FINAL STATE INTERACTIONS IN π^- MESIC DECAYS OF HYPERNUCLEI

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Final state interactions in the three and four body π^- mesic decays of hypernuclei are considered and a general formalism is developed which enables calculation of the angular and energy distribution. The distorted wave impulse approximation is applied to account for final state interactions. Spin effects, many channels, Coulomb interaction are discussed in detail. The paper contains only the derivation of the basic formulae, a comparison with experiment will be given in the second part of this work.

1. *Introduction*

Since the early days of research on hypernuclear structure there has been parallel interest [1], [2] in hypernuclear π^- mesic decay processes. The decays by π^- emission are known to usually undergo by several modes leading to multiparticle final states. The measurements of the branching ratios for various decay modes, energy distributions and angular correlations are extremely useful as they bear directly on the structure of hypernuclei. In particular, valuable information is provided for the determination of the spin values for hypernuclei (theoretical results are reviewed in Ref. [3] [4]; experimental data may be found in [5], [6], [7]).

Owing to the simple form of the leading, effective Λ-decay interaction, the decay rate is essentially given by the overlap between the Λ-hypernuclear wavefunction with the multiparticle final state function. The comparison of the decay rate with experiment is thus a test of our understanding of both — the effective ΛN and ΛNN forces which determine the Λ-hypernuclear wavefunction, and the final state interactions between nuclear fragments which may form various intermediate resonant states. A reliable evaluation of the decay rate with realistic hypernuclear wavefunctions, taking into account all of the final state interactions would be a very complicated many body problem of nuclear physics.

In order to calculate the total π^-decay rate various approximation schemes have been developed in the literature. The first ingredient of the overlap integral, the Λ-hypernuclear

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wavefunction, has been represented in a simplified form, sufficiently flexible, with parameters adjusted to the known Λ binding energy. In the evaluation of the total rate the summations over the final multinucleon states have been accomplished by means of the closure approximation (for details the reader is referred to Refs [3], [4].

The second physical factor, the final state interactions have been accounted for by several methods, by making use of the Watson theorem [8], by representing the interactions in the form of a square well potential [9], by using the zero range approximation [10], or by solving numerically the appropriate Schroedinger equation [11].

In this work we shall concentrate our attention on the decay spectra and no attempt will be made to calculate the lifetimes and branching ratios for various decay modes. Our goal is to formulate a theory of final state interactions capable of explaining the observed angular and energy distributions for three and four body decays, basing on an extremely simple model of the decaying hypernuclei. We shall assume that the hypernucleus is a two body (Λ+core), or a three body (Λ+ two clusters) system, ignoring all of the intrinsic structure of the "particles". On the other hand, much effort has been made in order to incorporate the final state interactions among the decay products. Our aim is to check how much of the experimental spectra can be explained with a comparatively simple hypernuclear wavefunction, but with all available information about the final state interaction used as the input. In our model the different decay modes are described as the exit channels of the reaction proton+core $\rightarrow$ anything.The pion, in these considerations, is regarded as a noninteracting particle, what may be justified by the fact that the phase shifts for π^{-}P scattering are small in the relevant energy range. In those cases when the core is not stable and no information about P+core reaction is available, we rely on the cluster model of the initial state. The philosophy of the present approach is to reduce a genuine many body problem to a few body problem which can be given a more refined treatment.

The hypernuclear wavefunction is assumed to be a product of correlation factors of Gaussian shape with parameters adjusted to the binding energy. This form enables to carry out analytically the overlap integrals even if the final state interaction is taken into account.

The final state wavefunction has the form of a distored wave. The analytic continuation of the corresponding Schroedinger equation from the asymptotic region inwards has been made by means of a number of auxiliary phenomenological potentials. The solution in the interaction region is essentially of the form of a plane wave multiplied by an enhancement factor which carries the information about the interaction. This form is very advantegeous in that it may be immediately generalized to a many channels situation. The inclusion of many exit channels does not introduce any additional complications in the wavefunction, all changes being contained merely in the generalized enhancement factors. The enhancement factors are determined by adjusting the appropriate scattering parameters to the data.

In principle, the four body decay cannot be treated along the above described lines, as that would require knowledge of the wavefunction describing a process of the type P+ core $\rightarrow$ three body state. Therefore, four body decays, if they compete with three body final states, may be only crudely estimated by introducing absorption in the three body channel, simulating the existence of four body channels.

Some of the four body decay modes, however, are still feasible in the present approach. We shall be concerned with those four body decays in which the core can be reasonably well represented as a system of two clusters. The four body decay is interpreted as a disintegration of the core into two clusters, caused by the decay of the boud Λ into the proton and the pion. The clusters, assumed to be structureless particles, are likely to interact strongly both with the final state proton, and among themselves. We shall develop an approximation scheme in which all three pairs of secondary interactions can be accounted for, including rescattering effects.

Spin effects will be considered with great care. The wavefunctions in the continuous spectrum for particles with spins are constructed in the form of spin operators acting on spinors. This formalism considerably simplifies various overlap integrals and spin summations, the latter being reduced to the standard calculations of traces.

In this paper we are going to present only our basic formulae, the detailed comparison with experiment will be given in the second part of this work.

2. *Phase space and kinematics for three body decays*

We shall consider three and four body decays. Let us remind then, at the begining, that for a N body decay 3 N-7 kinematical variables are required in order to provide a kinematically complete description of the decay. The number 3 N-7 appears as follows. We have $3N$ momenta but not all are independent as the energy-momentum conservation yields four constrains. Three further variables may always be eliminated by chosing the coordinate system in a convenient way by performing a suitable rotation. Hence, for a three body decay we have two variables, for a four body decay there are five variables, *etc.*

For a three body decay instead of the laboratory momenta $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_3$ it is convenient to introduce vectors $\boldsymbol{q}_{12}$, $\boldsymbol{k}_3$, $\boldsymbol{P}$ defined as

$$\boldsymbol{q}_{12} = \frac{m_2\boldsymbol{p}_1 - m_1\boldsymbol{p}_2}{m_1+m_2}, \tag{1a}$$

$$\boldsymbol{k}_3 = [m_3(\boldsymbol{p}_1+\boldsymbol{p}_2) - m_1+m_2)\,\boldsymbol{p}_3]/M, \tag{1b}$$

$$\boldsymbol{P} = \boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3;\ M = m_1+m_2+m_3, \tag{1c}$$

$\boldsymbol{q}_{12}$ is the relative momentum of the pair (1,2), $\boldsymbol{k}_3$ is the momentum canonically conjugated to the separation of particle 3 from the (1, 2) center of mass, and P is the total momentum of the system. In the laboratory frame (overall center of mass) $\boldsymbol{P} = 0$. The laboratory momenta are expressed by $\boldsymbol{q}_{12}$, $\boldsymbol{k}_3$ and $\boldsymbol{P}$ as follows

$$\boldsymbol{p}_1 = \boldsymbol{q}_{12} + \frac{m_1}{m_1+m_2}\boldsymbol{k}_3 + \frac{m_1}{M}\boldsymbol{P}, \tag{2a}$$

$$\boldsymbol{p}_2 = -\boldsymbol{q}_{12} + \frac{m_2}{m_1+m_2}\boldsymbol{k}_3 + \frac{m_2}{M}\boldsymbol{P}, \tag{2b}$$

$$\boldsymbol{p}_3 = -\boldsymbol{k}_3 + \frac{m_3}{M}\boldsymbol{P}. \tag{2c}$$

We shall also need the expressions for the total kinetic energy E_{kin} and the relative (2,3) momentum $\boldsymbol{q}_{23}$

$$E_{\text{kin}} = \sum_{i=1}^{3} \frac{\boldsymbol{p}_i^2}{2m_i} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{q_{12}^2}{2} + \left(\frac{1}{m_1+m_2} + \frac{1}{m_3}\right)\frac{k_3^2}{2} + \frac{1}{M}\frac{p^2}{2}, \tag{3}$$

$$\boldsymbol{q}_{23} = -\frac{m_3}{m_2+m_3}\boldsymbol{q}_{12} + \left[1 - \frac{m_1 m_3}{(m_1+m_2)(m_2+m_3)}\right]\boldsymbol{k}_3. \tag{4}$$

Henceforth we shall write $\boldsymbol{q}$ and $\boldsymbol{k}$ for $\boldsymbol{q}_{12}$ and $\boldsymbol{k}_3$ respectively.

We will consider now the decay hypernucleus $\to$ core $+P+\pi$ (the core, the proton, and the pion are labeled 1, 2 and 3, respectively) where the pion may need relativistic treatment. Using relativistic kinematics for the pion, we obtain

$$\begin{aligned} E_{\text{kin}} &= \frac{1}{2\mu_{cp}} q^2 + T\left[1 + \frac{T+2m_\pi}{2(m_p+m_c)}\right] \\ &\approx \frac{1}{2\mu_{cp}} q^2 + T\left(1 + \frac{m_\pi}{m_p+m_c}\right), \end{aligned} \tag{5}$$

where μ_{cp} is P-core reduced mass and T is pion kinetic energy,

$$T = (k^2+m^2)^{\frac{1}{2}} - m_\pi \approx \frac{1}{2m_\pi} k^2 + 0(k^4). \tag{6}$$

The three body phase space volume element is

$$\begin{aligned} d\varrho(3) &= \delta(E_{\text{kin}} - \varDelta E) d^3p_1, d^3p_2 d^3p_3 \\ &= \delta(E_{\text{kin}} - \varDelta E) d^3k d^3q, \end{aligned} \tag{7}$$

where $\varDelta E$ is the energy released in the decay. The decay is, as we know, described by two variables, and as our two kinematic variables we introduce s, the squared total energy of the pion, and t, the squared core recoil momentum

$$s = k^2 + m_\pi^2 = (T+m_\pi)^2, \tag{8a}$$

$$t = p_c^2 = \left(\boldsymbol{q} + \frac{m_c}{m_c+m_p}\boldsymbol{k}\right)^2. \tag{8b}$$

In terms of these variables the phase space volume element takes a very simple form

$$d\varrho(3) = \text{const}\ ds\, dt \Theta[-G(s, t)], \tag{9}$$

where the step function Θ has been introduced for indicating the limits of integration. The function $G(s, t)$ and the limits of integration are obtained from the condition that the cosine of the angle θ between vectors $\boldsymbol{q}$ and $\boldsymbol{k}$ is less, or equal to unity, *i.e.*,

$$\cos^2\theta = \left[t - q^2 - \left(\frac{m_c}{m_c+m_p}\right)^2 k^2\right]^2 \Bigg/ \left(\frac{2qkm_c}{m_c+m_p}\right)^2 \leqslant 1.$$

The above condition is cast into the form

$$G(s,t) \equiv \left[t-q^2-\left(\frac{m_c}{m_c+m_p}\right)^2 k^2\right]^2-\left(\frac{2qkm_c}{m_c+m_p}\right)^2 \leqslant 0,$$

where k^2 and q^2 are functions of s

$$k^2 = s-m_\pi^2 \tag{10}$$

$$q^2 = 2\mu_{cp}\left[\Delta E-\frac{s-m_\pi^2}{2(m_c+m_p)}-\sqrt{s}-m_\pi)\right]. \tag{11}$$

The condition $G(s, t) = 0$ gives the equation of the boundary curve in the (s, t) plane. A decay event is represented as a point in the (s, t) plane and all of the events should fall in the region of space surrounded by the boundary curve $G(s, t) = 0$. This curve is quadratic in t and the solution of the equation $G(s, t) = 0$, for a fixed s, can be obtained immediately:

$$t = t_\pm(s) = \left(q\pm\frac{m_c}{m_c+m_p}k\right)^2, \tag{12}$$

where k and q are given by (10) and (11). Formula (12) gives the limits of integration over t.

It is somewhat more difficult to determine the integration limits over s because $\sqrt{s}$ enters the expression for q^2. Although no simple analytic expression can be obtained for the integration limits $(s_\pm)$ over s, the boundary curve is determined completely by folding together the two pieces $t_+(s)$ and $t_-(s)$; thus, the integration limits $s_\pm$ may be read off from the plot.

3. *Plane wave impuls approximation (PWIA)*

Consider now the dynamics of the decay hypernucleus $\to$ core $+ P + \pi^-$. The simplest approximation one can make is to assume that the decay of the Λ-particle is quasifree, which differs from the free Λ decay by that the Λ is not stationary but has momentum distribution appropriate for the hypernuclear bound state. The core, which is assumed to be undistorted by the Λ, is only a spectator during the decay; the decay products are assumed to move freely. The effective Λ decay interaction leads to the hypernucleus decay matrix element

$$\mathcal{M}^\pm = P^{(\pm)}\int \psi_\Lambda^*(|\boldsymbol{r}_c-\boldsymbol{r}_\Lambda|)\left(S+P\frac{\boldsymbol{\sigma}\cdot\boldsymbol{q}_{\pi p}}{q_\Lambda}\right)e^{i\boldsymbol{p}_c\cdot\boldsymbol{r}_c}\,e^{i\boldsymbol{p}_p\cdot\boldsymbol{r}_p}\,e^{i\boldsymbol{p}_\pi\cdot\boldsymbol{r}_\pi}\times$$

$$\times\,\delta(\boldsymbol{r}_\Lambda-\boldsymbol{r}_p)\,\delta(\boldsymbol{r}_\Lambda-\boldsymbol{r}_\pi)\,\delta\left(\frac{m_c\boldsymbol{r}_c+m_p\boldsymbol{r}_p+m_\pi\boldsymbol{r}_\pi}{m_c+m_p+m_\pi}\right)d^3r_\Lambda d^3r_c d^3r_p d^3r_\pi, \tag{13}$$

where $P^{(\pm)}$ are spin projection operators, projecting the two possible spin values of the hypernucleus $J_N\pm\frac{1}{2}$ (J_N — spin of the core nucleus), ψ_Λ is the s-state wavefunction of the Λ-core relative motion, S and P are the s and p-wave amplitudes of the free Λ decay[1], q_Λ is

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[1] The P/S ratio is $P/S = 0.38\pm0.01$ and the relative phase is consistent with zero (*cf.* Ref. [14]).

the relative πp momentum for a free Λ decay, and $\boldsymbol{q}_{\pi p}$ is the relative πp momentum. The first two delta factors are necessary since the weak interaction is of the contact type. The third delta insures us that we are performing the calculation in the overall c.m. In order to simplify the integral in (13) it is a customary procedure to express $\boldsymbol{q}_{\pi p}$ in terms of $\boldsymbol{q}$ and $\boldsymbol{k}$ and neglect all terms of order of m_π/m_p. From formula (4) one has then $\boldsymbol{q}_{\pi p} \approx \boldsymbol{k}$, which means that in formula (13) terms $(m_\pi/m_p)\nabla\psi_\Lambda$ can be neglected, and the decay matrix element may be taken out of the integral. On performing the trivial integrations with delta functions one has

$$\mathcal{M}^{\pm} = P^{(\pm)}\left(S + P\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{q_\Lambda}\right)\int \psi_\Lambda^*(r)e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\, e^{i\frac{m_c}{m_c+m_p}\boldsymbol{k}\cdot\boldsymbol{r}}\, d^3r. \tag{14}$$

The overlap intergral in the above expression will be denoted by $Q^{(0)}$ and may be written as

$$Q^{(0)}(t) = \int \psi_\Lambda^*(r)e^{i\boldsymbol{p}_c\cdot\boldsymbol{r}}d^3r = \tilde{\psi}_\Lambda^*(|\boldsymbol{p}_c|), \tag{15}$$

where $\boldsymbol{p}_c$ is the core recoil momentum (*cf.* (2a)). Having in mind Eq. (8b), we see that in terms of PWIA the overlap integral (15) depends only on t and does not depend on s. The integral (15) is equal to the Fourier transform of the Λ wavefunction in the hypernucleus. The partial decay rate $\tau^{\pm}$ is obtained by squaring the modulus of $\mathcal{M}^{\pm}$ and summing over spins

$$\frac{\partial^2\tau^{\pm}}{\partial s\partial t} = \text{const}\ \Theta\,[-G(s,t)]\left(|S|^2 + |P|^2\frac{k^2}{q_\Lambda^2}\right)|Q^{(0)}(t)|^2. \tag{16}$$

Since the decay rate, in this approximation, does not make any distinction between the initial spin being $J_N-\frac{1}{2}$, or $J_N+\frac{1}{2}$, we shall drop the $\pm$ labels. As seen from formula (16), the rate is a product of three factors, the kinematical factor $\Theta[-G(s,t)]$, the Λ decay factor, and the core momentum distribution factor. This product form is typical for PWIA. The integration over s can be easily carried out and one arrives at the expression

$$\frac{\partial\tau}{\partial t} = \text{const}\int_{s_-(t)}^{s_+(t)}\left[|S|^2 + |P|^2\frac{s-m_\pi^2}{q_\Lambda^2}\right]ds|Q^{(0)}(t)|^2 = \text{const}\ K(t)\ |Q^{(0)}(t)|^2,$$

where the integral over s has been abbreviated by $K(t)$. With real $Q^{(0)}(t)$ is also real. Solving the above equation for $Q^{(0)}(t)$ one can compare with experiment the expression

$$Q^{(0)}_{\text{ex}}(t) = \text{const}\left[\frac{1}{K(t)}\frac{\partial\tau}{\partial t}\right]^{\frac{1}{2}}.$$

For all reasonable wavefunctions, $Q^{(0)}(t)$ should be a decreasing function of t, and any deviation from this behaviour (typically a local maximum) should be ascribed to a strong, final state interaction.

For p-shell hypernuclei, in the first approximation, $Q^{(0)}(t)$ will given by a uniform function — the same for all core nuclei from ^{4}He up to ^{16}O. This uniformity property follows from the uniformity of the Λ wavefunction which, as pointed out in Ref. [14], varies little with increasing mass number A, due to two opposing trends tending to compensate each other. Assuming that the Λ particle feels only the mean potential well provided by the core, the depth of the well can be regarded nearly constant and the range increases with A. It is argued in Ref. [12] that the increase of the range has the effect to push $\psi_\Lambda(r)$ towards larger values of r, but this tendency is largely reduced by the increase of the Λ binding energy B_Λ with A, which causes sharp fall-off of ψ_Λ in the region where the Λ-core potential goes to zero. Numerically calculated wavefunctions for hypernuclei with A ranging from $A = 8$ to $A = 13$ obtained by Gal *et al.* [12] show that these two effects nearly cancel each other and a single Gaussian term

$$\psi_\Lambda(r) = \text{const}\exp\left(-\frac{1}{2}\lambda r^2\right) \tag{17a}$$

($\lambda = 0.33\ \text{fm}^{-2}$) provides a fair approximation, and a two-term Gaussian form

$$\psi_\Lambda(r) = \text{const}\ (ye^{-\frac{1}{2}\lambda_1 r^2} + e^{-\frac{1}{2}\lambda_2 r^2}) \tag{17b}$$

($\lambda_1 = 0.495\ \text{fm}^2$, $\lambda_2 = 0.165\ \text{fm}^2$, $y = 3.0$) appears to be an excellent approximation of the mean Λ wavefunction.

Another simple approximation for ψ_Λ can be obtained by assuming that the Λ-core potential is of Fermi shape (Saxon-Woods potential), *viz.*,

$$V_{\Lambda\cdot\text{core}}(r) = -D\left[1+\exp\left(\frac{r-c}{a}\right)\right]^{-1}, \tag{18}$$

where $c = r_0 A^{\frac{1}{3}}$, $D = 27\text{–}35$ MeV, $r = 1.1\text{–}1.3$ fm, and $a = 0.54\text{–}0.68$ fm.

An approximate solution of the Schroedinger equation with the potential (18), for $l = 0$, can be obtained by introducing an equivalent square well potential [13] of depth D and a variable range R, chosen in such a way that the binding energy is always exactly equal to that inferred from the potential (18). The range R is then a function of the binding energy B_Λ,

$$R = c - a\Delta(B_\Lambda),$$

where

$$\Delta(B_\Lambda) = \frac{1}{\xi}\,[2\arg\Gamma(1+\eta+i\xi) - \arg\Gamma(1+2i\xi)],$$

with $\xi^2 = 2\mu a^2(D-B_\Lambda)$, $\eta^2 = 2\mu a^2 B_\Lambda$ and μ is the Λ-core reduced mass.

The wavefunction ψ_Λ is then approximated by the expression appropriate for the equivalent square well potential

$$r\psi_\Lambda(r) = \begin{cases} N\sin\left(\xi\dfrac{r}{a}\right), & r \leqslant R \\ Ne^{-\frac{\eta}{a}(r-R)}\sin\left(\xi\dfrac{R}{a}\right), & r \geqslant R \end{cases} \tag{17c}$$

where $N = \left[\frac{\eta}{2\pi}\frac{1}{1+\eta R}\right]^{\frac{1}{2}}$ for ψ_Λ normalized as $\int \psi^2 4\pi r^2 dr = 1$.

The overlap integral $Q^{(0)}(t)$ calculated with the wavefunctions (17a), (17b) and (17c) is given, respectively, by the expressions (18a), (18b) and (18c)

$$Q^{(0)}(t) = \text{const}\; e^{-\frac{t}{2\lambda}}, \tag{18a}$$

$$Q(t) = \text{const}\;(ye^{-\frac{t}{2\lambda_1}} + e^{-\frac{t}{2\lambda_2}}), \tag{18b}$$

$$Q^{(c)}(t) = \text{const}\;\frac{1}{\sqrt{t}}\left\{\frac{(\xi/a)\sin(\sqrt{t}\,R)\cos(\xi R/a) - \sqrt{t}\sin(\xi R/a)\cos(\sqrt{t}\,R)}{t-(\xi/a)^2} + \right.$$

$$\left. + \frac{\sin(\xi R/a)[(\eta/a)\sin(\sqrt{t}\,R) + \sqrt{t}\cos(\sqrt{t}\,R)]}{t+(\eta/a)^2}\right\}. \tag{18c}$$

With sufficiently large statistics also expression (16) can be compared with experiment. In the absence of strong final state interactions, for fixed core recoil (t = const) the rate should be a known linear function of s. This test is particularl yconvenient since no knowledge of ψ_Λ is required.

4. *Final state interactions* (*FSI*)

If the final state proton interacts strongly with the core nucleus in the expression for the overlap integral (14) the free relative motion represented by the plane wave exp $(i\boldsymbol{q}\cdot\boldsymbol{r})$ should be replaced by the exact wavefunction $u_q(\boldsymbol{r})$ which takes into account the proton-core interaction. The function $u_q(\boldsymbol{r})$ is a solution of the Schroedinger equation

$$\left[-\frac{1}{2\mu}\nabla^2 + V_{\text{p-core}}(r)\right]u_q(\boldsymbol{r}) = \frac{q^2}{2\mu}u_q(\boldsymbol{r}),$$

where the p-core potential $V_{\text{p-core}}$ will be, in general, spin dependent. Since $u_q(\boldsymbol{r})$ is the solution of the scattering problem, it has the asymptotic behaviour

$$u_q(\boldsymbol{r}) \to e^{i\boldsymbol{q}\boldsymbol{r}} + Te^{iqr}(1/r), \tag{19}$$

where T is the scattering amplitude and can be, as usual, written in terms of phase shifts and mixing parameters. The asymptotic form of $u_q(\boldsymbol{r})$ may be used as an approximate expression for $u_q(\boldsymbol{r})$ for all $\boldsymbol{r}$'s. This approximation, used for calculating the overlap integral, is known as the zero range approximation, or, if the overlap integral is written in momentum space, as the on-shell approximation. The overlap integral will be then a sum of two terms, the PWIA term (15), and a correction for FSI proportional to T. The phase shifts may be taken from experiment and the correction term is completely determined.

It is clear that the overlap integral will in this case be a function of both variables, s and t. The rate will contain, besides the PWIA term, an interference term proportional to T. This term is s-dependent and can strongly modify the t spectrum (after integrating over s).

The interference term may, *e.g.* produce a maximum in the t spectrum, or cause an increase of the rate for large t. Although the zero range approximation leads to predictions which seem to be quite reasonable, it is, in general, not very accurate. The wavefunction ψ_Λ damps the integral strongly for large r and, in fact, the contribution from the asymptotic region where formula (19) is valid, is negligible. The largest contribution comes from the region of small r, *i.e.* from the region where the asymptotic expression is not yet applicable. The zero range approximation would work if the range of the proton-core force were much smaller than the dimension of the hypernucleus. Since they are roughly equal, the zero range approximation is not likely to be correct. We shall work out another approximation based on the assumption that the ranges of the Λ-core and p-core forces are nearly equal, as the corresponding potentials essentially follow the nucleon distribution within the core nucleus. This point of view is in the spirit of the optical model approach.

Spin complications will be considered in the next section and now, for simplicity, we shall assume that the p-core potential does not depend on spin, but may be l dependent. In other words, we introduce as many different potentials, as the number of phase shifts which give a non-negligible contribution in the energy range considered. In practice, it should be sufficient to neglect all phase shifts for $l > 2$, though there is no difficulty to take more partial waves into account.

Adding and subtracting from $u_{\boldsymbol{q}}(\boldsymbol{r})$ the plane wave $e^{i\boldsymbol{q}\cdot\boldsymbol{r}}$ we define a new function $\bar{u}_{\boldsymbol{q}}(\boldsymbol{r})$

$$u_{\boldsymbol{q}}(\boldsymbol{r}) = e^{i\boldsymbol{q}\cdot\boldsymbol{r}}+[u_{\boldsymbol{q}}(\boldsymbol{r})-e^{i\boldsymbol{q}\cdot\boldsymbol{r}}] = e^{i\boldsymbol{q}\cdot\boldsymbol{r}}+\bar{u}_{\boldsymbol{q}}(\boldsymbol{r}). \tag{20}$$

Expanding into spherical harmonics, one has

$$u_{\boldsymbol{q}}(\boldsymbol{r}) = 4\pi \sum_{l,m} i^l f_l\,(r)\, Y_{lm}(\hat{r})\, Y^*_{lm}\,(\hat{q}), \tag{21a}$$

$$\bar{u}_{\boldsymbol{q}}(\boldsymbol{r}) = 4\pi \sum_{l,m} i^l \bar{f}_l(r)\, Y_{lm}(\hat{r})\, Y^*_{lm}(\hat{q}), \tag{21b}$$

$$\bar{f}_l(r) = f_l(r) - j_l(qr), \tag{22}$$

where $f_l(r)$ is the solutions of the radial Schroedinger equation and the asymptotic behaviour of $f_l(r)$ is

$$f_l(r) \to j_l(qr) + i h_l^{(-)}(qr) e^{i\delta_l} \sin \delta_l, \tag{23}$$

where δ_l is the l-th wave phase shift, $j_l(qr)$ and $h_l^{(-)}(qr)$ are spherical Bessel and Hankel functions normalized as in Ref. [18]. Notice, that the expansion for $\bar{u}_{\boldsymbol{q}}(\boldsymbol{r})$, in contrast to that for $u_{\boldsymbol{q}}(\boldsymbol{r})$, contains only a few terms since begining with some $l \geqslant l_{\max}$ all the phase shifts become negligible and $\bar{f}_l(r) \approx 0$. In order to be able to analytically perform the integration over r in the overlap integral, we find it practical to approximate the interaction by a number of square well potentials with depths D_l and ranges R_l, adjusted to reproduce the phase shifts. The parameters D_l and R_l for $l = 0, 1, 2, \ldots$ are determined by fitting the experimental l-th phase shift using the expression

$$\tan \delta_l = - \frac{q j_l'(qR_l) j_l(\varkappa_l R_l) - \varkappa_l j_l(qR_l) j_l'(\varkappa_l R_l)}{q n_l'(qR_l) j_l(\varkappa_l R_l) - \varkappa_l n_l(qR_l) j_l'(\varkappa_l R_l)}, \tag{24}$$

where $\varkappa_l^2 = q^2+2\mu D_l$, $n_l(qr)$ is spherical the Neuman function (*cf.* Ref. [18]), and differentiation is with respect to the argument.

The analytic expression $f_l(r)$ is readily obtained

$$f_l(r) = j_l(\varkappa_l r)\left[\frac{j_l(q_l R)+\tan\delta_l\, n_l(qR_l)}{j_l(\varkappa_l R_l)}\right]\cos\delta_l e^{\delta_l}, \quad r \leqslant R_l \tag{25a}$$

$$f_l(r) = [j_l(qr)+\tan\delta_l n_l(qr)]\cos\delta_l e^{i\delta_l}, \; r \geqslant R_l. \tag{25b}$$

The amplitude which multiplies $j_l(\varkappa_l r)$ may be regarded as an enhancement factor. For $qR_l \gg 1$ this factor becomes essentially the Watson [15] factor $\dfrac{\sin\delta_l e^{i\delta_l}}{q^{2l+1}}$.

The overlap integral

$$Q = \int \psi_\Lambda^*(r) u_{\boldsymbol{q}}(\boldsymbol{r}) e^{i\boldsymbol{k}'\cdot\boldsymbol{r}} d^3r, \leftrightarrow \boldsymbol{k}' = \frac{m_c}{m_c+m_p}\boldsymbol{k}, \tag{26}$$

using (20), may be written as a sum of two terms

$$Q = Q^{(0)}+Q^{(1)}, \tag{27}$$

where $Q^{(0)}$ is given by the PWIA expression (15) and $Q^{(1)}$ is the correction for FSI

$$Q^{(1)} = \int \psi_\Lambda^*(r)\bar{u}_{\boldsymbol{q}}(\boldsymbol{r})e^{i\boldsymbol{k}'\cdot\boldsymbol{r}}d^3r = \sum_{l=0}^{l_{\max}} (2l+1)4\pi c_l^{(1)} P_l(\cos\Theta), \tag{28}$$

where

$$c_l^{(1)} = \int_0^\infty \psi_\Lambda^*(r)\bar{f}_l(r) j_l(k'r) r^2 dr. \tag{29}$$

Since the square of the overlap integral Q enters the integrand in the integration over s and t in the expression for the partial rate, it is desirable to have an analytic formula for Q. This can be achieved by representing ψ_Λ as a superposition of Gaussians and taking for $f_l(r)$ the expression (25a). In fact, for $r \geqslant R_l$ we should have used expression (25b), but due to the sharp fall-off of ψ_Λ the contribution from the region $r \geqslant R_l$ is small and the error introduced by using formula (25a) will be also small. This approximation will be frequently employed in subsequent sections and may be called the distorted wave impulse approximation (DWIA). The radial overlap integrals c_l, using DWIA and Gaussian ψ_Λ, are superpositions of integrals of the form

$$\int e^{-\lambda r^2} j_l(ar) j_l(br) r^2 dr = \frac{1}{4}\sqrt{\frac{\pi}{\lambda^3}} \exp\left(-\frac{a^2+b^2}{4\lambda}\right) i_l\left(\frac{ab}{2\lambda}\right). \tag{30}$$

5. *Spin effects*

In this section we will generalize the previous results by taking spin effects into account. The plane wave $e^{i\boldsymbol{qr}}$ will be replaced by a suitable operator $u(\boldsymbol{S},\boldsymbol{q},\boldsymbol{r})$ acting on spinors describing the spins of the core and the proton. If a system of two colliding particles has the total spin $\boldsymbol{S}$, the wavefunction with definite momentum $\boldsymbol{q}$ is

$$\Psi_{m_S}^S(\boldsymbol{q},\boldsymbol{r}) = u(\boldsymbol{S},\boldsymbol{q},\boldsymbol{r})\chi_{m_S}^S = \sum_{J,L,M_L} i^L f_L^J(r) \Pi^{LSJ} Y_{LM_L}^*(\hat{q}) Y_{LM_L}(\hat{r})\chi_{m_S}^S, \tag{31}$$

where $\chi^S_{m_S}$ is the $(2S+1)$ component spinor, f^J_L are the radial wavefunctions with definite JLS, Π^{LSJ} is the projection operator $\sum_M |JMLS\rangle\langle JMLS|$ which acting on $Y_{Lm_L}(\hat{r})\chi^S_{m_S}$ projects a state with definite JLS. This operator can be written as [16]

$$\Pi^{LSJ} = \prod_{j\neq J} \frac{\boldsymbol{S}\cdot\boldsymbol{L}-\Omega(j)}{\Omega(J)-\Omega(j)}, \tag{32}$$

where $\Omega(j) = \frac{1}{2}[j(j+1)-L(L+1)-S(S+1)]$, and $\boldsymbol{L}$ is the angular momentum operator $\boldsymbol{L} = -i\,\hat{\boldsymbol{r}}\times\frac{\partial}{\partial\hat{\boldsymbol{r}}}$ acting on the spherical harmonics $Y_{LM_L}(\boldsymbol{r})$ in formula (31).

It is evident from formula (32) that the operator Π^{LSJ} is a polynomial of order $2S$ in $\boldsymbol{S}\cdot\boldsymbol{L}$, and therefore the operator $u(\boldsymbol{S},\boldsymbol{q},\boldsymbol{r})$ defined by expression (31) will also be a polynomial and may be written as

$$\begin{aligned} u(\boldsymbol{S},\boldsymbol{q},\boldsymbol{r}) &= \sum_{L=0}^{\infty} i^L[g^{(1)}_L(qr)+g^{(2)}_L(qr)\boldsymbol{S}\cdot\boldsymbol{L}+g^{(3)}_L(qr)(\boldsymbol{S}\cdot\boldsymbol{L})^2+\ldots \\ &\quad + g^{(2S+1)}_L(qr)(\boldsymbol{S}\cdot\boldsymbol{L})^{2S}]P_L(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{r}}) \\ &\equiv \Phi^{(1)}(\boldsymbol{q},\boldsymbol{r})+(\boldsymbol{S}\cdot\boldsymbol{L})\Phi^{(2)}(\boldsymbol{q},\boldsymbol{r})+(\boldsymbol{S}\cdot\boldsymbol{L})^2\Phi^{(3)}(\boldsymbol{q},\boldsymbol{r})+\ldots(\boldsymbol{S}\cdot\boldsymbol{L})^{2S}\Phi^{(2S+1)}(\boldsymbol{q},\boldsymbol{r}), \end{aligned} \tag{33}$$

where the functions $g^{(i)}_L$ are superpositions of f^J_L functions. The functions $g^{(i)}_L$ for $S=\frac{1}{2}, 1, \frac{3}{2}$ have been explicitly evaluated in Appendix A.

Although the form (33) of the wavefunction may have appeared rather complicated at first sight, it proves to be very convenient for two reasons. The first is that the spin summations of the products of spin dependent expressions are quite simple and can be accomplished by taking the traces of spin operators. The second advantage lies with the operator $\boldsymbol{L}$ acting on $P_L(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{r}})$. The differentiation over $\hat{\boldsymbol{r}}$ may be switched to differentiation over $\hat{\boldsymbol{q}}$, *viz.*,

$$\boldsymbol{L}P_L(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{r}}) = -i\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{q}}P'_L(q\boldsymbol{r}) = i\hat{q}\times\frac{\partial}{\partial\hat{\boldsymbol{q}}}P_L(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{r}}) = -\boldsymbol{L}_q P_L(\hat{q}\cdot\hat{\boldsymbol{r}}).$$

Thus, in expression (33) the operator $\boldsymbol{S}\cdot\boldsymbol{L}$ is replaced by $-\boldsymbol{S}\boldsymbol{L}_L$ and can be eventually taken out of the overlap integral.

Formula (33) is easily generalized to the case when the colliding particles have spins S_1 and S_2, respectively. The corresponding operator $w(\boldsymbol{S}_1,\boldsymbol{S}_2,\boldsymbol{q},\boldsymbol{r})$ which acts on the direct products of spinors is a sum of terms of the form (33), each multiplied by spin projection operator $\Pi^{S_1S_2S}$ projecting the total spin S:

$$w(\boldsymbol{S}_1,\boldsymbol{S}_2,\boldsymbol{q},\boldsymbol{r}) = \sum_{S=|S_1-S_2|}^{S_1+S_2} u(\boldsymbol{S},\boldsymbol{q},\boldsymbol{r})\Pi^{S_1S_2S}. \tag{34}$$

In particular, for $S_1=\frac{1}{2}$ and $S_2=J_N$, we have

$$\begin{aligned} w\left(\frac{1}{2}\boldsymbol{\sigma},\boldsymbol{J}_N,\boldsymbol{q},\boldsymbol{r}\right) &= [\Phi^{(1)}_+ +(\boldsymbol{I}\cdot\boldsymbol{L})\Phi^{(2)}_+ +\ldots+(\boldsymbol{I}\cdot\boldsymbol{L})^{2J_N+1}\Phi^{(2J_N+2)}_+]P^+ + \\ &+[\Phi^{(1)}_-(\boldsymbol{I}\cdot\boldsymbol{L})\Phi^{(2)}_- +\ldots+(\boldsymbol{I}\cdot\boldsymbol{L})^{2J_N-1}\Phi^{(2J_N)}]\,P^{(-)} \equiv u_+P^{(+)}+u_-P^{(-)}, \end{aligned} \tag{35}$$

where $\boldsymbol{I} = \boldsymbol{J}_N + \frac{1}{2}\sigma$, the $\pm$ sign refers to $I = J_N \pm \frac{1}{2}$, and $P^{(\pm)}$ are the spin projection operators

$$P^{(+)} = \frac{J_N + 1 + \boldsymbol{\sigma} \cdot \boldsymbol{J}_N}{2J_N + 1}; \quad P^{(-)} = \frac{J_N - \boldsymbol{\sigma} \cdot \boldsymbol{J}_N}{2J_N + 1}. \tag{36}$$

Similarly as in the preceding section, we add and deduct from $w(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{q}, \boldsymbol{r})$ the plane wave $e^{i\boldsymbol{qr}}$

$$w(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{q}, \boldsymbol{r}) = e^{i\boldsymbol{qr}} + \bar{u}_+ P^{(+)} + \bar{u}_- P^{(-)}. \tag{37}$$

The functions $\bar{u}_J^{\pm}$ are obtained from $u_{\pm}$ by replacing $f_L^J \to \bar{f}_L^J = f_L^J - j_L$.

The appearance of $2S+1$ functions $\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(2S+1)}$ reflects the presence of spin--orbit splitting. For central forces we have $2S+1$ fold degeneracy and only $\Phi^{(1)}$ will be different from zero. The functions $\Phi^{(n)}$ for $n > 1$ may be regarded as corrections due to spin-orbit forces which vanish if we are to neglect spin-orbit interaction.

Consider now a three body decay rate (27) corrected for FSI. The transition matrix element is

$$\mathscr{M}^{\pm} = \text{const } P^{(\pm)} DQ, \tag{38}$$

where $D = S + P\,\dfrac{\sigma \cdot \boldsymbol{k}}{q_\Lambda}$, and

$$Q = \int \psi_\Lambda^*(r) e^{i\boldsymbol{k}' \cdot \boldsymbol{r}} w(\tfrac{1}{2}\sigma, \boldsymbol{J}_N, \boldsymbol{q}, \boldsymbol{r}) d^3r. \tag{39}$$

Inserting expression (37) for $w(\frac{1}{2}\boldsymbol{\sigma}, \boldsymbol{J}_N, \boldsymbol{q}, \boldsymbol{r})$ enables the overlap integral (39) to be written as

$$Q = Q_+ P^{(+)} + Q_- P^{(-)}, \tag{40}$$

$$Q_{\pm} = \int \psi_\Lambda^*(r) e^{i\boldsymbol{k}' \cdot \boldsymbol{r}} u(\boldsymbol{I}, \boldsymbol{q}, \boldsymbol{r}) d^3r.$$

The transition rate is obtained by averaging $|\mathscr{M}^{\pm}|^2$ over spins

$$\begin{aligned}\frac{\partial^2 \tau^{\pm}}{\partial t \partial s} &= \text{const} \sum_{\text{spin}} |\mathscr{M}^{\pm}|^2 \\ &= \text{const } \mathrm{Tr}_{J_N} \mathrm{tr}_\sigma [P^{(\pm)} DQQ^\dagger D^\dagger P^{(\pm)}] \\ &= \text{const } \mathrm{Tr}_{J_N} \mathrm{tr}_\sigma [D^\dagger P^{(\pm)} D(Q_+ Q_+^\dagger P^{(+)} + Q_- Q_-^\dagger P^{(-)})].\end{aligned} \tag{41}$$

With two independent vectors $\boldsymbol{q}$ and $\boldsymbol{k}$ there is no pseudoscalar quantity available and, therefore, only scalars can survive spin summation. Thus,

$$\begin{aligned}\frac{\partial^2 \tau^{\pm}}{\partial t \partial s} = \text{const} \Bigg\{ \left[|S|^2 + |P|^2 \left(\frac{k}{q_\Lambda} \right)^2 \right] \mathrm{Tr}_{J_N} \mathrm{tr}_\sigma [P^{(\pm)} Q_{\pm} Q_{\pm}^\dagger] \pm \\ \pm \frac{2}{2J_N + 1} |P|^2 \left(\frac{k}{q_\Lambda} \right)^2 \mathrm{Tr}_{J_N} \mathrm{tr}_\sigma [(J_N + 1 + \boldsymbol{\sigma} \cdot \hat{\boldsymbol{q}}\, \boldsymbol{J}_N \cdot \hat{\boldsymbol{q}}) P^{(-)} Q_- Q_-^\dagger + \\ + (-J_N + \boldsymbol{\sigma} \cdot \hat{\boldsymbol{q}}\, \boldsymbol{J}_N \cdot \hat{\boldsymbol{q}}) P^{(+)} Q_+ Q_+^\dagger] \Bigg\}.\end{aligned} \tag{42}$$

In constrast to formula (16), this expression does make a distinction between the initial spin being $J_N+\frac{1}{2}$, or $J_N-\frac{1}{2}$. In particular, neglecting spin corrections due to spin-orbit interaction in $Q_{\pm}$, one has

$$\frac{\partial^2\tau^+}{\partial t\partial s} = \text{const}\left\{\left[|S|^2+|P|^2\left(\frac{k}{q_\Lambda}\right)^2\right]|Q_+|^2+\right.$$

$$\left.+\frac{4}{3}\frac{J_N}{2J_N+1}|P|^2\left(\frac{k}{q_\Lambda}\right)^2[|Q_-|^2-|Q_+|^2]\right\}; \tag{43a}$$

$$\frac{\partial\tau^-}{\partial t\partial s} = \text{const}\left\{\left[|S|^2+|P|^2\left(\frac{k}{q_\Lambda}\right)^2\right]|Q_-|^2-\right.$$

$$\left.-\frac{4}{3}\frac{J_N+1}{2J_N+1}|P|^2\left(\frac{k}{q_\Lambda}\right)^2(|Q_-|^2-|Q_+|^2)\right\}. \tag{43b}$$

Writting $Q_{\pm}$ as

$$Q_{\pm} = Q^{(0)}(t)+Q^{(1)}_{\pm}(s, t)$$

one can calculate explicitly the FSI corrections to PWIA. The dependence on s and t is now, in general, quite complicated and does not factorize. The expressions for the $\pm$ rates will be different provided that the p-core forces are spin dependent.

Another example where the trace calculations are also very simple is the case of a spinless core. The spin of the hypernucleus in then $\frac{1}{2}$ and the overlap integral Q can be written as (*cf.* Appendix A)

$$Q = Q_+ = Q^{(0)}+Q^{(1)}+(-\tfrac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{L}_q)Q^{(2)}, \tag{44}$$

$$Q^{(i)} = \int \psi^*_\Lambda(r)e^{i\boldsymbol{k}'\cdot\boldsymbol{r}}\overline{\Phi}^{(i)}(\boldsymbol{q}, \boldsymbol{r})d^3r.$$

The last term in (44) can be further simplified

$$-\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{L}_qQ^{(2)} = i\,\frac{1}{2}\,\boldsymbol{\sigma}\cdot\left(\hat{\boldsymbol{q}}\times\frac{\partial}{\partial\hat{\boldsymbol{q}}}\right)Q^{(2)} = i\,\frac{1}{2}\,\boldsymbol{\sigma}\cdot(\hat{\boldsymbol{q}}\times\boldsymbol{k})\,\frac{dQ^{(2)}}{d\cos\Theta}.$$

On evaluating the traces the decay rate is

$$\frac{\partial^2\tau}{\partial t\partial s} = \text{const}\left[|S|^2+|P|^2\left(\frac{k}{q_\Lambda}\right)^2\right]\left[|Q^{(0)}+Q^{(1)}|^2+\sin^2\Theta\left|\frac{dQ^{(2)}}{d\cos\Theta}\right|^2\right],$$

where $Q^{(0)}$ is given by the PWIA expression (15) and the FSI corrections $Q^{(1)}$ and $Q^{(2)}$ when expanded in Legendre polynomials are of the form

$$Q^{(i)} = 4\pi\sum_{L=0}^{L_{\max}}(-)^L c^{(i)}_L P_L(\cos\Theta),\quad i = 1, 2;$$

$$c^{(i)}_L = (L+1)\int_0^\infty \psi^*_\Lambda(r)j_L(k'r)\overline{f}^+_L(qr)r^2dr\;+$$

$$+ L \int_0^\infty \psi_\Lambda^*(r) j_L(k'r) \bar{f}_L^-(qr) r^2 dr,$$

$$c_L^{(2)} = \int_0^\infty \psi_\Lambda^*(r) j_L(k'r) [f_L^+(qr) - f_L^-(qr)] r^2 dr.$$

Using the DWIA and Gaussian ψ_Λ one can easily obtain analytic expressions for the radial integrals (*cf.* formula (30)).

6. *Many-channel case*

In ths section we shall discuss a many-channel three body decay situation. We will consider all two body final states which may be reached from the initial state P+core and are energetically accessible. For N open channels, both, the functions $\Phi^{(i)}$, and the operator w from the preceding section, become matrices in the channel space. To simplify matters let us, for the time being, disregard spin effects. We define the final state wavefunction ψ_j, $j = 2, 3, \ldots, N$, as a solution of the many channel Schroedinger equation. Separating out the angular part, one obtains the radial matrix equation

$$(d + \varkappa_j^2)\psi_j + \sum_s 2\mu_j V'_{js}\psi_s = 0, \tag{45}$$

where

$$d = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2},$$

$$V'_{ij} = V_{ij}(1 - \delta_{ij}),$$

$$\varkappa_j^2 = p_j^2 + 2\mu_j V_{jj},$$

$$\psi_j(0) = 0,$$

p_j and μ_j are j-th channel momentum and reduced mass, and V_{ij} is a potential matrix which may, in general, be l dependent. Similarly as before we assume that V_{ij} are square wells of depths $\bar{V}_{ij}$ and equal ranges in all channels. To simplify notation we shal drop the index l.

The solution which vanishes at the origin is

$$\psi_n = A_n j(\lambda r). \tag{46}$$

Inserting (46) into (45) one is left with a set of linear, homogeneous equations for A_n,

$$(\varkappa_n^2 - \lambda^2)A_n + \sum_s 2\mu_n V'_{ns} A_s = 0, \tag{47}$$

and the condition for the existence of nontrivial solutions for A_n yields the equation for λ

$$\mathrm{Det}\, \|(\varkappa_n^2 - \lambda^2)\delta_{mn} + 2\mu_m \bar{V}'_{mn}\| = 0. \tag{48}$$

Denoting the roots of Eq. (48) by $\lambda_1, \lambda_2, \ldots, \lambda_N$ a general solution ψ_n can be written as

$$\psi_n = \sum_{\alpha=1}^{N} A_n^\alpha j(\lambda_\alpha r). \tag{49}$$

We have N^2 constants A_n^α, which are determined as follows. First, we divide A_n^α into diagonal and off-diagonal parts

$$A_n^n = C_n,\ A_n^m = E_n^m C_n,\ m \neq n.$$

The amplitudes C_n can be calculated from the boundary condition at infinity, and the off-diagonal elements E_n^m are readily obtained from the requirement that (49) is a solution of (45). One obtains for E_n^m a set of linear, inhomogeneous equations

$$(\varkappa_n^2 - \lambda_m^2)E_m^m + \sum_{s \neq m} 2\mu_n \overline{V}'_{ns} E_s^m = -2\mu_n \overline{V}'_{nm};\ n \neq m.$$

For $r > R$ we have N different column matrices $\psi_n^{(\alpha)}$ to represent the solutions with the ingoing wave in channel α, where $\alpha = 1, 2, 3, \ldots, N$. The asymptotic form of $\psi_n^{(\alpha)}$ is

$$\psi_n^{(\alpha)} = \sqrt{\varrho_n/\varrho_\alpha}[j(p_n r)\delta_{n\alpha} + ih^{(-)}(p_n r)T'_{n\alpha}], \tag{50}$$

where $\varrho_n = \mu_n p_n$ and $T'_{n\alpha}$ is the reduced T matrix for the $\alpha \to n$ transition with such a normalization that the cross-section for the $\alpha \to n$ process is

$$\sigma(a \to n) = \frac{4\pi}{p_\alpha^2}|T'_{n\alpha}|^2.$$

Matching (49) and (50) at $r = R$ one has N equations for the N constants C_N. The solutions are functions of $T'_{n\alpha}$ and, therefore, the amplitudes C_n may be regarded as many channel generalizations of the enhancement factor introduced in Section 4. Explicit calculation for a two channel case is presented in Appendix B. For a given partial wave l, we have $\frac{1}{2}N(N+1)+1$ parameters $\overline{V}_{ij}$ and R) to be fitted from the scattering data.

7. *Coulomb interaction*

It is not difficult to extend the formalism introduced in the preceding sections to include the Coulomb final state interaction. To this end we assume that in the region where the strong interaction is operative we can neglect the Coulomb interaction. In the outer region the plane wave ought to be replaced by the appropriate Coulomb wavefunction. Let $F_l(qr)$ and $G_l(qr)$ be, respectively, the regular and irregular solutions of the radial Schroedinger equation with the Coulomb potential (the particles are assumed to have no spatial extension, and the charges are Z_1e^2 and Z_2e^2). We adopt here the definitions and normalizations of F_l and G_l as in Ref. [17]. It is convenient to take linear combinations of F_l and G_l and construct the following functions

$$x_l(qr) = [F_l(qr)\cos\sigma_l - G_l(qr)\sin\sigma_l]\ (1/qr),$$

$$y_l(qr) = [F_l(qr)\sin\sigma_l + G(qr)\cos\sigma_l]\ (1/qr),$$

$$z_l^{(\pm)}(qr) = [F_l(qr) \pm iG_l(qr)]e^{\pm i\sigma_l}(1/qr),$$

where σ_l is the Coulomb phase shift $\sigma_l = \arg(l+1+i\eta)$, $\eta = Z_1Z_2\alpha\mu/q$, α is the fine structure constant, and μ is the reduced mass. In the absence of the Coulomb interaction the functions x_l, y_l, $z_l^{(\pm)}$ go over into j_l, n_l, $h_l^{(\pm)}$, respectively. The asymptotic behaviour of x_l, y_l, $z^{(\pm)}$ is

$$x_l(qr) \rightarrow \sin\left(qr - \frac{\pi l}{2} - \eta \ln 2qr\right)(1/qr),$$

$$y_l(qr) \rightarrow \cos\left(qr - \frac{\pi l}{2} - \eta \ln 2qr\right)(1/qr),$$

$$z_l^{(\pm)}(qr) \rightarrow \pm i \exp\left[\mp i\left(qr - \frac{\pi l}{2} - \eta \ln 2qr\right)\right](1/qr).$$

In the region where the strong interaction is no longer present and the particles interact only *via* the Coulomb potential, the radial solution of the Schroedinger equation is

$$f_l(qr) = x_l(qr) + \tfrac{1}{2}(e^{2i\delta_l + 2i\sigma_l} - 1) z_l^{(-)}(qr).$$

The above expression is a generalization of formula (23) and may be written as

$$f_l(qr) = x_l(qr) + \tfrac{1}{2}(e^{2i\sigma_l} - 1) z_l^{(-)}(qr) + \tfrac{1}{2} e^{2i\sigma_l}(e^{2i\delta_l} - 1) z_l^{(-)}(qr).$$

In the asymptotic region the first term will represent the ingoing distorted wave, whereas the second and third terms will represent the outgoing wave scattered by the Coulomb and nuclear potential respectively.

The prescription for accounting for the Coulomb interaction is the following. Change δ_l into $\delta_l + \sigma_l$, and change the functions j_l, n_l, $h_l^{(\pm)}$ into x_l, y_l, $z_l^{(\pm)}$, respectively. For example, the radial wave function (25a), which has been used for evaluating the overlap integral, in the presence of the Coulomb interaction has the form

$$f_l(qr) = j_l(\varkappa_l r)\,\frac{x_l(qR_l) + \tan(\delta_l + \sigma_l) y_l(qR_l)}{j_l(\varkappa_l R_l)}\; e^{i\delta_l + i\sigma_l} \cos(\delta_l + \sigma_l).$$

Hence, the inclusion of the Coulomb final state interaction changes merely the enhancement factors and does not lead to any complications.

8. *Four body kinematics and phase space*

As we have already mentioned, there are five independent kinematical variables to provide a complete description of a four body decay. We find it convenient to define two sets of momenta in the overall c.m. system. The first set is suitable for situations where one is willing to account for the interaction of only one pair of particles, and to treat the two remaining ones as free. The second set will be useful when one particle is assumed to move freely and the three remaining particles interact among themselves. In both cases we have three independent momenta (after making use of momentum conservation) and a particular choice of the axes leaves us eventually with five kinematical variables, such as angles and energies.

Set I. Let the only interacting particles be 1 and 2. As the independent momenta we take $\boldsymbol{q}_{12}$, the relative momentum of 1 and 2; $\boldsymbol{q}_{34}$, the relative momentum of 3 and 4; and $\boldsymbol{q}$, the relative momentum of both pairs. They are expressed in terms of the laboratory momenta as follows

$$\boldsymbol{q}_{12}=\frac{m_2\boldsymbol{p}_1-m_1\boldsymbol{p}_2}{m_1+m_2},\quad \boldsymbol{q}_{34}=\frac{m_4\boldsymbol{p}_3-m_3\boldsymbol{p}_4}{m_3+m_4},$$

$$\boldsymbol{q}=\boldsymbol{p}_1+\boldsymbol{p}_2=-\boldsymbol{p}_3-\boldsymbol{p}_4. \tag{51}$$

The total kinetic energy is

$$E_{\text{kin}}=\sum_{i=1}^{4}\frac{p_i^2}{2m_i}=T_{12}+T_{34}+T_r,$$

$$T_{12}=\frac{1}{2\mu_{12}}q_{12}^2,\quad T_{34}=\frac{1}{2\mu_{34}}q_{34}^2, \tag{52}$$

$$T_r=\frac{1}{2}\left[\frac{1}{m_1+m_2}+\frac{1}{m_3+m_4}\right]q^2\equiv\frac{1}{2\mu_{12\,34}}q^2.$$

We take our Z axis along the direction of $\boldsymbol{q}$, and the XZ plane is identical with the $(\boldsymbol{q},\boldsymbol{q}_{12})$ plane. With this choice three angles are needed to determine the directions $\hat{\boldsymbol{q}}$, $\hat{\boldsymbol{q}}_{12}$, $\hat{\boldsymbol{q}}_{34}$, *viz.*,

$$\begin{aligned}\hat{\boldsymbol{q}}_{12}&=[\sin\theta_{12},0,\cos\Theta_{12}],\\ \hat{\boldsymbol{q}}&=[0,0,1],\\ \hat{\boldsymbol{q}}_{34}&=[\cos\varphi\sin\Theta_{34},\sin\varphi\sin\Theta_{34},\cos\Theta_{34}].\end{aligned} \tag{53}$$

The angle φ is then the angle between normals to the $(\boldsymbol{q}_{12},\boldsymbol{q})$ and $(\boldsymbol{q}_{34},\boldsymbol{q})$ planes.

As the independent kinematical variables one can take any two of the relative energies and the three angles. The phase space volume element is

$$\begin{aligned}d\varrho(4)&=d^3q_{12}\,d^3q_{34}\,d^3q\,\delta(E_{\text{kin}}-\Delta E)\\ &=\text{const}\,[T_{12}\,T_{34}\,T_r]^{\frac{1}{2}}\,\delta\,(T_{12}+T_{34}+T_r-\Delta E)\,dT_{12}\,dT_{34}\,dT_r\,d\cos\Theta_{12}\,d\Omega_{34}.\end{aligned} \tag{54}$$

Set II. Let particle 4 be the noninteracting one. As independent momenta we take $\boldsymbol{p}_4$, $\boldsymbol{q}_{12}$, and the momentum canonically conjugated with the separation of particle 3 from the (1,2) center of mass. The transition from Set I to Set II is accomplished by using the following relations:

$$\boldsymbol{q}_{34}=-\frac{m_4}{m_3+m_4}\boldsymbol{k}_3-\frac{m_3(m_1+m_2+m_3+m_4)}{(m_3+m_4)(m_1+m_2+m_3)}\boldsymbol{p}_4,$$

$$\boldsymbol{q}=\boldsymbol{k}_3-\frac{m_1+m_2}{m_1+m_2+m_3}\boldsymbol{p}_4. \tag{55}$$

The total kinetic energy is

$$E_{\text{kin}} = T_{12} + T_{12,3} + T_{123,4},$$

where

$$T_{12,3} = \frac{1}{2}\left[\frac{1}{m_1+m_2} + \frac{1}{m_3}\right] k_3^2 \equiv \frac{1}{2\mu_{12,3}} k_3^2,$$

$$T_{123,4} = \frac{1}{2}\left[\frac{1}{m_1+m_2+m_3} + \frac{1}{m_4}\right] p_4^2 \equiv \frac{1}{2\mu_{123,4}} p_4^2. \tag{56}$$

As the kinematic variables use may be made of any two of the three relative energies and the three angles defined as

$$\hat{\boldsymbol{q}}_{12} = [\sin\vartheta_{12}, 0, \cos\vartheta_{12}],$$

$$\hat{\boldsymbol{k}}_3 = [0, 0, 1], \tag{57}$$

$$\hat{\boldsymbol{p}}_4 = [\cos\varphi_4 \sin\Theta_4, \sin\varphi_4 \sin\Theta_4, \cos\Theta_4].$$

The phase space volume element is

$$d\varrho(4) = \text{const}\,[T_{12}\, T_{12,3}\, T_{123,4}]^{\frac{1}{2}} dT_{12}\, dT_{12,3}\, dT_{123,4} \times$$

$$\times\, \delta(T_{12} + T_{12,3} + T_{123,4} - \Delta E)\, d\cos\vartheta_{12}\, d\Omega_4. \tag{58}$$

9. Four body decay rate — cluster model approach

In this section we shall consider a four body decay of a hypernucleus: hypernucleus $\rightarrow X + Y + P + \pi^-$, where X and Y may be either light nuclei or nucleons. We shall assume that the decay of the Λ particle causes a disintegration of the core nucleus into the clusters X and Y. In terms of the cluster model the initial state is viewed as a three body system $(XY\Lambda)$ characterized by the wavefunction ψ_Λ which depends upon the three relative separations. The position vectors of the clusters and the Λ will be denoted $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_\Lambda$, respectively, and the proton and the pion will be labeled 3 and 4. The transition matrix element for the four body decay in terms of PWIA is

$$\mathcal{M}^{\pm} = P^{(\pm)}\left(S + P\frac{\boldsymbol{\sigma}\cdot\boldsymbol{q}_{34}}{q_\Lambda}\right) Q^{(0)}, \tag{59}$$

$$Q^{(0)} = \int \psi_\Lambda^*(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_\Lambda) \exp\left(i \sum_{j=1}^{4} \boldsymbol{r}_j\cdot\boldsymbol{p}_j\right) \delta(\boldsymbol{r}_\Lambda - \boldsymbol{r}_3) \times$$

$$\times\, \delta(\boldsymbol{r}_\Lambda - \boldsymbol{r}_4)\, \delta\left(\frac{\sum_{i=1}^{4} m_i \boldsymbol{r}_i}{\sum_{i=1}^{4} m_i}\right) d^3r_1 d^3r_2 d^3r_3 d^3r_4 d^3r_\Lambda. \tag{60}$$

Introducing ϱ — the separation vector of particle 3 relative to the (1, 2) center of mass — and performing the trivial integrations one obtains for $Q^{(0)}$ the following expression:

$$Q^{(0)} = \int \psi_\Lambda^*(\boldsymbol{r}, \varrho)\, e^{i\boldsymbol{q}_{12}\boldsymbol{r}}\, e^{i\boldsymbol{q}\varrho}\, d^3r d^3\varrho = \tilde{\psi}_\Lambda^*(\boldsymbol{q}_{12}, \boldsymbol{q}). \tag{61}$$

Hence, similarly as for a three body decay, the overlap integral (61) is a (double) Fourier transform of ψ_Λ. The rate does not depend on spin addition and may be written as

$$\frac{\partial^3 \tau}{\partial T_{12} \partial T_{34} \partial \cos \Theta_{12}} = \text{const} \left[|S|^2 + |P|^2 \left(\frac{q_{34}}{q_\Lambda} \right)^2 \right] |\tilde{\psi}_\Lambda(\boldsymbol{q}_{12}, \boldsymbol{q})|^2 \times$$
$$\times [T_{12}(\Delta E - T_{12} - T_{34}) T_{34}]^{\frac{1}{2}}. \tag{62}$$

In general, the right hand side will depend on Θ_{12}, except for those cases when in the initial state the relative motion allows only s-states. The two other angles (φ and Θ_{34}) are redudant and have been integrated out in (62).

The next step in incorporating final state interactions, is to assume that the two clusters may interact, whereas both the proton and the pion do not. The overlap integral (60) is then generalized by replacing the exponential $e^{i\boldsymbol{q}_{12}\boldsymbol{r}}$ in the integrand of formula (60) by the corresponding wavefunction representing the relative motion of the interacting clusters. In this approximation also the rate does not depend on φ and Θ_{34} provided the clusters are spinless. If at least one of the clusters has non zero spin the rate may contain pseudoscalar terms proportional to $\hat{\boldsymbol{q}} \cdot (\hat{\boldsymbol{q}}_{12} \times \hat{\boldsymbol{q}}_{34})$. As is well known, the parity nonconservation in the Λ decay allows both the scalars and the pseudoscalars to appear in the expression for the partial rate. In a three body decay, in contrast with the present situation, we simply did not have a pseudoscalar at our disposal. In a four body decay there are three independent vectors available and one can construct a pseudoscalar which can survive spin summations. The overlap integrals are functions of T_{12}, T_{34} and $\cos \Theta_{12}$. They do not depend on $\hat{\boldsymbol{q}}_{34}$ and the only place where $\boldsymbol{q}_{34}$ appears is the pseudoscalar quantity $\hat{\boldsymbol{q}} \cdot (\hat{\boldsymbol{q}}_{12} \times \hat{\boldsymbol{q}}_{34})$. Since the latter expression is linear in $\hat{\boldsymbol{q}}_{34}$ it must vanish after integrating over $d\Omega_{34}$. As we shall see later on, the pseudoscalar correlation terms are sensitive to the initial spin of the system and supply a useful test for the spin determination of the hypernuclei.

Usually, it should be expected that the proton is likely to interact with the clusters and, therefore the above approximation is not well justified. It is, however, a good starting point for a more adequate treatment in which only the pion is assumed to move freely, the remaining three particles interacting among themselves. Consider then a three particle system, labeled 1, 2, 3, embedded in a four particle system with noninteracting particle 4. For the three body system we can write the Schroedinger equation

$$\left[-\frac{1}{2\mu_{12}} \nabla_r^2 - \frac{1}{2\mu_{12,3}} \nabla_\varrho^2 - \frac{1}{2} \frac{1}{m_1+m_2+m_3} \nabla_{\boldsymbol{R}_3}^2 + V_{12}(r) + V_{23}\left(\left| \varrho - \frac{m_1}{m_1+m_2} \boldsymbol{r} \right| \right) + \right.$$
$$\left. + V_{31}\left(\left| \varrho + \frac{m_2}{m_1+m_2} \boldsymbol{r} \right| \right) \right] \Psi(\boldsymbol{r}, \varrho, \boldsymbol{R}_3) = \left[\frac{q_{12}^2}{2\mu_{12}} + \frac{k_3^2}{2\mu_{12,3}} + \frac{1}{2} \frac{P_3^2}{m_1+m_2+m_3} \right] \Psi(\boldsymbol{r}, \varrho, \boldsymbol{R}_3)$$
$$\tag{63}$$

where $\boldsymbol{P}_3$ and $\boldsymbol{R}_3$ are the total momentum and center of mass vectors, respectively.

$$\boldsymbol{P}_3 = \sum_{i=1}^{3} \boldsymbol{p}_i = -\boldsymbol{p}_4,\ \boldsymbol{R}_3 = \sum_{i=1}^{3} m_i r \Big/ \sum_{i=1}^{3} m_i.$$

Our approximation consists in representing Ψ as

$$\Psi(\boldsymbol{r}, \varrho, \boldsymbol{R}_3) \sim e^{i\boldsymbol{P}_3\cdot\boldsymbol{R}_3}\, u_{\boldsymbol{q}_{12}}(\boldsymbol{r})\, w_{\boldsymbol{k},\boldsymbol{q}_{12}}(\varrho), \tag{64}$$

where $u_{\boldsymbol{q}_{12}}(\boldsymbol{r})$ and $w_{\boldsymbol{k}_3,\boldsymbol{q}_{12}}(\varrho)$ are solutions of the equations

$$\left[-\frac{1}{2\mu_{12}}\,\nabla_{\boldsymbol{r}}^2 + V_{12}(r)\right] u_{\boldsymbol{q}_{12}}(\boldsymbol{r}) = \frac{q_{12}^2}{2\mu_{12}}\, u_{\boldsymbol{q}_{12}}(\boldsymbol{r}), \tag{65}$$

$$\left[-\frac{1}{2\mu_{12,3}}\,\nabla_{\varrho}^2 + V_{\text{eff}}(\varrho, \boldsymbol{q}_{12})\right] w_{\boldsymbol{k}_3,\boldsymbol{q}_{12}}(\varrho) = \frac{k_3^2}{2\mu_{12,3}}\, w_{\boldsymbol{k}_3,\boldsymbol{q}_{12}}(\varrho) \tag{66}$$

where

$$V_{\text{eff}}(\varrho, \boldsymbol{q}_{12}) = \int u_{\boldsymbol{q}_{12}}^*(\boldsymbol{r}) \left[V_{23}\left(\left|\varrho - \frac{m_1}{m_1+m_2}\,\boldsymbol{r}\right|\right) + V_{31}\left(\left|\varrho + \frac{m_2}{m_1+m_2}\,\boldsymbol{r}\right|\right)\right] u_{\boldsymbol{q}_{12}}(\boldsymbol{r}) d^3r.$$

In practice, we shall approximate V_{eff} by an energy independent local potential $\overline{V}(\varrho)$ of a given shape, adjusted to fit the scattering or binding energy data for a $3+(1,2)$ system. In other words, we shall apply DWIA twice, for $u_{\boldsymbol{q}_{12}}(\boldsymbol{r})$, and for $w_{\boldsymbol{k}_3}(\varrho)$. To emphasize the fact that V_{eff} has been replaced by $\overline{V}$, we shall consequently write $w_{\boldsymbol{k}_3}(\varrho)$ instead of $w_{\boldsymbol{k}_3,\boldsymbol{q}_{12}}(\varrho)$.

The overlap integral takes the form

$$Q = \int \psi_\Lambda^*(\boldsymbol{r}, \varrho)\, e^{-i\boldsymbol{p}\cdot\varrho} u_{\boldsymbol{q}}(\boldsymbol{r})\, w_{\boldsymbol{k}}(\varrho)\, d^3r d^3\varrho,$$

where $\boldsymbol{p} = \dfrac{m_1+m_2}{m_1+m_2+m_3}\,\boldsymbol{p}_4$ and we have discarded the subscript 1, 2 and 3 for q and k, respectively.

In order to simplify the discussion, let us, for the time being, ignore the spin complications. Writing $u_{\boldsymbol{q}}(\boldsymbol{r})$ and $w_{\boldsymbol{k}}(\varrho)$ as

$$u_{\boldsymbol{q}}(\boldsymbol{r}) = u_{\boldsymbol{q}}^{(0)}(\boldsymbol{r}) + u_{\boldsymbol{q}}^{(1)}(\boldsymbol{r}) \equiv e^{i\boldsymbol{q}\cdot\boldsymbol{r}} + [u_{\boldsymbol{q}}(\boldsymbol{r}) - e^{i\boldsymbol{q}\cdot\boldsymbol{r}}], \tag{68}$$

$$w_{\boldsymbol{k}}(\varrho) = w_{\boldsymbol{k}}^{(0)}(\varrho) + w_{\boldsymbol{k}}^{(1)}(\varrho) \equiv e^{i\boldsymbol{k}\cdot\varrho} + [w_{\boldsymbol{k}}(\varrho) - e^{i\boldsymbol{k}\cdot\varrho}], \tag{69}$$

the overlap integral may be written in the form

$$Q = Q^{[0,0]} + Q^{[1,0]} + Q^{[0,1]} + Q^{[1,1]} \tag{70}$$

where

$$Q^{[i,j]} = \int \psi_\Lambda^*(\boldsymbol{r}, \varrho) e^{-i\boldsymbol{p}\cdot\varrho}\, u_{\boldsymbol{q}}^{(i)}(\boldsymbol{r})\, w_{\boldsymbol{k}}^{(j)}(\varrho)\, d^3r d^3\varrho. \tag{71}$$

The first term $Q^{[0,0]}$ is the PWIA expression (61) where all final state interactions are neglected, the second term $Q^{[1,0]}$ represents the correction for the cluster-cluster interaction, the third term $Q^{[0,1]}$ represents the interaction of the proton with both clusters, and the fourth term $Q^{[1,1]}$ is a rescattering correction.

The angular integration in (71) can be easily performed using the expansions in spherical harmonics,

$$\psi_\Lambda(\boldsymbol{r}, \boldsymbol{\varrho}) = 4\pi \sum_{L,M} i^L \varphi_L(r, \varrho)\, Y_{LM}(\hat{\boldsymbol{r}})\, Y^*_{LM}(\hat{\boldsymbol{\varrho}}), \tag{72}$$

$$u_{\boldsymbol{q}}(\boldsymbol{r}) = 4\pi \sum_{L,M} i^L f_L(qr)\, Y_{LM}(\hat{\boldsymbol{r}})\, Y^*_{LM}(\hat{\boldsymbol{q}}). \tag{73}$$

$$w_{\boldsymbol{k}}(\boldsymbol{\varrho}) = 4\pi \sum_{L,M} i^L F_L(k\varrho)\, Y_{LM}(\hat{\varrho})\, Y^*_{LM}(\hat{\boldsymbol{k}}). \tag{74}$$

For the overlap integral (71) one obtains the following expression:

$$Q^{i,} = \sum_{L,l,\lambda} I^{[i,j]}_{Ll\lambda}\, W_{Ll\lambda}(\hat{\boldsymbol{q}}, \hat{\boldsymbol{p}}, \hat{\boldsymbol{k}}), \tag{75}$$

where

$$I^{[i,j]}_{Ll\lambda} = (4\pi)^2\, i^{\lambda-l}\,(2L+1)\,(2l+1)\,(2\lambda+1)\times$$

$$\times \int \varphi^*_L(r, \varrho)\, f^{(i)}_L(qr)\, j_l(p\varrho)\, F^{(j)}_\lambda(k\varrho)\, r^2 dr\, \varrho^2 d\varrho, \tag{76}$$

$$f^{(0)}_L(qr) \equiv j_L(qr), f^{(1)}_L(qr) \equiv f_L(qr) - j_L(qr),$$

$$F^{(0)}_L\,(k\varrho) \equiv j_\lambda(k\varrho), F^{(1)}_\lambda(k\varrho) \equiv F_\lambda(k\varrho) - j_\lambda(k\varrho);$$

$$W_{Ll\lambda}(\hat{\boldsymbol{q}}, \hat{\boldsymbol{p}}, \hat{\boldsymbol{k}}) = \left(\frac{4\pi}{2L+1}\right)^{\frac{1}{2}} \left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}} \left(\frac{4\pi}{2\lambda+1}\right)^{\frac{1}{2}} \times$$

$$\times \begin{pmatrix} L & l & \lambda \\ 0 & 0 & 0 \end{pmatrix} \sum_{M,m,\mu} \begin{pmatrix} L & l & \lambda \\ M & m & \mu \end{pmatrix} Y_{LM}(\hat{\boldsymbol{q}})\, Y_{LM}(\hat{\boldsymbol{p}}\, Y_{\lambda\mu}(\hat{\boldsymbol{k}}). \tag{77}$$

Owing to the presence of the $3j$ symbols L, l, λ must fulfil the triangle inequality and the sum $L+l+\lambda$ must be an even number. In the correction terms (75) the summations over L and λ run from zero to some $L_{\max}$ and $\lambda_{\max}$. The sum over l is does not go to infinity either but terminates, due to the triangle inequality, as $l_{\max} = L_{\max} + \lambda_{\max}$. The correlation function $W_{Ll\lambda}(\hat{\boldsymbol{q}}, \hat{\boldsymbol{p}}, \hat{\boldsymbol{k}})$ can be also written in a form very convenient for computations:

$$W_{Ll\lambda}(\hat{\boldsymbol{q}}, \hat{\boldsymbol{p}}, \hat{\boldsymbol{k}}) = \frac{1}{4\pi} \int P_L(\hat{\boldsymbol{q}} \cdot \hat{\boldsymbol{x}}) P_l(\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{x}}) P_\lambda(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{x}}) d\Omega_x. \tag{78}$$

Some properties of the correlation function are collected in Appendix C. The radial double overlap integral $I^{[i,j]}_{Ll\lambda}$ can be evaluated analytically when $\psi_\Lambda(\boldsymbol{r}, \boldsymbol{\varrho})$ is assumed to be a product of three Gaussian functions representing the correlations between the particles which constitute the hypernucleus and if the distorted wave expressions are taken for $f_L(qr)$ and $F_\lambda(k\varrho)$. The details of the calculation are given in Appendix D.

There is no formal difficulty in generalizing the above results by including spins, Coulomb interaction and more channels. We shall briefly discuss the case when the clusters have spins $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ and add up to form the total spin J_N of the core. The wavefunctions $u_{\boldsymbol{q}}(\boldsymbol{r})$ and $w_{\boldsymbol{k}}(\varrho)$ are then replaced by operators in spin space

$$u_{\boldsymbol{q}}(\boldsymbol{r}) = [u_{\boldsymbol{q}}^{(0)} + u_{\boldsymbol{q}}^{(1)} + (-\boldsymbol{J}_N \cdot \boldsymbol{L}_{\boldsymbol{q}}) u_{\boldsymbol{q}}^{(2)} + \ldots + (-\boldsymbol{J}_N \cdot \boldsymbol{L}_{\boldsymbol{q}})^{2J_N} u_{\boldsymbol{q}}^{(2J_N+1)}] \Pi^{S_1 S_2 J_N}; \tag{79}$$

$$w_{\boldsymbol{k}}(\varrho) = w_{\boldsymbol{k}}^{(0)} + [w_{\boldsymbol{k},+}^{(1)} + (-\boldsymbol{I} \cdot \boldsymbol{L}_{\boldsymbol{k}})\, w_{\boldsymbol{k},+}^{(2)} + \ldots + (-\boldsymbol{I} \cdot \boldsymbol{L}_{\boldsymbol{k}})^{2J_N+1}\, w_{\boldsymbol{k},+}^{(2J_N+2)}]\, P^{(+)} + [w_{\boldsymbol{k},-}^{(1)} + (-\boldsymbol{I} \cdot \boldsymbol{L}_{\boldsymbol{k}})\, w_{\boldsymbol{k},-}^{(2)} + \ldots + (-\boldsymbol{I} \cdot \boldsymbol{L}_{\boldsymbol{k}})^{2J_N-1}\, w_{\boldsymbol{k},-}^{(2J_N)}]\, P^{(-)}, \tag{80}$$

where $\boldsymbol{J}_N = \boldsymbol{S}_1 + \boldsymbol{S}_2$, $\boldsymbol{I} = \boldsymbol{J}_N + \frac{1}{2}\boldsymbol{\sigma}$.

The overlap integral will receive various spin corrections in addition to those which have been already accounted for in formula (70). The terms $(-\boldsymbol{J}_N \cdot \boldsymbol{L}_{\boldsymbol{q}})$ and $(-\boldsymbol{I} \cdot \boldsymbol{L}_{\boldsymbol{k}})$ multiplied by $\boldsymbol{\sigma}\hat{\boldsymbol{p}}\, P^{(\pm)}$, in general, will give a nonvanishing contribution in the summation over spins. This contribution will be proportional to the pseudoscalar $\pm\hat{\boldsymbol{q}} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{p}})$, where the sign $\pm$ corresponds to $I = J_N \pm \frac{1}{2}$. The decay rate integrated over all kinematical variables except φ will be of the form

$$\frac{\partial \tau^{\pm}}{\partial \varphi} = \text{const}\, [A^{\pm} \pm B^{\pm} \sin \varphi], \tag{81}$$

where $A^{\pm}$ and $B^{\pm}$ are numbers. This formula, when confronted with experimental distributions, may prove to be quite helpful for establishing the spin value of the hypernucleus.

APPENDIX A

The wavefunctions with detemined momentum and spin

The wavefunction with determined momentum and spin $\boldsymbol{S}$ is represented as an operator $u_{\boldsymbol{q}}(\boldsymbol{r})$ acting on a $(2S+1)$ component spinor χ. The operator $u_{\boldsymbol{q}}(\boldsymbol{r})$ is given by

$$u_{\boldsymbol{q}}(\boldsymbol{r}) = \sum_{L=0}^{\infty} i^L\, [g_L^{(1)}(qr) + g_L^{(2)}(qr)\, (\boldsymbol{S} \cdot \boldsymbol{L}) + \ldots + g_L^{(2S+1)}(qr)\, (\boldsymbol{S} \cdot \boldsymbol{L})^{2S}]\, P_L(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{q}});$$

a. $S = \frac{1}{2}$

$$g_L^{(1)} = (L+1)\, f_L^+ + L f_L^-,$$

$$g_L^{(2)} = f_L^+ - f_L^-,$$

where $\pm$ denotes $J = L \pm \frac{1}{2}$.

b. $S = 1$

$$g_L^{(1)} = (2L+1)\, f_L^0 + f_L^+ - f_L^-,$$

$$g_L^{(2)} = \frac{L+2}{L+1} f_L^+ - \frac{L-1}{L} f_L^- - \frac{2L+1}{L(L+1)} f_L^0,$$

$$g_L^{(3)} = \frac{1}{L+1} f_L^+ + \frac{1}{L} f_L^- - \frac{2L+1}{L(L+1)} f_L^0,$$

where $\pm$ and 0 correspond to $J = L\pm 1$ and $J = L$, respectively

c. $S = \frac{3}{2}$

$$g_L^{(1)} = \frac{(L+4)(2L+3)}{2(L+1)(L+2)(L+3)} f_L^{+\frac{3}{2}} + \frac{(L+4)(2L+3)}{2(L+1)(L+3)} f_L^{+\frac{1}{2}} - \frac{3L(2L+3)}{2(L^2-1)(L+2)} f_L^{-\frac{1}{2}} + \frac{L+4}{2(L^2-1)} f_L^{-\frac{3}{2}};$$

$$g_L^{(2)} = \frac{2L^2+20L+33}{3(L+1)(L+2)(L+3)} f_L^{+\frac{3}{2}} + \frac{(L+2)(L-6)}{L(L+1)(L+3)} f_L^{+\frac{1}{2}} - \frac{2L^2-q}{(L^2-1)(L+2)} f_L^{-\frac{1}{2}} + \frac{(L-2)(L+6)}{3L(L^2-1)} f_L^{-\frac{3}{2}};$$

$$g_L^{(3)} = \frac{2(3L+10)}{3(L+1)(L+2)(L+3)} f_L^{+\frac{3}{2}} - \frac{2(2L+7)}{L(L+1)(L+3)} f_L^{+\frac{1}{2}} + \frac{2(L+6)}{(L^2-1)(L+2)} f_L^{-\frac{1}{2}} - \frac{14}{3L(L^2-1)} f_L^{-\frac{3}{2}};$$

$$g_L^{(4)} = \frac{4}{3(L+1)(L+2)(L+3)} f_L^{+\frac{3}{2}} - \frac{4}{L(L+1)(L+3)} f_L^{+\frac{1}{2}} + \frac{4}{(L^2-1)(L^2+2)} f_L^{-\frac{1}{2}} - \frac{4}{3L(L^2-1)} f_L^{-\frac{3}{2}};$$

where $\pm\frac{3}{2}$ and $\pm\frac{1}{2}$ correspond to $J = L\pm\frac{3}{2}$ and $J = L\pm\frac{1}{2}$, respectively.

APPENDIX B

Explicit formulae for generalized enhancement factors in the case of two channel scattering

The equation for λ (*cf.* (48)),

$$\begin{vmatrix} \varkappa_1^2-\lambda^2 & 2\mu_1 \bar{V}_{12} \\ 2\mu_2 \bar{V}_{12} & \varkappa_2^2-\lambda^2 \end{vmatrix} = 0$$

has the following solutions

$$\lambda_{1,2}^2 = \tfrac{1}{2}(\varkappa_1^2+\varkappa_2^2) \pm [\tfrac{1}{4}(\varkappa_1^2-\varkappa_2^2)^2 + 4\mu_1\mu_2\bar{V}_{12}]^{\frac{1}{2}}.$$

ψ_1 and ψ_2 can be written as

$$\psi_1 = C_1 j(\lambda_1 r) - C_2 \frac{2\mu_1\bar{V}_{12}}{\varkappa_1^2-\lambda_2^2} j(\lambda_2 r),$$

$$\psi_2 = -C_1 \frac{2\mu_2\bar{V}_{12}}{\varkappa_2^2-\lambda_1^2} j(\lambda_1 r) + C_2 j(\lambda_2 r),$$

where the two complex amplitudes C_1 and C_2 may regarded as generalized enhancement factors. C_1 and C_2 can be expressed in terms of the reduced T' matrix elements

$$C_1 = \frac{1}{j(\lambda_1 R)} \frac{\varkappa_1^2-\lambda_2^2}{\lambda_1^2-\lambda_2^2}\left[A + B\frac{2\mu_1\bar{V}_{12}}{\varkappa_1^2-\lambda_2^2}\right],$$

$$C_2 = -\frac{1}{j(\lambda_2 R)} \frac{\varkappa_2^2-\lambda_1^2}{\lambda_1^2-\lambda_2^2}\left[B + A\frac{2\mu_2\bar{V}_{12}}{\varkappa_2^2-\lambda_1^2}\right],$$

where

$$A = \psi_1^{(1)}(R) = j(p_1 R) + ih^{(-)}(p_1 k)\, T'_{11},$$

$$B = \psi_2^{(1)}(R) = \sqrt{\varrho_2/\varrho_1}\, [ih^{(-)}(p_2 R) T'_{21}],$$

if the ingoing wave is in channel 1, or

$$A = \psi_1^{(2)}(R) = \sqrt{\varrho_1/\varrho_2}\, [ih^{(-)}(p_1 R)\, T'_{12}],$$

$$B = \psi_2^{(2)}(R) = j(p_2 R) + ih^{(-)}(p_2 R)\, T'_{22},$$

if the ingoing wave is in channel 2.

APPENDIX C

Some properties of the correlation function $W_{Ll\lambda}(\hat{\boldsymbol{q}}, \hat{\boldsymbol{p}}, \hat{\boldsymbol{k}})$

(*i*) For $\lambda = 0$ W_{Ll0} is a Legendre polynomial

$$W_{Ll0}(\hat{\boldsymbol{q}}, \hat{\boldsymbol{p}}, \hat{\boldsymbol{k}}) = \delta_{Ll} \frac{1}{2L+1} P_L(\hat{\boldsymbol{q}} \cdot \hat{\boldsymbol{p}});$$

(*ii*) Symmetry properties

$$W_{Ll\lambda}(\hat{\boldsymbol{q}}, \hat{\boldsymbol{p}}, \hat{\boldsymbol{k}}) = W_{lL\lambda}(\hat{\boldsymbol{p}}, \hat{\boldsymbol{q}}, \hat{\boldsymbol{k}}) = W_{L\lambda l}(\hat{\boldsymbol{q}}, \hat{\boldsymbol{k}}, \hat{\boldsymbol{p}}),$$

$$W_{Ll\lambda}(\hat{\boldsymbol{q}}, \hat{\boldsymbol{p}}, \hat{\boldsymbol{k}}) = 0 \text{ if } L+l+\lambda = \text{odd number.}$$

(*iii*) The integral (78) can be easily evaluated using the following expressions

$$\int \hat{x}_{\mu_1} \hat{x}_{\mu_2} \dots \hat{x}_{\mu_{2N+1}}\, d\Omega_x = 0,$$

$$\int \hat{x}_{\mu_1} \hat{x}_{\mu_2} \dots \hat{x}_{\mu_{2N+1}}\, d\Omega_x = \frac{1}{(2N+1)!!} \begin{bmatrix} \text{A sum of all possible products} \\ \text{of Kronecker's deltas.} \end{bmatrix}$$

APPENDIX D

The radial integral

If $\psi_\Lambda(\boldsymbol{r}, \boldsymbol{\varrho})$ is a superposition of Gaussians, the radial overlap integral will be a sum of integrals of the type

$$X_{Ll\lambda} = \int_0^\infty e^{-\alpha^2 x^2} J_{l+\frac{1}{2}}(ax)\, J_{l+\frac{1}{2}}(bx)\, J_{\lambda+\frac{1}{2}}(cx)\, x^{\frac{1}{2}}\, dx.$$

Expanding the product of Bessel functions $J_{L+\frac{1}{2}}(ax)\,J_{l+\frac{1}{2}}(bx)$ in a power series one can evaluate the integral and arrives at the expression

$$X_{Ll\lambda} = \frac{1}{2}\,a^{-\frac{3}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma\left(\frac{(L+l+\lambda+3+2n)}{2}\right)}{\Gamma(L+\frac{3}{2}+n)\Gamma(l+\frac{3}{2})\Gamma(\lambda+\frac{3}{2})} \times$$

$$\times \left(\frac{a}{2a}\right)^{L+\frac{1}{2}+2n} \left(\frac{b}{2a}\right)^{l+\frac{1}{2}} \left(\frac{c}{2a}\right)^{\lambda+\frac{1}{2}} \times$$

$$\times F\left(-n,\,-n-L-\frac{1}{2},\,L+\frac{3}{2};\,\frac{b^2}{a^2}\right)\,{}_1F_1\left(\frac{L+l+\lambda+3+2n}{2},\,\lambda+\frac{3}{2},\,-\frac{c^2}{4a^2}\right).$$

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