ON THE ENERGY-MOMENTUM COMPLEX OF THE GRAVITATIONAL FIELD

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By considering the Riemann space of the general relativity theory as a space imbedded in a pseudo-Euclidean space $E_{10}(1, 9)$, new formalism may be constructed which allows to solve satisfactorily the energy-momentum complex problem of the geneal relativity theory and other interesting problems.

1.

It is known that the energy-momentum complex T^i_k of the general relativity theory must fulfil following conditions:

1. T^i_k is an affinite tensor density depending algebraically on the gravitational field variables and their derivatives and moreover, it satisfies the divergence condition:

$$T^i_{k,i} = \frac{\partial T^i_k}{\partial x^i} = 0. \tag{1.1}$$

2. For a closed system in an asymptotically flat space-time, the quantities

$$P_i = \frac{1}{c} \iiint\limits_{x^4 = \text{const.}} T^4_i dx^1 dx^2 dx^3 \tag{1.2}$$

aquire values which do not depend on time and transform as the covariant components of a 4-vector under the linear transformations of the space-time coordinates.

3. The four quantities $T^k = T^k_4$ transform as the 4-vector density under the space transformation of the following form

$$'x^a = f^a(x^b),\ 'x^4 = x^4\ (a, b, = 1, 2, 3). \tag{1.3}$$

These conditions should be fulfilled in order for the energy of the gravitational field to be localised.

Different energy-momentum tensor densities of the gravitational field have been given

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by Einstein [1], Møller [2] and by Landau and Lifshitz [3]. But those tensor densities do not fulfil all of the above mentioned conditions.

For example, the tensor density of Einstein fulfils only the first two conditions and Møller's complex fulfils in general only conditions 1 and 3.

By analysing these difficulties, Møller has shown [4] that the energy-momentum complex T^i_k fulfils all of the above mentioned conditions if and only if the Lagrangian $\mathcal{L}$ of the gravitational field satisfies the two following conditions:

a) $\mathcal{L}$ depends algebraically on the field variables and their derivatives and moreover is a homogenous quadratic function of the latter.

b) $\mathcal{L}$ is a true scalar density under arbitrary transformation of the space-time coordinates.

If we consider the components of the metric tensor as field variables, no such a simple Lagrangian can exist. However, as mentioned by Møller [5] such a Lagrangian exist if the tetrad formalism of the general relativity theory is used, in which one uses the tetrad quantities h^a_i; instead of the metric tensor g_{ik} for constructing the metric tensor.

By using this formalism, Møller constructed the energy-momentum complex T^i_k satisfying the three conditions 1, 2 and 3; but in his theory a new difficulty arises. To overcome this dificulty, Möller introduced additional conditions to the field equations and these conditions are not unique.

2.

In this part, following the point of view of Levi-Civita, we imbedd isometrically the Riemann space of the general relativity theory into a pseudo-Euclidean space.

Friedman [6] has proved that one may imbedd analytically and isometrically an arbitrary riemannian manifold $V_n(p, q)$ with the analytical metric into a pseudo-Euclidean space $E_m(r, s)$ where $m = \frac{1}{2}n(n+1)$ and the two integral numbers r, s satisfy the conditions $r \geqslant p$, $s \geqslant q$.

The Riemann space of the general relativity theory is of $V_4(1, 3)$ type and therefore, in general, the pseudo-Euclidean space is of $E_{10}(1, 9)$ type.

Let $\overset{\circ}{g}_{\mu\nu}$ be the metric tensor of the pseudo-Euclidean space with the following signature

$$\overset{\circ}{g}_{00} = -\overset{\circ}{g}_{11} = \cdots = -\overset{\circ}{g}_{99} = 1$$
$$\overset{\circ}{g}_{\mu\nu} = 0 \quad \text{if} \quad \mu \neq \nu \tag{2.1}$$

we assume that the Riemann space $V_4(1, 3)$ is given by ten following equations

$$z^\mu = f^\mu(x^i)$$
$$i = 0, 1, 2, 3 \tag{2.2}$$

from which we obtain the metric tensor of the Riemann space

$$g_{ik} = \overset{\circ}{g}_{\mu\nu} \frac{\partial f^\mu}{\partial x^i} \cdot \frac{\partial f^\nu}{\partial x^k}. \tag{2.3}$$

Let g^{ik} be the inverse of g_{ik}, that is

$$g^{ik} g_{kj} = \delta^i_j.$$

Let us define

$$h_i^\mu \stackrel{\text{def}}{=} \frac{\partial f^\mu}{\partial x^i}$$

$$h_{\mu i} = \overset{\circ}{g}_{\mu\nu} h_i^\nu$$

$$h^{\mu i} = g^{ik} h_k^\mu$$

$$h_\mu^i = g^{ik} h_{\mu k}.$$

we obtain

$$h_\mu^i h_k^\mu = \delta_k^i$$

$$h_\nu^k h_k^p = \delta_\nu^k$$

$$h^{\mu i} h_\mu^k = g^{ik}$$

and

$$h_{i\mu} h_k^\mu = g_{ik}.$$

It is easily seen that the metric tensor g_{ik} is invariant under generalized Lorentz transformations in $E_{10}(1,9)$:

$$h_i^\mu \to \Omega_\nu^\mu h_i^\nu \tag{2.4}$$

where Ω_ν^μ are the transformations parameters which do not depend on the coordinates.

As is well known, in the tetrad formalism (4) g_{ik} is invariant under local Lorentz-rotations of the form

$$h_i^a \to \omega_b^a(x) h_i^b$$
$$a, b = 0, 1, 2, 3, \tag{2.5}$$

where $\omega_b^a(x)$ are the rotations parameters depending on the local coordinates x^i.

This is the fundamental difference between ocurs and Møller's tetrad formalism.

Let $\mathscr{L}$ and $\mathscr{L}_{(m)}$ be the Lagrangian of the gravitational field and of the matter respectively. They are functions of h_i^μ and of the derivatives of the latter.

The variational principle

$$\delta \int (\mathscr{L} + \mathscr{L}_{(m)}) dx = 0$$

yields the following field equations

$$\frac{\delta \mathscr{L}}{\delta h_i^\nu} + \frac{\delta \mathscr{L}_{(m)}}{\delta h_i^\nu} = 0$$

Let us denote

$$\frac{\delta \mathscr{L}_{(m)}}{\delta h_i^\nu} = \mathfrak{T}_\nu^i$$

or

$$h_k^\nu \frac{\delta \mathscr{L}_{(m)}}{\delta h_i^\nu} = \mathfrak{T}_k^i.$$

The field equations become

$$h^{\nu}_{k}\frac{\delta\mathcal{L}}{\delta h^{\nu}_{i}}=-\mathfrak{T}^{i}_{k}. \tag{2.6}$$

Clearly we can write expressions for $\mathfrak{T}^{i}_{k}$ as follows

$$\mathfrak{T}^{i}_{\mu}=\frac{\partial}{\partial x^{l}}\frac{\partial\mathcal{L}_{(m)}}{\partial h^{\mu}_{i,l}}-\frac{\partial\mathcal{L}_{(m)}}{\partial h^{\mu}_{i}}.$$

Our main problem is to find the gravitational field Lagrangian $\mathcal{L}$ which satisfies the two conditions a) and b) mentioned at § *1*.

In this formalism, the scalar curvature R may be written in the following form

$$R=G+W^{i}_{,i} \tag{2.7}$$

where W^{i} is a vector density and G has the following form

$$G=\sqrt{-g}(\gamma_{ijk}\gamma^{kji}-\Phi_{r}\Phi^{r}, \tag{2.8}$$

where

$$\gamma_{ijk}=h_{\mu i}h^{\mu}_{j;k}=\gamma_{jik}$$
$$\Phi_{r}=\gamma^{i}_{ri} \tag{2.9}$$

we can now choose the Lagrangian density $\mathcal{L}$ of the gravitational field as follows:

$$\mathcal{L}=\frac{1}{2x}G. \tag{2.10}$$

It is clear that this Lagrangian is analogous to Møller's [4]. Using this Lagrangian we obtain the following field equations:

$$\frac{\delta\mathcal{L}}{\delta h^{\mu}_{i}}=\frac{\delta R}{\delta g^{rs}}\cdot\frac{\delta g^{rs}}{\delta h^{\mu}_{i}}=-G_{rs}(h^{r}_{\mu}g^{si}+h^{s}_{\mu}g^{ri})$$

or

$$h^{\mu}_{k}\frac{\delta\mathcal{L}}{\delta h^{\mu}_{i}}=-2\,G^{i}_{k}$$

and therefore

$$G^{i}_{k}=x\mathfrak{T}^{i}_{k} \tag{2.11}$$

which is simply Einstein's equation.

The Einstein energy-momentum complex becomes:

$$T^{i}_{k}=\mathfrak{T}^{i}_{k}+\Theta^{i}_{k},$$

where

$$\theta^{i}_{k}=\frac{\partial\mathcal{L}}{\partial h^{\mu}_{l\,i}}h^{\mu}_{l,k}-\delta^{i}_{k}\mathcal{L}.$$

From the condition that $\mathscr{L}$ is a true scalar density, *i.e.* from

$$\left(h_i^\mu \frac{\delta \mathscr{L}}{\delta h_k^\mu}\right)_{,k} - \frac{\delta \mathscr{L}}{\delta h_l^\mu} h_{l,i}^\mu = 0$$

we can deduce that the energy-momentum complex T_k^i satisfies all the above mentioned conditions 1, 2 and 3 at §*1*.

A specific feature of this formalism is that the Lagrangian $\mathscr{L}$ defined by (2.7) and by (2.8) as well as T_k^i and the field equations (2.11) are invariant under the generalized Lorentz-rotation (2.4).

Therefore it is possible that we may not need to add additional conditions to the field equations.

In the tetrad formalism the Lagrangian $\mathscr{L}$ is only invariant under local Lorentz-rotations with the constant rotation parameters depending on the coordinates at each point. Such Lorentz-rotations do not exist in our formalism.

3.

Let us now examine the relations between g_{ik} and h_i^μ. In the case of a weak gravitational field

$$g_{ik} = \overset{\circ}{g}_{ik} + \eta_{ik}(x) \tag{3.1}$$

where $\eta_{ik}(x)$ are small. On performing, if necessary, a coordinate transformation, we may choose

$$h_i^\mu = \delta_i^\mu + \frac{1}{2}\eta_i^\mu(x). \tag{3.2}$$

In Møller's tetrad formalism, we obtain relations analogous to (3.2), only if the coordinate system is harmonic. Equations (2.2) defining $V_4(1,3)$ should then have the following form:

$$f^\mu(x^i) = L^\mu(x^i) + \varepsilon^\mu(x^i) \tag{3.3}$$

where $L^\mu(x^i)$ are linear functions of x^i and $\varepsilon^\mu(x^i)$ are small differentiable functions.

Geometrically, the functions $f^\mu(x^i)$ differ from the equations of a four-dimensional hyperplane by very small quantities.

In the case when the Riemann space $V_4(1,3)$ is of Friedmann's type with the metric given by

$$d\sigma^2 = A(s)(c^2dt^2 - d\vec{r}^2) \tag{3.4}$$

where

$$s = \sqrt{c^2t^2 - r^2},$$

the equations (2.2) are simply the parameter equations of a four-dimensional sphere in $E_{10}(1,9)$. Hence we may easily define the quantities h_i^μ.

It is well known that in this case, such a pseudo-Euclidean space needs to have only five dimensions.

Instead of equations (2.2) we have the following equation defining the Friedmann space:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_0^2 = a^2(x_0) \tag{3.5}$$

Hence, quantities h_i^μ are defined uniquely.

As the next case, let us study the Schwarzchild metric

$$d\sigma^2 = \left(c^2 - \frac{2\,km}{r}\right) dt^2 - r^2 \left(\sin^2\theta d\varphi^2 + d\theta^2 - \frac{dr^2}{1 - \frac{2\,km}{c^2 r}}\right. \tag{3.6}$$

we obtain

$$h_i^\mu = \sqrt{g_{ii}\delta_i^\mu}. \tag{3.7}$$

Let us finally study the conditions at infinity:

$$g_{ik} \approx \overset{\circ}{g}_{ik}$$

we have then

$$f^\mu(x^i) \approx L^\mu(x^i)$$

where $L^\mu(x^i)$ are linear functions.

4.

The problem of spinor equations in the gravitational fields has been studied by many authors [7], [8] and [9] by means of the tetrad formalism.

We shall now present spinor equation in our formalism. Some of the results are presented in [10]. There are two methods of establishing the spinor equations in the gravitational field. One of these has been developed by Arbuzov and Filippov [11] in analogy with Ivanenko-Fock method [7], we shall present here the other method.

Let ψ be a spinor in the tangent space of $V_4(1, 3)$ we assume that there exists at any point a spinor φ induced by ψ and connected with ψ by the following relation

$$\psi = S\varphi \tag{4.1}$$

where S is a matrix called the inducing matrix.

From the relation between the vector of the tangent space and its inducing vector

$$h_i^\mu V_\mu = v_i$$

we infer that

$$f_i^\mu \frac{\partial\psi}{\partial z^\mu} = S\left(\frac{\partial}{\partial x^i} - G_i\right)\varphi \tag{4.2}$$

where

$$G_i = -S^{-1}\frac{\partial S}{\partial x^i}$$

with S^{-1} defined by

$$SS^{-1} = 1.$$

It is easily seen that the operator

$$D_i = \frac{\partial}{\partial x^i} - G_i$$

is the covariant derivative of a spinor in $V_4(1,3)$. We call G_i the spinor connexion. We shall now study its properties. Let Γ^μ be the Dirac matrices in $E_{10}(1,9)$:

$$\Gamma^\mu\Gamma^\nu+\Gamma^\nu\Gamma^\mu=2\overset{\circ}{g}{}^{\mu\nu} \tag{4.3}$$

and γ^i be the Dirac matrices in $V_4(1,3)$

$$\gamma^i\gamma^k+\gamma^k\gamma^i=2g^{ik}, \tag{4.4}$$

We can easily see that

$$h_i^\mu\Gamma_\mu=S\gamma_iS^{-1}$$

or

$$\Gamma_\mu=h_\mu^iS\gamma_iS^{-1}$$

from which we have

$$O=\frac{\partial\Gamma^\mu}{\partial z^\nu}=h_\nu^k\frac{\partial}{\partial x^k}(h_i^\mu S\gamma^iS^{-1})$$

or

$$h_i^\mu\frac{\partial\gamma^i}{\partial x^k}+\frac{\partial h_i^\mu}{\partial x^k}\gamma^i+h_i^\mu\gamma^i\frac{\partial S^{-1}}{\partial x^k}S+h_i^\mu S^{-1}\frac{\partial S}{\partial x^k}\gamma^i=0$$

and therefore, we obtain

$$\frac{\partial\gamma^j}{\partial x^k}+h_\mu^j\frac{\partial h_j^\mu}{\partial x^k}\gamma^{j'}+\gamma^j\frac{\partial S^{-1}}{\partial x^k}S+S^{-1}\frac{\partial S}{\partial x^k}\gamma^j=0.$$

It is easily seen that

$$G_k=-S^{-1}\frac{\partial S}{\partial x^k}=\frac{\partial S^{-1}}{\partial x^k}S$$

we finally obtain the following equations:

$$\frac{\partial\gamma^i}{\partial x^k}+\Gamma_{jk}^i\gamma^j+\gamma^iG_k-G_k\gamma^i=0 \tag{4.6}$$

which allow to define G_k.

We can easily find that the quantities

$$G_k=\frac{1}{4}g_{ab}\left[\frac{\partial h_i^\varrho}{\partial x^k}h_\varrho^b-\Gamma_{ik}^b\right]S^{ai}+a_k\cdot 1$$

are the solutions of (4.6), where $S^{ab}=\frac{1}{2}(\gamma^a\gamma^b-\gamma^b\gamma^a)$ and a^k are arbitrary constants.

Let us now study the transformation properties of G_k under the transformations of the space-time coordinates.

Let us assume the following spinor transformations

$$'\psi=\Sigma\psi$$

$$'\varphi=\sigma\varphi.$$

Therefore

$$'\psi = 'S'\varphi = \Sigma S\varphi$$

hence

$$'S = \Sigma S\sigma^{-1}$$

which gives us

$$'S^{-1}\frac{\partial' S}{\partial' x^k} = \frac{\partial x^i}{\partial' x^k}\sigma S^{-1}\frac{\partial S}{\partial x^i}\sigma^{-1} + \frac{\partial x^i}{\partial' x^k}\sigma S^{-1}\frac{\partial \sigma^{-1}}{\partial x^i}\sigma^{-1}$$

we have finally

$$'G_k = \frac{\partial x^i}{\partial' x^k}\sigma G_i\sigma^{-1} + \frac{\partial x^i}{\partial' x^k}\sigma g_i\sigma^{-1} \tag{4.7}$$

where we have denoted

$$g_i = S^{-1}S\frac{\partial \sigma^{-1}}{\partial x^i}\,.$$

5.

To conclude we suggest that our formalism may be useful in studying some problems of the general relativity theory.

On the basis of above results we can conclude that this formalism can enable us to construct satisfactorily the energy-momentum complex to fulfil all conditions of § *1*. Moreover the connexion between the metric tensor g_{ik} and the h_i^μ quantities has been found correctly.

Finally, the spinor equations in the gravitational field may be established in a simple way with the aid of this formalism.

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