \tilde{K}^{yx}_{\mu\nu}(q'q), \tag{4.41}$$

and, obviously, $\mathscr{I}^{s(t)}_{\nu\mu} = \mathscr{I}^{s(t)}_{\mu\nu}$, the form (4.39), because of its symmetry, is particularly suited for numerical solution.

The system of coupled integral equations (4.33) or (4.39) has to be solved numerically. In this way we obtain the functions $s_\nu(q)$, $t_\nu(q)$, $T_\nu(q)$ which then may be used to construct the wave function of the three-nucleon system. From Eq. (4.1), we have

$$|\Psi\rangle = D^{-1}\sum_{i=1}^{3}|\Phi_i\rangle \tag{4.42}$$

We may represent $|\Psi\rangle$ in any of the three bases. Let us, *e. g.*, use the momentum states of the first base, $|\boldsymbol{qk}\rangle_1$, and define the wave function in the momentum space, and the state vector in the spin and isospin space, $\boldsymbol{\Psi}(\boldsymbol{qk})$:

$$\boldsymbol{\Psi}(\boldsymbol{qk}) = {}_1\langle \boldsymbol{qk}|\Psi\rangle \tag{4.43}$$

With the help of Eqs (3.7), (4.10), we get

$$\boldsymbol{\Psi}(\boldsymbol{qk}) = \sum_{i=1}^{3}{}_1\langle \boldsymbol{qk}|D^{-1}|\Phi_i\rangle = \mathscr{D}^{-1}(qk)\sum_{i=1}^{3}{}_1\langle \boldsymbol{qk}|\Phi_i\rangle, \tag{4.44}$$

and by applying Eqs (3.6), we get finally:

$$\boldsymbol{\Psi}(\boldsymbol{qk}) = \mathscr{D}^{-1}(qk)\ \{\boldsymbol{\Phi}_1(\boldsymbol{qk}) + \boldsymbol{\Phi}_2(\boldsymbol{q}_2\boldsymbol{k}_2) + \boldsymbol{\Phi}_3(\boldsymbol{q}_3\boldsymbol{k}_2)\}, \tag{4.45}$$

where

$$\boldsymbol{q}_2 = -\frac{1}{2}\boldsymbol{q} + \boldsymbol{k}, \quad \boldsymbol{q}_3 = -\frac{1}{2}\boldsymbol{q} - \boldsymbol{k},$$

$$\boldsymbol{k}_2 = -\frac{3}{4}\boldsymbol{q} - \frac{1}{2}\boldsymbol{k}, \quad \boldsymbol{k}_3 = \frac{3}{4}\boldsymbol{q} - \frac{1}{2}\boldsymbol{k}, \tag{4.46}$$

with $\boldsymbol{\Phi}_i$ given by Eqs (4.12), (4.27) and (4.28). It should be pointed out that the total spin $J(\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{I}^{\mathrm{tot}})$ of the state $\boldsymbol{\Psi}$ is $J = 1/2$.

Now, let us discuss the distinction between the case of determining the bound state of the three-nucleon system and the case of determining the *n-d* scattering amplitude, both

of them in the state $J = 1/2$. In the first case, we have to determine the value of $E = E_T$ for which the system of homogeneous integral equations (4.33) or (4.39) has a nonvanishing solution. For this value of the three-nucleon (triton) energy, E_T, we then solve the system of equations (4.33) or (4.39) to determine the corresponding ground state wave function, Eq. (4.45). Details of the procedure will be described in the next Section 5.

Let us now discuss the case of the n-d scattering (see Ref. [2]). Here, we have the initial and final states energies, E_0 and E_f:

$$E_0 = -\varepsilon_D + q_0^2/2\tilde{\mu},\ E_f = -\varepsilon_D + q_f^2/2\tilde{\mu}, \tag{4.47}$$

where $\tilde{\mu} = 2m/3$ is the n-d reduced mass, and $\boldsymbol{q}_0, \boldsymbol{q}_f$ are the momenta of the incoming and scattered neutron.

If the nucleon 1 is considered to be the neutron which undergoes the scattering, then the scattering amplitude f (in the CMS system) is:

$$f(\boldsymbol{q}_0 \to \boldsymbol{q}_f) = -\frac{\tilde{\mu}}{2\pi} \langle \tilde{\Psi}^0_{\boldsymbol{q}_f} | \hat{V}_2 + \hat{V}_3 | \Psi_{\boldsymbol{q}_0} \rangle |_{q_0 = q_f}$$

$$= \frac{1}{4\pi} \lim_{q_f \to q_0} (q_f^2 - q_0^2) \int \frac{d\boldsymbol{k}}{(2\pi)^3}\ {}_1\langle 1 | \hat{\varphi}_1^D(\boldsymbol{k})^\dagger \Psi(-\boldsymbol{q}_f \boldsymbol{k}), \tag{4.48}$$

where $|\tilde{\Psi}^0_{\boldsymbol{q}_f}\rangle$ in the $|\boldsymbol{q}\boldsymbol{k}\rangle_1$ representation has the form:

$${}_1\langle \boldsymbol{q}\boldsymbol{k} | \tilde{\Psi}^0_{\boldsymbol{q}_f} \rangle = \tilde{\boldsymbol{\Psi}}^0_{\boldsymbol{q}_f}(\boldsymbol{q}\boldsymbol{k}) = (2\pi)^3\, \delta(\boldsymbol{q} + \boldsymbol{q}_f)\, \hat{\varphi}_1^D(\boldsymbol{k}) | 1 \gg_1, \tag{4.49}$$

and satisfies the equation:

$$(K + \hat{V}_1)\, |\tilde{\Psi}^0_{\boldsymbol{q}_f}\rangle = E_f\, |\tilde{\Psi}^0_{\boldsymbol{q}_f}\rangle. \tag{4.50}$$

The $\hat{\varphi}_1^D$ in Eq. (4.49) is defined by Eq. (2.13) with the subscript 1 indicating the spin and isospin variables of the 1-st pair of nucleons forming the deuteron. Notice that $|\tilde{\Psi}^0_{\boldsymbol{q}_f}\rangle$ is not antisymmetric in all the coordinates of the three nucleons and this is indicated by the tilde.

The state $|\Psi_{\boldsymbol{q}_0}\rangle$ is the solution of Eq. (4.1) for $E = E_0$, Eq. (4.47), with the boundary condition:

$${}_1\langle \boldsymbol{q}\boldsymbol{k} | \Psi_{\boldsymbol{q}_0} \rangle = \boldsymbol{\Psi}_{\boldsymbol{q}_0}(\boldsymbol{q}\boldsymbol{k}) = \boldsymbol{\Psi}^0_{\boldsymbol{q}_0}(\boldsymbol{q}\boldsymbol{k}) + \text{outgoing wave}, \tag{4.51}$$

where $\boldsymbol{\Psi}^0_{\boldsymbol{q}_0}$ is given by Eq. (4.49) with $\boldsymbol{q}_f$ being replaced by $\boldsymbol{q}_0$, and with additional terms arising from antisymmetrization. The last step in Eq. (4.48) may be checked easily with the help of Eqs (4.49), (4.50), and the equation satisfied by $|\Psi_{\boldsymbol{q}_0}\rangle$.

By comparing Eq. (4.49) for $\boldsymbol{q}_f = \boldsymbol{q}_0$ with the general form of $\boldsymbol{\Psi}$, Eq. (4.45), we notice that to get $|\Psi^0_{\boldsymbol{q}_0}\rangle$ we must put:

$$s_\nu^0(\boldsymbol{q}) = T_\nu^0(\boldsymbol{q}) = 0,\ t_\nu^0(\boldsymbol{q}) = N(2\pi)^3\, \delta(\boldsymbol{q} + \boldsymbol{q}_0)\, c_\nu, \tag{4.52}$$

where c_ν are defined by Eqs (2.15), (2.18), and N is the deuteron normalization constant, Eq. (2.19). Notice that s_ν^0, t_ν^0, T_ν^0, Eq. (4.52), inserted into Eq. (4.45), produce automatically a completely antisymmetric state $|\Psi^0_{\boldsymbol{q}_0}\rangle$. Notice also, that in the case of the n-d scattering with the initial neutron momentum $\boldsymbol{q}_0$ the functions s_ν, t_ν, T_ν which appeare in our Ansatz, Eqs (4.27–4.28), depend on the direction of $\boldsymbol{q}$ (with respect to $\boldsymbol{q}_0$). This complicates the

derivation of the equations for the functions $s_\nu(\boldsymbol{q})$, $t_\nu(\boldsymbol{q})$, $T_\nu(\boldsymbol{q})$. The problem, however, does not arise in the case $q_0 = 0$, to which we shall restrict ourselves in this paper.

In constructing the state $|\Psi_{q_0}\rangle$, we must use the functions s_ν, t_ν, T_ν of such a form that the boundary condition (4.51) be satisfied. Thus we put:

$$\begin{aligned} s_\nu(\boldsymbol{q}) &= N\{-4\pi a_\nu^s(\boldsymbol{q})/[q^2-q_0^2-i0]\}, \\ t_\nu(\boldsymbol{q}) &= N\{(2\pi)^3\,\delta(\boldsymbol{q}+\boldsymbol{q}_0)\,c_\nu - 4\pi a_\nu^t(\boldsymbol{q})/[q^2-q_0^2-i0]\}, \\ T_\nu(\boldsymbol{q}) &= N\{-4\pi a_\nu^T(\boldsymbol{q})/[q^2-q_0^2-i0]\}. \end{aligned} \tag{4.53}$$

It is easy to show that Eq. (4.48) with Ψ constructed with the help of the functions s_ν, t_ν, T_ν of Eq. (4.53) gives:[3]

$$f(\boldsymbol{q}_0 - \boldsymbol{q}_f) = -\sum_\nu A_\nu a_\nu^t(-\boldsymbol{q}_f)|_{q_f=q_0}, \tag{4.54}$$

where

$$A_\nu = N^2 \sum_\nu c_\nu \mathcal{N}_{\mu\nu}, \tag{4.55}$$

with $\mathcal{N}_{\mu\nu}$ defined by Eq. (2.20).

In the case of zero energy neutrons, $q_0 = 0$, we come back to the case when the functions s_ν, t_ν, T_ν depend on q only, and thus satisfy Eqs (4.33). Eqs (4.53) may now be written in the simple form:

$$\begin{aligned} s_\nu(q) &= N\{-4\pi a_\nu^s(q)/q^2\}, \\ t_\nu(q) &= N\{2\pi^2 c_\nu\,\delta(q)/q^2 - 4\pi a_\nu^t(q)/q^2\}, \\ T_\nu(q) &= N\{-4\pi a_\nu^T(q)/q^2\}. \end{aligned} \tag{4.56}$$

Since our $|\Psi_{q_0}\rangle$ is a state with $J = 1/2$, we have:

$$-\lim_{q_0\to 0} f(\boldsymbol{q}_0 - \boldsymbol{q}_f) = {}^2a, \tag{4.57}$$

where ${}^2a$ is the doublet n-d scattering length. Thus, from Eq. (4.54), we have:

$${}^2a = \sum_\nu A_\nu a_\nu^t(0). \tag{4.58}$$

To get the equations for the functions $a_\nu^s(q)$, $a_\nu^t(q)$, $a_\nu^T(q)$, we insert the expressions (4.56) for s_ν, t_ν, T_ν into Eq. (4.33) and obtain the following system of inhomogeneous integral equations for the functions $a(q)$:

$$\sum_\mu \left\{ \frac{1}{q^2}\,[\mathcal{I}_{\nu\mu}^s(q) - \delta_{\nu\mu}/\lambda_\nu^s] a_\mu^s(q) + \int_0^\infty dq'\,[K_{\nu\mu}^{ss}(qq')\,a_\mu^s(q') + \right.$$

$$\left. + K_{\nu\mu}^{st}(qq') a_\mu^t(q') + K_{\nu\mu}^{sT}(qq') a_\mu^T(q')] \right\} = \frac{1}{2}\,\pi \sum_\mu K_{\nu\mu}^{st}(q0) c_\mu, \tag{4.59s}$$

---

[3] Only the part $\mathcal{D}^{-1}\Phi_1$ of Ψ constructed according to Eq. (4.45) gives a nonvanishing contribution to Eq. (4.54).

$$\sum_{\mu}\left\{\frac{1}{q^2}\left[\mathscr{F}^t_{\nu\mu}(q)-\delta_{\nu\mu}/\lambda^t_\nu\right]a^t_\mu(q)+\int_0^\infty dq'\left[K^{ts}_{\nu\mu}(qq')a^s_\mu(q')+\right.\right.$$

$$\left.\left.+K^{tt}_{\nu\mu}(qq')a^t_\mu(q')+K^{tT}_{\nu\mu}(qq')a^T_\mu(q')\right]\right\}=\frac{1}{2}\pi\sum_{\mu}K^{tt}_{\nu\mu}(q0)c_\mu, \tag{4.59t}$$

$$\sum_{\mu}\left\{\frac{1}{q^2}\left[\mathscr{F}^t_{\nu\mu}(q)-\delta_{\nu\mu}/\lambda^t_\nu\right]a^T_\mu(q)+\int_0^\infty dq'\left[K^{Ts}_{\nu\mu}(qq')a^s_\mu(q')+\right.\right.$$

$$\left.\left.+K^{Tt}_{\nu\mu}(qq')a^t_\mu(q')+K^{TT}_{\nu\mu}(qq')a^T_\mu(q')\right]\right\}=\frac{1}{2}\pi\sum_{\mu}K^{Tt}_{\nu\mu}(q0)c_\mu. \tag{4.59T}$$

The δ part of t_ν does not contribute to the left hand side of Eq. (4.59t), which is a consequence of Eq. (2.15) (notice that $\mathscr{F}^t_{\nu\mu}(0)=E_{\nu\mu}$). This is the mathematical reason for the possibility and necessity of imposing the boundary condition (4.51). Let us notice that by multiplying Eq. (4.59t) by q^2 we get:

$$\sum_{\mu}\left[\mathscr{F}^t_{\nu\mu}(0)-\delta_{\nu\mu}/\lambda^t_\nu\right]a^t_\mu(0)=0. \tag{4.60}$$

In the case of the deuteron wave function coefficients, c_ν, being determined uniquely, Eqs (4.60), (2.15) imply that

$$a^t_\mu(0)=\text{const.}\times c_\nu, \tag{4.61}$$

and Eq. (4.58), in turn shows that const. $=\,^2a$. Thus we have:

$$a^t_\mu(0)=c_\mu\,{}^2a. \tag{4.62}$$

Equations analogous to Eqs (4.60–4.61) hold for $a^T_\mu(0)$. In this case, however, the structure of the equations (4.59) causes that the constant in the analogue of Eq. (4.61) vanishes.

To determine the n-d doublet scattering length, ^2a, we have to solve the system of coupled inhomogeneous equations (4.59) for the functions $a^t_\nu(q)$, $a^t_\nu(q)$, $a^T_\nu(q)$. The doublet scattering length has then to be calculated from Eq. (4.58) (this requires knowledge of the solution of the system of linear equations (2.15)).

5. *Numerical procedure*

The numerical calculation of the ground state energy of the triton, E_T, and of the doublet n-d scattering length, ^2a, has been performed for the two-body potential described in Section 2, Eqs (2.4), (2.5), (2.10). In this case we have $N_t=1$, and we shall drop the subscripts ν, $\mu=1$ at all the triplet state quantities like t_ν, T_ν, $\mathscr{F}^t_{\mu\nu}$, a^t_ν, a^T_ν. Notice that in the special case $N_t=1$, we have only one coefficient $c_1=1$ in Eq. (2.14), and Eq. (4.62) is simplified to:

$$a^t(0)=\,^2a. \tag{5.1}$$

The quantities $\mathscr{I}^s_{\nu\mu}(q)$ and $\mathscr{I}^t_{\nu\mu}(q)$, Eq. (4.31), may be calculated analytically with the result:

$$\mathscr{I}^s_{11}(q) = -\frac{m}{8\pi k_0(\beta_s+k_0)^2}\, e^{-2\beta_s r_c}\left[1+\frac{k_0}{\beta_s}-e^{-2k_0 r_c}\right], \tag{5.2}$$

$$\mathscr{I}^s_{22}(q) = -\frac{m}{8\pi k_0 r_c^2}\,[1-e^{-2k_0 r_c}], \tag{5.3}$$

$$\mathscr{I}^s_{12}(q) = \mathscr{I}^s_{21}(q) = -\frac{m}{8\pi k_0 r_c(\beta_s+k_0)}\, e^{-\beta_s r_c}\,[1-e^{-2k_0 r_c}], \tag{5.4}$$

$$\mathscr{I}^t(q) = -\frac{m}{8\pi}\left\{\frac{1}{\beta_t(k_0+\beta_t)^2}+\frac{t_Y^2(5k_0^2+4k_0\gamma+\gamma^2)}{8\gamma(k_0+\gamma)^4}\right\}, \tag{5.5}$$

where

$$k_0 = \sqrt{\frac{3}{4}q^2 - Em}\ . \tag{5.6}$$

The integrals over the cosine of the angle between $\boldsymbol{q}$ and $\boldsymbol{q}'$ in the expression (4.34) for $K^{xy}_{\nu\mu}(qq')$ have been calculated numerically by means of the Simpson rule with an integration step of 0.1, although some of the integrations could have been performed analitically.

The integrals over q' in the systems of the integral equations (4.39) and (4 59) have been approximated by sums with the help of the Simpson rule, with the following meshes:

$$q' = 0.0\ (0.06)\ 1.08\ (0.15)\ 1.68\ (0.30)\ 2.88\ (0.60)\ 6.48\ \mathrm{fm}^{-1} \tag{5.7}$$

in the case of Eqs (4.39), and

$$q' = 0.0\ (0.03)\ 0.12\ (0.06)\ 0.48\ (0.15)\ 1.08\ (0.30)\ 2.28\ (0.60)\ 4.68\ (1.20)\ 11.88\ \mathrm{fm}^{-1} \tag{5.8}$$

in the case of Eqs (4.59). Both the meshes have been established empirically, *i. e.*, making them denser and shifting further the upper limits did not change the results for E_T and ${}^2a$ within the desired accuracy.

In this way we have reduced the system of integral equations (4.39) determining the triton ground state energy and wave function to a system of $4\times 33 = 132$ homogeneous linear equations with symmetric coefficients. If we take all the terms of Eqs (4.39) on the left hand side we may write these equations in the form of an eigenvalue problem:

$$\Omega\Gamma = \Lambda\Gamma \tag{5.9}$$

where Γ is a vector of the dimension 132 whose components are the values of the functions s_1, s_2, t, T in the mesh points, and Ω is a symmetric matrix of order 132. The solution representing the triton bound state corresponds to the eigenvalue $\Lambda = 0$. Thus, we have calculated the eigenvalues of (5.9) as functions of energy, $\Lambda = \Lambda(E)$, with the result that only one of the eigenvalues vanishes in a reasonable range of energies, E. This value of E represents the triton ground state energy, E_T, *i. e.*, we have

$$\Lambda(E_T) = 0. \tag{5.10}$$

The corresponding eigenvector of (5.9) gives the triton functions s_1, s_2, t, T. For determining the eigenvalues and eigenvectors of the big symmetric matrix Ω, we have used the program BIGMAT based on the Householder method. In a few cases we have checked our results by calculating directly the determinant of the homogeneous system of equations, $\Omega T = 0$, by means of the Gauss method, and by solving the system of equations for the value of $E = E_T$ for which the determinant vanishes.

In the case of Eqs (4.59), determining the scattering length, with the mesh (5.8) we have reduced these equations to a system of $4 \times 29 = 116$ linear inhomogeneous equations. Whereas $a_1^s(0) = a_2^s(0) = a^T(0) = 0$, special care had to be taken with the crucial value of $a^t(0) = {}^2a$. On the left hand side of Eq. (4.59t) we have the term

$$\frac{1}{q^2}\left[\mathscr{I}^t(q) - \frac{1}{\lambda^t}\right] a^t(q). \tag{5.11}$$

For $q = 0$ we have

$$\left[\mathscr{I}^t(0) - \frac{1}{\lambda^t}\right] = 0. \tag{5.12}$$

which is in the case of $N_t = 1$ the eigenvalue equation for the deuteron. Hence, for $q = 0$ we have replaced the term (5.11) by

$$\left[\frac{\partial}{\partial q^2}\,\mathscr{I}^t(q)\right]_{q=0} a^t(0). \tag{5.13}$$

Furthermore, instead of simply using the value of λ^t of Eq. (2.11), we had to calculate λ^t in our program from Eq. (5.12) to get reliable accuracy of our results. The system of 116 inhomogeneous linear equations (with the special treatment of the term (5.11) for $q = 0$) has been solved with the help of the Gauss method.

All the numerical calculations have been performed on the IBM 7044 computer of the Centro di Calcolo dell'Universita di Trieste.

6. *Results and discussion*

All the results of the present calculation are collected[4] in Table II. The dependence of the calculated values of the triton energy and of the neutron-deuteron doublet scattering length on the two-body potentials used is shown graphically in Figs 3 and 4, respectively. The functions $\tilde{s}_1$, $\tilde{s}_2$, $\tilde{t}$, $\tilde{T}$ and the functions a_1^s, a_2^s, a^t, a^T in the case of our best two-body potential are shown in Figs 5 and 6, respectively.

As may be seen from Fig. 3, the binding energy of the triton, $B = -E_T$, is diminished by both the tensor force and the hard shell repulsion. The effect, however, of the two factors is not additive. In the presence of the tensor force the hard shell repulsion is less effective

[4] Our pure central force result for $r_c = 0$, $E_T = -11.88$ MeV should coincide with the corresponding result of [12] which, however, is $E_T = -11.6$ MeV. A similar small discrepancy with the result of [12] has been noticed by Jaffe and Reiner [28]. This would suggest that the accuracy of the results of [12], obtained on a small computer, was not sufficient.

TABLE II

The values of E_T and ^2a calculated with and without tensor forces for different values of the hard shall radius, r_c

Pure central forces			With tensor forces	
r_c(fm)	E_T(MeV)	^2a(fm)	E_T(MeV)	^2a(fm)
0.00	−11.88	−1.51	−9.58	−0.15
0.10	−11.19	−0.93	−9.28	+0.15
0.15	−10.84	−0.66	−9.11	+0.30
0.25	−10.20	−0.18	−8.81	+0.58
0.40	−9.43	+0.43	—	—

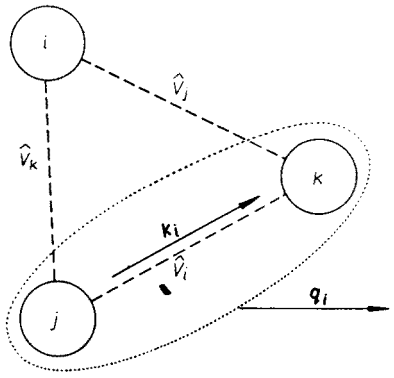

Fig. 2

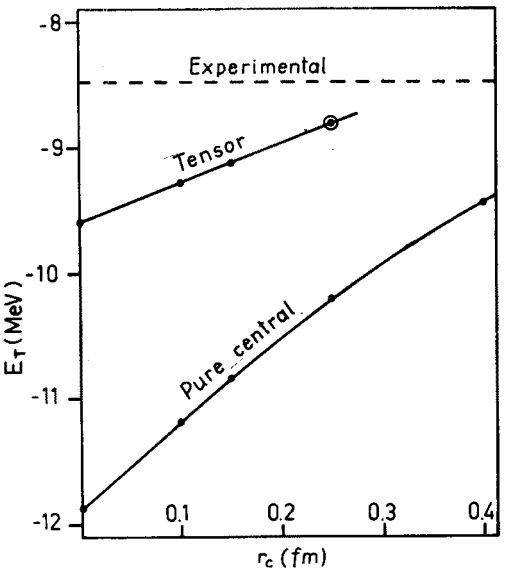

Fig. 3

Fig. 2. Momenta in the i-th natural system

Fig. 3. The value of E_T obtained with and without tensor forces as a function of the hard shell radius, r_c. Dots denote the calculated points. The dot within the circle corresponds to our best potential. The broken line indicates the experimental value of E_T

in diminishing B than it is in the absence of the tensor force. And *vice versa*, the effectiveness of the tensor force in diminishing B is reduced by the presence of the hard shell repulsion. Similar is the situation with the doublet scattering length, ^2a. As may be seen from Fig. 4, both the tensor force and the hard shell repulsion increase the resulting value of ^2a. Again, however, the effect of the two factors is not additive: the total effect is smaller than the sum of the two effects.

It is encouraging to see that the best results $E_T = -8.81$ MeV, is obtained with our best potential which contains the tensor force and a hard shell repulsion with the hard shell

radius $r_c = 0.25$ fm in the singlet state. Compared with the experimental value, $E_T = -8.48$ MeV, we get an overbinding of 0.33 MeV. There are several possibilities to account for the small difference, $\Delta = 0.33$ MeV, even without mentioning the role of the two-body interaction in states different from the $^1S_0$ and $^3S_1+^3D_1$ states considered in the present calculation[5]. Let us mention some of them:

(1) *The ambiguity in the strength of the tensor force.* Similarly as in other nuclear structure calculations, this ambiguity has its consequences in the ^{3}H problem. A measure of the strength of the tensor force is the probability of the D state in the deuteron, P_D. Our triplet potential leads to $P_D = 4\%$. According to Levinger [29], the minimum value of

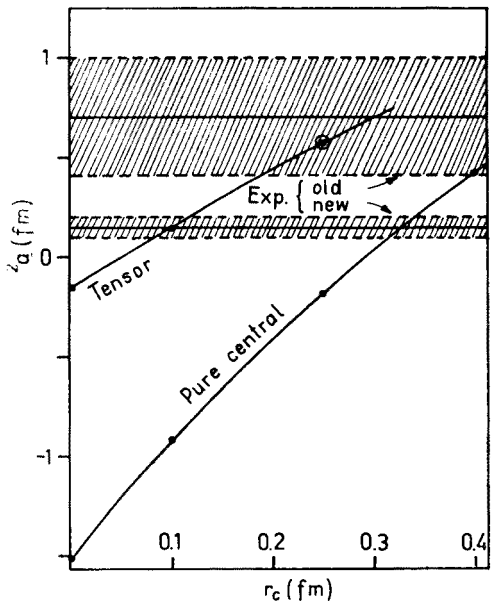

Fig. 4. The value of ^2a obtained with and without tensor forces as a function of the hard shell radius, r_c. Dots denote the calculated points. The dot within the circle corresponds to our best potential. The new [42] and the old [40] experimental values of ^2a are represented by the shaded areas

P_D consistent with the deuteron binding energy and quadrupole moment and with the effective range is $P_D = 0.45\%$, whereas the upper limit is probably about 7%. Within this range of admissible values of P_D we certainly could get a value of E_T which would coincide with the experimental value.

(2) *The ambiguity in the singlet effective range, r_{0s}.* Its value is probably between 2.5 and 2.7 fm (see, *e. g.*, [30]). According to [31], an increase in r_{0s} from our value of 2.5 to 2.7 fm causes a decrease in B by 0.44 MeV in the case of $r_c = 0$. In the presence of a repulsive core this decrease would probably be smaller [32], [33].

(3) *The short range repulsion in the triplet state.* This repulsion, not considered in the present calculation, certainly would reduce the ^{3}H binding (see [34]). A comparison of our

[5] According to [11] the neglected interaction in the ^1P, ^3P and ^1D states contributes energy of the order of only 0.01 MeV to E_T.

pure central force results with the results of [12] and [28] (where, however, a slightly different form of v_2 has been used) suggests that the reduction should not exceed a few tenths of one MeV (see also [35]).

(4) *Charge dependence of nuclear forces.* According to [12], as the result of the charge dependence one would expect in our pure central force case a reduction in binding of about 0.8 MeV, althoug in the presence of the tensor force the reduction might be smaller. It should be stressed, however, that it is difficult to separate from the reduction obtained in [12] the part due to the charge dependence from the part due to an increase in the singlet effective range.

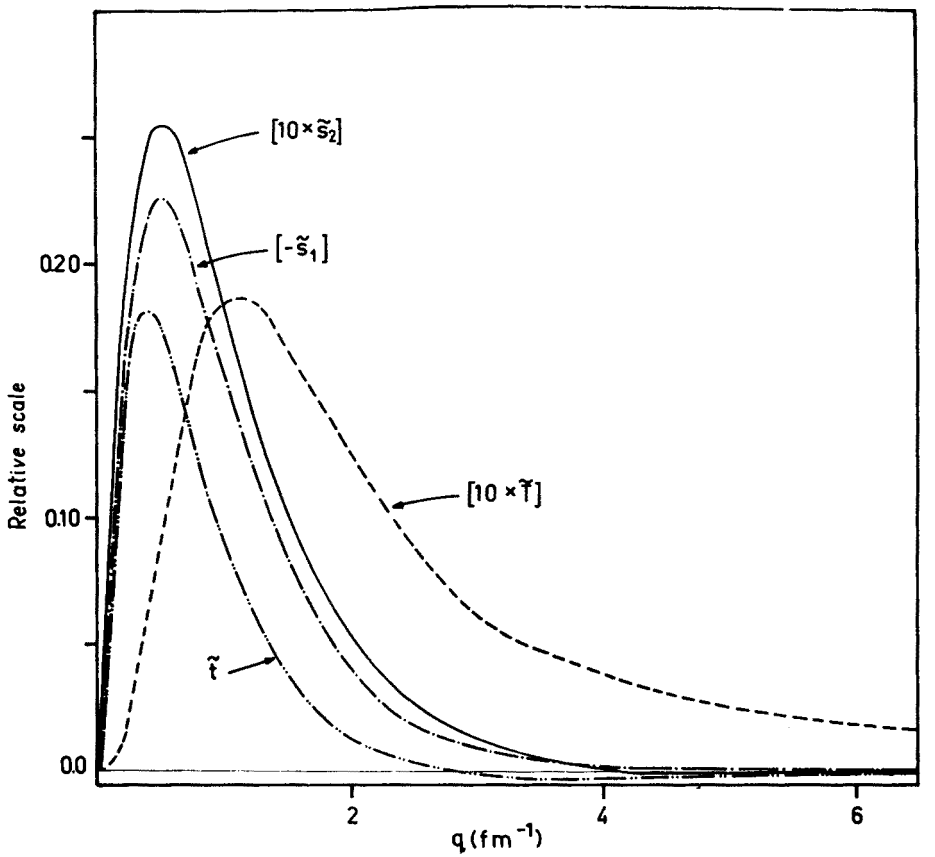

Fig. 5. The functions $\tilde{s}_1$, $\tilde{s}_2$, $\tilde{t}$, $\tilde{T}$ in the case of our best potential ($r_s = 0.25$ fm, central+tensor)

(5) *Off-shell effects.* By this we mean the ambiguity in choosing the two-body potential subject to the restriction that it should reproduce the experimental phase shifts and the properties of the deuteron. As shown in [36] on a model of triton, one may obtain different values of E_T by using different phase shift equivalent potentials. Thus, we may hope that by playing with the shape of the potential we could get a coincidence of the calculated value of E_T with the experimental value, without changing the fit to the two-body data.

(6) *Relativistic effects and three-body forces.* Some estimates [37] suggest that relativistic effects should increase the ^{3}H binding by about 0.25 MeV, althoug other estimates [38] suggest that they should reduce it by about 5%. Obviously, our knowledge of possible three--body forces and of their effect on E_T is extremely poor (see, *e. g.*, [39]).

Less satisfactory is the situation with the doublet scattering length, ^2a. For a long time, the old value,

$$^2a = 0.7 \pm 0.3 \text{ fm}, \tag{6.1}$$

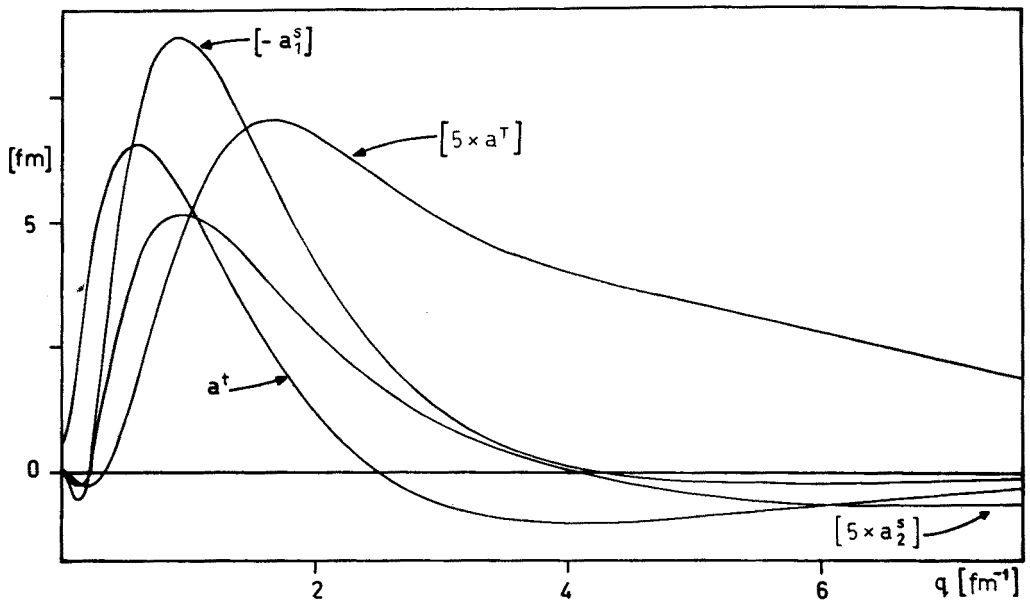

Fig. 6. The functions a_1^s, a_2^s, a^t, a^T in the case of our best potential ($r_c = 0.25$ fm, central+tensor)

determined by Hurst and Alcock [40] had been accepted as the experimental value of ^2a (the ambiguity of the two values of ^2a of [40] was resolved in [41]). The recent analysis of van Oers and Seagrave [42] has led to a new experimental value[6],

$$^2a = 0.15 \pm 0.05 \text{ fm}. \tag{6.2}$$

As may be seen from Table II and Fig. 4, we are able to reproduce both values of ^2a. However, with our best two-body potential, which also gives the best value for E_T, we get the value $^2a = 0.58$ fm which agrees with the old value (6.1) and disagrees with the new value (6.2). The situation is best represented in Fig. 7, in which — following Phillips [43] — we have plotted E_T *versus* ^2a.

If we accept the new experimental value of ^2a, we are faced with the impossibility of reproducing both E_T and ^2a with our two-body potential. In the discussion of our results for E_T we have pointed out several methods of getting the calculated value of E_T closer

[6] We quote this value after Ref. [1].

to the experimental value. Most of these methods would not work in improving the simultaneous agreement of E_T and ${}^2a$ with experiment. To get an idea of the effect of increasing the strength of the tensor force we have drawn straight broken lines through the points on Fig. 7 calculated for the same value of r_c with and without the tensor force. We see that all these lines point towards the old experimental value of ${}^2a$. This suggests that we cannot solve the E_T—${}^2a$ problem by changing the strength of the tensor force, as had been pointed out already by Phillips [43]. Exactly the same situation is encountered in the dependence of the position of the calculated points on the E_T—${}^2a$ plot on the magnitude of the singlet effective range r_{0s} [43]. A similar situation seems to occur with the off-shell effects. Namely, by investigating the position on the E_T—${}^2a$ plot of the points calculated with different forms of the separable two-body potentials, one finds that all the points stay close to a straight line (the Phillips

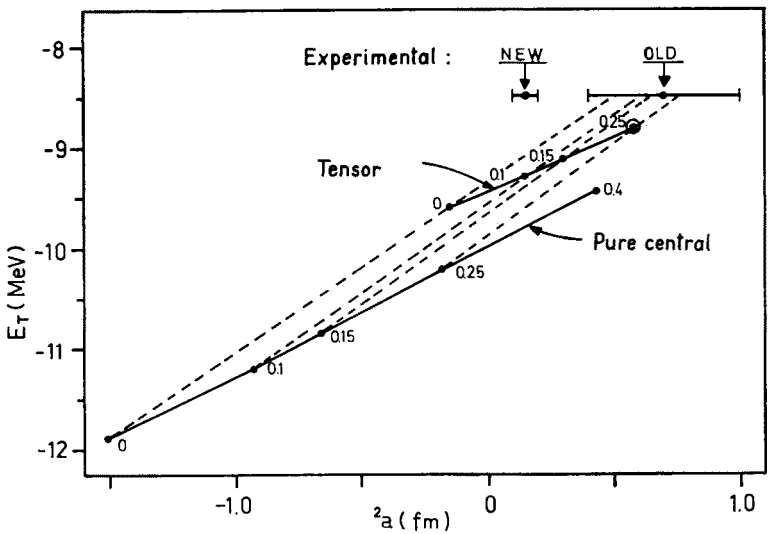

Fig. 7. The values of E_T and ${}^2a$ obtained with and without tensor forces. Dots denote the calculated points. The corresponding value of the hard shell radius, r_c, is indicated at each point (in fm). The dot within the circle corresponds to our best potential

line) which goes through the old experimental point. This conclusion has been confirmed recently by Brady, Harms, Laroze and Levinger [44] and is further supported by the collection of results by Shrenk, Gupta and Mitra [33].

The interesting question is whether the same situation persists in the case of local two-body potentials. Unfortunately, the error involved in the variational calculations with realistic local potentials does not allow one to answer this question on the ground of the variational results [5]. On the other hand, an exact calculation of ${}^2a$ and E_T with a realistic local potential is very difficult and not many results of such calculations have been obtained so far. However, the local potential results of Belyaev, Wrzecionko and Zubarev [45], and of Malfiet and Tjon [46] (mentioned in Ref. [44]) also stay on the Phillips line.

In looking for the solution of the E_T—${}^2a$ problem, let us discuss the effect of the short range repulsion in the triplet state. As mentioned before, this repulsion is expected to

have only a small effect on E_T, in particular in the presence of the tensor force. How would it affect ^2a? A direct calculation with a separable potential would certainly be possible. However, looking at the dependence of the position of the calculated points in Fig. 7 on the hard shell radius in the singlet state one might expect a similar dependence on the short range repulsion in the triplet state. This expectation is supported by the results obtained by Tjon [35]. Consequently, one should expect that the short range repulsion in the triplet state would shift the points in the E_T—^2a plot even more away from the new experimental point.

The fact that all the results obtained so far for E_T and ^2a stay on the Phillips line should not prevent us from investigating other forms of the two-body interaction. For instance, by inserting addidional terms of the two-body potential inside the hard shell one may leave the fit to the two-body data unchanged, but change the off-shell behaviour in such a way as to get off the Phillips line [47]. Also the role of higher partial waves, and of the spin-orbit force should be investigated. Furthermore, the effect of the charge dependence of nuclear forces deserves an investigation (the difference in the n—n and n—p singlet effective range might have a different effect on E_T and on ^2a). If these investigations would not lead to a solution of the E_T—^2a problem, and if the new experimental value of ^2a is confirmed by further experiments, then it might be an indication that the three-body forces and the relativistic effects play an essential role in the three-nucleon system.

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APPENDIX

Here, we collect the identities and relations which we have used in deriving Eqs (43.3) from Eqs (4.29) and (4.30). In all the equations we have $i \neq j \neq k \neq i$. Furthermore, we use the notation: $\hat{b} = \boldsymbol{b}/b$ (where $\boldsymbol{b}$ is an arbitrary vector), $A_k^{\sigma t} = \frac{1}{4}(3+\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j)$, $\boldsymbol{a}$ and $\boldsymbol{c}$ are the vectors defined in Eq. (4.32).

$$\sum_{k\neq i} \varepsilon_i\varepsilon_k |I\gg_k = \frac{1}{2}\,|0\gg_i - \frac{3}{2}\,|1\gg_i. \tag{A.1}$$

$$\sum_{k\neq i} \varepsilon_i\varepsilon_k \hat{S}_k(\boldsymbol{b})|1\rangle\rangle_k = -\hat{S}_i(\boldsymbol{b})|1\rangle\rangle_i. \tag{A.2}$$

$$\hat{A}_i^t \hat{S}_k(\boldsymbol{b})|1\rangle\rangle_k = 0, \tag{A.3}$$

which is a special consequence of the fact that by acting with $\hat{S}_k(\boldsymbol{b})$ on any three-nucleon spin state with $I^{\mathrm{tot}} = 1/2$ we get a state with $I^{\mathrm{tot}} = 3/2$.

$$\frac{1}{2\pi}\int d\varnothing\, \hat{S}_i(\boldsymbol{b}) = Q(\boldsymbol{bq})\hat{S}_i(\boldsymbol{q}), \tag{A.4}$$

where $\varnothing$ is the azimuthal angle of $\boldsymbol{b}$ measured around the direction of $\boldsymbol{q}$.

$$\frac{1}{2\pi}\int d\varnothing' \sum_{k\neq i} \varepsilon_i\varepsilon_k \hat{S}_i(\boldsymbol{c})\, [(\boldsymbol{\sigma}_i+\boldsymbol{\sigma}_j)\,(\boldsymbol{a}\times\boldsymbol{q}')]|1\gg_k = 0 \tag{A.5}$$

where $\varnothing'$ is the azimuthal angle of $\boldsymbol{q}'$ measured around the direction of $\boldsymbol{q}$.

$$\hat{S}_k(\boldsymbol{p})\hat{S}_k(\boldsymbol{q}) = -\hat{S}_k(\boldsymbol{p}) - \hat{S}_k(\boldsymbol{q}) - 2[1-Q(\boldsymbol{pq})]\hat{S}_k(\boldsymbol{p}\times\boldsymbol{q}) + \\ +8Q(\boldsymbol{pq})\hat{A}_k^{\alpha\tau} + 9i(\hat{p}\,\hat{q})\,(\boldsymbol{\sigma}_i+\boldsymbol{\sigma}_j)\,(\hat{p}\times\hat{q}). \tag{A.6}$$

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