ON THE CALCULATION OF D_Λ IN THE PHASE-SHIFT APPROXIMATION

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The accuracy of the phase-shift approximation in calculating the well-depth of a Λ-particle in nuclear matter, D_Λ, is investigated. A model case of a simple Λ-N potential is considered. The value of D_Λ calculated in the phase-shift approximation is about 10 MeV higher than the value obtained by the complete K-matrix method. This indicates that the phase-shift approximation is too rough for application in the D_Λ problem.

1. *Introduction*

The well-depth D_Λ of a Λ-particle in nuclear matter is of importance and interest because it gives some informations about the Λ-nucleon interaction.

The most accurate method of calculating D_Λ is probably the K-matrix method based on the Brueckner theory [1], [2]. Recently, an approximation to the K-matrix method, namely, the phase-shift approximation (PSA) has been applied by Bhaduri and Law in calculating D_Λ [3]. However, in the case of pure nuclear matter the phase-shift approximation is known to fail [4]. The purpose of the present paper is to investigate the applicability of the PSA in the D_Λ problem.

To check the accuracy of the PSA we consider a model case, in which we assume the ΛN interaction to be represented by one of the simple spin-independent potentials considered by Downs and Ware [5]:

$$V_{\Lambda N} = \begin{cases} \infty & \text{for} \quad r \leqslant r_c, \\ -V_0 \exp\left[-3.5412(r-r_c)/b\right] & \text{for} \quad r > r_c, \end{cases} \tag{1}$$

with $r_c = 0.4$ fm, $b = 1.1$ fm, $V_0 = 330.9$ MeV.

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In this case we calculate D_Λ in the PSA. On the other hand, the "exact" result for D_Λ, *i. e.*, the result obtained with the K-matrix method, is known for $V_{\Lambda N}$, Eq. (1) [1].

The results obtained indicate a large difference (of about 25%) between the PSA and the "exact" result for D_Λ. This seems to indicate that the PSA is too rough to be applied in the D_Λ problem. The calculational method is presented in Section 2, and it follows in essence the procedure applied by Bhaduri and Law [3]. The results obtained are presented and discussed in Section 3.

2. *Calculation of D_Λ in the PSA*

In nuclear matter, according to the present state of the Brueckner theory, the potential energy of a Λ-particle is

$$-D_\Lambda = \frac{4}{(2\pi)^3}\frac{1}{\beta^3}\int_0^{\beta k_F} \langle \bar{\boldsymbol{k}}|t^N|\bar{\boldsymbol{k}}\rangle d_3k, \tag{2}$$

where $\beta = m_\Lambda/(m_\Lambda+m_N)(m_\Lambda$ and m_N are the masses of the Λ-particle and the nucleon), k_F is the Fermi momentum of the nucleons, $\boldsymbol{k}$ is the relative momentum, and t^N is the ΛN reaction matrix inside nuclear matter.

Expressing t^N in terms of t^F (the reaction matrix for free ΛN scattering) one gets:

$$\begin{aligned} t^N &= t^F + t^F\left(\frac{P}{e_0} - \frac{Q'}{e_N}\right)t^N \\ &= t^F + t^F\left(\frac{P}{e_0} - \frac{Q'}{e_N}\right)t^F + \text{higher order terms}, \end{aligned} \tag{3}$$

where P is the principal value operator, Q' is the exclusion principle operator, e_0 and e_N are the energy denominators, which will be specified later. Combining Eqs (2) and (3) one gets $D_\Lambda^{(1)}$, the first-order term of D_Λ in the PSA,

$$D_\Lambda^{(1)} = -\frac{4}{(2\pi)^3}\frac{1}{\beta^3}\int_0^{\beta k_F} \langle \boldsymbol{k}|t^F|\boldsymbol{k}\rangle d_3k \tag{4}$$

and the second-order term,

$$D_\Lambda^{(2)} = -\frac{4}{(2\pi)^3}\frac{1}{\beta^3}\int_0^{\beta k_F} \left\langle \boldsymbol{k}\,\middle|\, t^F\left(\frac{P}{e_0} - \frac{Q'}{e_N}\right)t^F \,\middle|\, \boldsymbol{k}\right\rangle d_3k. \tag{5}$$

To evaluate $D_\Lambda^{(1)}$ one has to know the diagonal matrix elements of the reaction matrix t^F. These matrix elements can be expressed in terms of δ_l, the l-partial wave phase-shifts generated by the ΛN potential,

$$\langle \boldsymbol{k}|t^F|\boldsymbol{k}\rangle = -4\pi\frac{\hbar^2}{2\mu_{\Lambda N}}\frac{1}{k}\sum_l (2l+1)\,\delta_l, \tag{6}$$

where $\mu_{\Lambda N}$ is the ΛN reduced mass.

Inserting (6) into Eq. (4) one obtains $D_{A,l}^{(1)}$, the l-partial wave contribution to the first-order term of D_A,

$$D_{A,l}^{(1)} = \frac{8}{\pi}\frac{1}{\beta^3}\frac{\hbar^2}{2\mu_{AN}}\int_0^{\beta k_F}(2l+1)\,\delta_l k dk. \tag{7}$$

In our calculation we restrict ourselves to the first three phase-shifts, $l = 0, 1, 2$. In our model case the phase-shifts are obtained by the numerical solution of the Schödinger equation with V_{AN}, Eq. (1). The values of $\delta_0(k)$, $\delta_1(k)$, $\delta_2(k)$ thus obtained are shown in Fig. 1. Using these values of δ_l and taking $k_F = 1.366\ \text{fm}^{-1}$ we have performed numerically the integrations in Eq. (7). The resulting values of $D_{A,l}^{(1)}$ are presented in Table I.

The calculation of $D_A^{(2)}$ is more complicated. Let us denote by $\boldsymbol{q}$ the momentum transfer of the interacting lambda and nucleon whose momenta are initially $\boldsymbol{k}_A$ and $\boldsymbol{k}_1$. Then, the

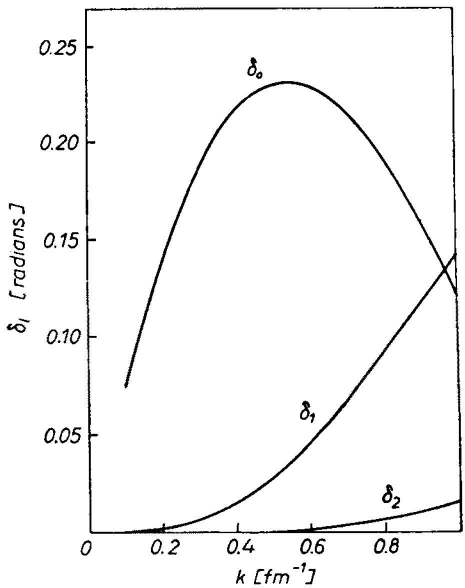

Fig. 1. The phase-shifts (in radians) generated by the V_{AN} potential, Eq. (1), as functions of the AN relative momentum, k

TABLE I

The results of D_A (in MeV) obtained in the phase-shift approximation and the exact results of Ref. [1]

l	First-order term	Second-order term	Exact [1]
0	34.7	−4.4	21.2
1	16.9	—	14.0
2	1.2	—	1.1
Total	52.8	∼−7.1	36.3

energy denominator e_0 is defined by

$$e_0 = \frac{\hbar^2(\boldsymbol{k}_1+\boldsymbol{q})^2}{2m_N} + \frac{\hbar^2(\boldsymbol{k}_\Lambda-\boldsymbol{q})^2}{2m_\Lambda} - \frac{\hbar^2k_1^2}{2m_N} - \frac{\hbar^2k_\Lambda^2}{2m_\Lambda} = \frac{\hbar^2}{2\mu_{\Lambda N}}(q^2+2\boldsymbol{kq}). \tag{8}$$

To determine the energy denominator e_N we assume that the nucleon and the Λ-particle are moving initially in a potential well U_N and U_Λ respectively and they are free after scattering, *i. e.*, we assume the single nucleon and lambda potentials to be equal to zero in the intermediate states. Then[1]

$$e_N = \frac{\hbar^2}{2\mu_{\Lambda N}}(q^2+2\boldsymbol{kq}) - U_N - U_\Lambda. \tag{9}$$

We approximate U_N by a quadratic function

$$U_N = Ak_1^2 + C, \tag{10}$$

with the constants A and C fixed by the two requirements [1]:

$$\bar{\varepsilon}_0 + \tfrac{1}{2}\bar{U}_N = \tfrac{1}{2}(\bar{\varepsilon}_0+\bar{\varepsilon}_N) = \varepsilon_{\mathrm{vol}}, \tag{11}$$

$$\varepsilon_N(k_F) = \varepsilon_{\mathrm{vol}}, \tag{12}$$

where ε_0 denotes the nucleon kinetic energy, $\varepsilon_N = \varepsilon_0 + U_N$, the bars denote average values over the Fermi sea, and $\varepsilon_{\mathrm{vol}} = -15.8$ MeV is the energy per nucleon in nuclear matter. Solving Eqs (11) and (12) we obtain

$$A = 63.407 \text{ MeV fm}^2, \quad C = -113.33 \text{ MeV}.$$

Taking $U_\Lambda = -36.3$ MeV (which is the "exact" value obtained in Ref, [1]) we may write Eq. (9) in a modified form

$$e_N = \frac{\hbar^2}{2\mu_{\Lambda N}}(q^2+2\boldsymbol{kq}+\Delta-\nu k^2), \tag{13}$$

with $\Delta = 3.917$ fm^{-2}, $\nu = 5.6254$.
Now, the second-order contribution to D_Λ is given by:

$$D_\Lambda^{(2)} = -\frac{4}{(2\pi)^3}\frac{1}{\beta^3}\frac{1}{(2\pi)^3}\int\limits_0^{\beta k_F} d_3k \int\limits_0^\infty d_3q |\langle \boldsymbol{k}|t^F|\boldsymbol{k}+\boldsymbol{q}\rangle|^2 \left(\frac{P}{e_0(\boldsymbol{k},\boldsymbol{q})} - \frac{Q'}{e_N(\boldsymbol{k},\boldsymbol{q})}\right). \tag{14}$$

The main problem is to evaluate off-the-momentum-shell matrix elements $\langle \boldsymbol{k}|t^F|\boldsymbol{k}+\boldsymbol{q}\rangle$. Assuming that t^F is local and energy-independent, *i. e.*,

$$\langle \boldsymbol{k}|t^F|\boldsymbol{k}+\boldsymbol{q}\rangle \approx \int\limits_0^\infty t^F(r)\, e^{i\boldsymbol{qr}} d_3r \equiv t^F(\boldsymbol{q}), \tag{15}$$

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[1] The gap in the single lambda particle spectrum has been neglected in Ref. [3].

and replacing $t^F(q)$ by $t_0^F(q)$ (the s-wave contribution is predominant) one arrives at the following formula:

$$t_0^F(q) = -\frac{16\pi}{q}\frac{\hbar^2}{2\mu_{\Lambda N}}\frac{d}{dq}\{\int_0^\infty dk\, k\delta_0(k)\,[\delta(2k-q)+\delta(2k+q)]\}. \tag{16}$$

Now, the total ΛN cross-section σ_T may be expressed in terms of δ_l, *i. e.*,

$$\sigma_T = \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2\delta_l \equiv \sum_l \sigma_l, \tag{17}$$

where σ_1 is the l-partial wave cross-section. Hence $t_0^F(q)$ may also be written as a function of σ_0, the s-wave cross-section,

$$t_0^F(q) = -\frac{2\pi}{q}\frac{\hbar^2}{2\mu_{\Lambda N}}\frac{d}{dq}\left[q^2\sqrt{\frac{\sigma_0(q/2)}{4\pi}}\right]. \tag{18}$$

Changing the parameter Δ to

$$\delta = \frac{\Delta}{(\beta k_F)^2},$$

and performing part of the integrations in Eq. (14) we obtain:

$$D_\Lambda^{(2)} = -\frac{8}{(2\pi)^4}\frac{2\mu_{\Lambda N}}{\hbar^2}\frac{(\beta k_F)^4}{\beta^3}\int_0^\infty dq|t^F(\beta k_F q)|^2\eta(q), \tag{19}$$

with

$$\eta(q) = q[f_<(q)-f\,(q)] \text{ for } \beta q \leqslant 2, \tag{20}$$

$$qf_>(q) \text{ for } \beta q > 2,$$

where

$$f(q) = \frac{1}{16}\left[(4-q^2)\ln\frac{2+q}{|2-q|}+4q\right], \tag{21}$$

$$f_<(q) = \frac{1}{2}\int_{1-\beta q}^{1} kdk\ln\left[\frac{q^2+2kq-\nu k^2+\delta}{q^2(1-\beta)-\left(\nu+\frac{1}{\beta}\right)k^2+\left(\delta+\frac{1}{\beta}\right)}\right], \tag{22}$$

$$f_>(q) = \frac{1}{2}\int_0^1 kdk\ln\left[1+\frac{4kq}{q^2-2kq-\nu k^2+\delta}\right]. \tag{23}$$

The cross-sections σ_0 and σ_T calculated with the phase-shifts obtained previously are shown in Figs 2 and 3. These cross-sections are reproduced by the following forms

$$\sigma_0 = 7.31\exp(-3.5\,k^{1.8}), \tag{24}$$

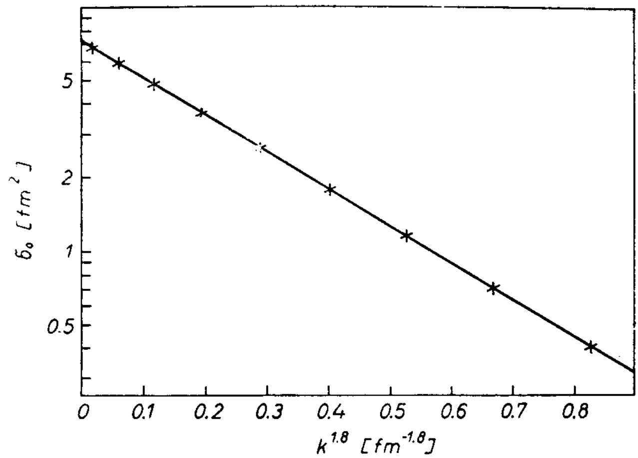

Fig. 2. The s-wave cross-section given by Eq. (24). Crosses represent values calculated with the δ_0 phase-shift

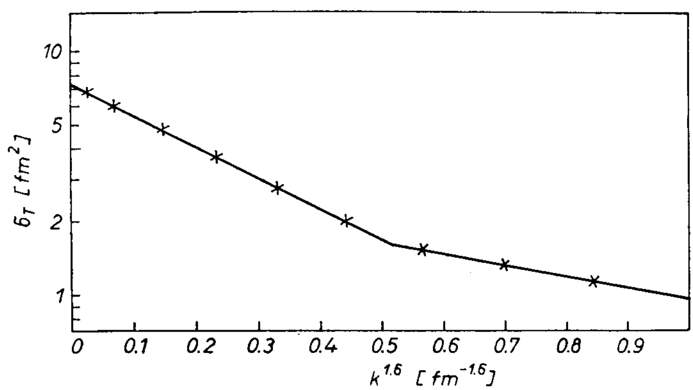

Fig. 3. The total cross-section given by Eq. (25a, b). Crosses represent values calculated with the phase-shifts δ_0 δ_1, δ_2, Eq. (17)

and

$$\sigma_T = \begin{cases} 7.5 \exp(-3k^{1.6}) & \text{for } k \leqslant 0.66 \text{ fm}^{-1}, \quad (25a) \\ 2.67 \exp(-k^{1.6}) & \text{for } k > 0.66 \text{ fm}^{-1}. \quad (25b) \end{cases}$$

Inserting expression (24) into Eq. (18) we have

$$t^F(q) \sim 2\sqrt{7.31}\,[1-0.2643\,q^{1.8}] \exp(-0.2937\,q^{1.8}). \tag{26}$$

Inserting (25a) into Eq. (18) we get

$$t^F(q) \sim 2\sqrt{7.5}\,[1-0.2455\,q^{1.6}] \exp(-0.3069\,q^{1.6}), \tag{27a}$$

and inserting (25b) into Eq. (18) we obtain

$$t^F(q) \sim 2\sqrt{2.67}\,[1-0.0818\,q^{1.6}] \exp(-0.1023\,q^{1.6}). \tag{27b}$$

Introducing expressions (26) and (27a, b) into Eq. (19) and performing the q-integration numerically we find $D^{(2)}_{\Lambda,0}$(the s-wave contribution to the second-order term of D_Λ) and $D^{(2)}_{\Lambda,T}$ (obtained by replacing σ_0 by σ_T in Eq. (18)).

3. *Discussion*

The results of our calculation are shown in Table I, which also contains the "exact" results of Ref. [1]. Our total first-order contribution, $D^{(1)}_\Lambda = 52.8$ MeV, *i. e.* it is 16.5 MeV higher than the "exact" result, $D_\Lambda = 36.3$ MeV.

Strictly speaking, the second-order contribution, $D^{(2)}_\Lambda$, goes beyond the PSA. Only by introducing drastic approximations (locality and energy independence of t^F) is it possible to express the second-order contribution through the ΛN phase-shifts. Actually, Bhaduri and Law [3] express the second-order contribution through the ΛN total cross-section and obtain in this way, in our notation, $D^{(2)}_{\Lambda,T}$. In the spirit of the approach of Ref. [3], the value of $D^{(2)}_{\Lambda,T} = -7.1$ MeV is supposed to approximate the second-order contribution. This reduces the PSA value of the Λ potential depth from 52.8 MeV to 45.7 MeV, which is still higher than the "exact" value of D_Λ by 9.4 MeV. If, on the other hand, we assume that $D^{(2)}_\Lambda$ may be approximated by $D^{(2)}_{\Lambda,0} = -4.4$ MeV we are lead to a difference of 12.1 MeV between the "exact" and approximate value of D_Λ. In any case we see that the PSA procedure of Bhaduri and Law [3] applied to our model case leads to a value of D_Λ which is about 10 MeV higher than the "exact" result. Now, the difference between the D_Λ values calculated with the contemporary, phenomenological ΛN potentials and the D_Λ values estimated empirically is probably less than 10 MeV [1]. For this reason it seems to us that the PSA is too rough to be applied in the D_Λ problem. On the other hand, as seen from Table I, the PSA may be useful for estimating the higher partial wave contribution to D_Λ.

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