CYLINDRICAL SYMMETRY IN EINSTEIN'S UNIFIED FIELD THEORY. I

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The static, cylindrically symmetric solutions of Einstein's unified field equations are derived, in the case when only an electric field is present. It is shown that, in general, the strong field equations lead to untenable physical conclusions.

1. Introduction

The solutions of the field equations in the nonsymmetric unified field theory (Einstein 1945, Einstein and Straus 1946, and Einstein 1951) hitherto discovered, fall into three classes. They are either static, spherically symmetric (Papapetrou 1948, Wyman 1950, Bonnor 1952 and others), plane-symmetric (Rao 1958, Sarkar 1966) or wave-like (Takeno 1958, Ingraham 1950, Vaidya 1961). None of these have met with marked success in leading to empirically verifiable predictions of the kind which enhanced the acceptance of General Relativity as a theory of the gravitational field, better than the classical model of Newton.

There are several obvious reasons for the difficulties in relating Einstein's theory to physics. First, of course, is the purely mathematical complexity of the equations themselves. Secondly, there is a considerable difficulty in interpreting a given solution because is it not clear either what should be regarded as the "metric" tensor (which ought to be symmetric and need not be the symmetric part $g_{\underline{\mu\nu}}$ of Einstein's nonsymmetric fundamental tensor $g_{\mu\nu}$ — we let Greek indices go from 1 to 4), or how to define the electromagnetic field (which might not be the skew part $g_{\underset{\vee}{\mu\nu}}$ of $g_{\mu\nu}$, *e.g.* Hlavaty 1957). There is doubt also how one should define symmetry itself and more will be said on this point in the next section. Similarly, we encounter trouble in postulating boundary conditions because we cannot easily conceive what happens "at infinity", and we do not know *a priori* with what kind of objects we are dealing locally. Einstein's own insistence that only everywhere nonsingular solutions should be regarded as meaningful does not help and there is also something radically wrong with the only postulate of the theory which has a clear physical content, namely the concept of Hermitian symmetry or Transposition Invariance (Klotz 1970).

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All this does not contribute to rendering Einstein's (or, for that matter, any other) unified field theory credible as a model of the macroscopic world structure. On the other hand, it may be argued that General Relativity itself requires construction of such a theory, even if it should prove a failure providing we knew where the fault would lie (Klotz 1969). In particular, no avenue should be left unexplored through which new solutions of the field equations can be found.

In this and the subsequent articles, we propose to derive a number of solutions in the case of static, cylindrical symmetry. It may be felt that the latter symmetry is more interesting in unified field theory than spherical since, celestial bodies apart, should stars carry a residual charge, it is more likely to be developed into an account of electric current flow and perhaps of a locally observable effect. Such an extension would involve seeking time dependent solutions and although we cannot attempt it now, one can regard our results as a first step towards it.

One final remark should be made before we can proceed. Up to a point we shall discuss concurrently the so-called strong (whose compatibility has never been proved in general) and weak (which have been derived from an invariant variational principle by Einstein and Straus and are therefore compatible whatever symmetry restrictions may be imposed) field equations. Indeed, one of us (Russell 1970) shows that in the particular case to be discussed, the two sets of equations coincide.

In this article we shall discuss the derivation of general solutions reserving the particular cases to the second paper. We shall find that the former are valid only for the strong field equations.

2. *Cylindrical symmetry in a unified field*

In General Relativity, definition of symmetry does not create serious problems because a Riemannian manifold can be always embedded in a higher dimensional, Euclidean space. Although the group G of coordinate transformations X under which the equations of the nonsymmetric unified field theory are invariant, is the same, the embedding property must be now assumed.

$$x'^{\mu} = x'^{\mu}(x^{\nu}). \tag{2.1}$$

This means that the space is to be thought of as possessing a symmetry in as much as it might do when viewed in a Euclidean space.

Let $y(x^{\mu})$ be a mathematical object, possibly the space itself. Then if

$$y'(x'^{\mu}) = y(x'^{\mu}) \tag{2.2}$$

for an element of G, we say that this element is a symmetry of y and the set of all such elements (a sub-group of G) is called the symmetry group of y.

Suppose that a tensor $A_{\mu\nu}$ possesses a symmetry for infinitesimal transformations

$$x'^{\mu} = x^{\mu} + \varepsilon\xi^{\mu}, \tag{2.3}$$

where ξ^μ are known functions of x^μ and $\varepsilon^2 \ll \varepsilon$. The law of tensor transformation then gives

$$A'_{\mu\nu}(x'^\lambda) = A_{\mu\nu}(x^\lambda) - \varepsilon\xi^\sigma{}_{,\mu}A_{\sigma\nu} - \varepsilon\xi^\sigma{}_{,\nu}A_{\mu\sigma} + 0(\varepsilon^2),$$

and, using Taylor's expansion

$$A'_{\mu\nu}(x'^\lambda) = A'_{\mu\nu}(x^\lambda) + \varepsilon A'_{\mu\nu,\sigma}\xi^\sigma + 0(\varepsilon^2).$$

Hence, from (2.2)

$$A_{\mu\nu,\sigma}\xi^\sigma + \xi^\sigma{}_{,\mu}A_{\sigma\nu} + \xi^\sigma{}_{,\nu}A_{\mu\sigma} = 0, \tag{2.4}$$

since $y'(x^\mu) = y(x^\mu)$ also.

Here, as throughout, comma denotes ordinary partial differentiation with respect to the coordinates, and the summation convention over repeated indices is used. A semi-colon will denote a covariant differentiation for a suitably specified affine connection $\Gamma^\lambda_{\mu\nu}$ which replaces the Riemannian-Christoffel brackets $\{^{\lambda}_{\mu\nu}\}$. Equation (2.4) represents the condition that $A_{\mu\nu}$ be symmetric under (2.3). In the Riemannian case, when $A_{\mu\nu}$ is the metric $g_{\mu\nu}$ of the space, it reduces to Killings' equations (10 in number)

$$\xi_{\mu;\nu} + \xi_{\mu;\nu} = 0$$

because the connection is given by $\{^{\lambda}_{\mu\nu}\}$.
However, in Einstein's nonsymmetric theory, the connection is defined by

$$(g_{\underset{+}{\mu}\,\underset{-}{\nu};\lambda} \equiv) g_{\mu\nu,\lambda} - \Gamma^\sigma_{\mu\lambda}g_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu}g_{\mu\sigma} = 0. \tag{2.5}$$

Hence, putting $A_{\mu\nu} = g_{\mu\nu}$ in (2.4) gives

$$\Gamma^\varrho_{\mu\sigma}g_{\varrho\nu}\xi^\sigma + \Gamma^\varrho_{\sigma\nu}g_{\mu\varrho}\xi^\sigma + \xi^\sigma{}_{,\mu}g_{\sigma\nu} + \xi^\sigma{}_{,\nu}g_{\mu\sigma} = 0,$$

that is

$$g_{\mu\varrho}(\xi^\varrho{}_{,\nu} + \Gamma^\varrho_{\sigma\nu}\xi^\sigma) + g_{\varrho\nu}(\xi^\varrho{}_{,\mu} + \Gamma^\varrho_{\mu\sigma}\xi^\sigma) = 0,$$

or, in Einstein's notation,

$$g_{\mu\varrho}\xi^{\varrho}_{+;\nu} + g_{\varrho\nu}\xi^{\varrho}_{-;\mu} = 0 \tag{2.6}$$

and we cannot conveniently lower the indices inside the semi-colon (see, for example, Einstein 1950).

In the case of cylindrical symmetry we require $y(x^\mu)$ to be invariant under arbitrary, and in particular, infinitesimal rotations (given by equation (2.3)) about a fixed axis of symmetry. In other words, a hoop of radius 'a' which is perpendicular to, and centred on the axis must transform into itself under (2.3).

Let (r, θ, ψ) be polar coordinates chosen so that the axis is given by

$$\theta = 0.$$

The metric on the hoop is then

$$ds^2 = a^2 d\psi^2,$$

and will be invariant if, in (2.4)

$$A_{\mu\nu} = 0\ (\mu, \nu \neq 3, 3), \qquad A_{33} = a^2.$$

A rotation along the hoop is given by $\xi^1 = \xi^2 = \xi^4 = 0$, so that $\xi^3 = \text{const}$, and therefore for a nonsymmetric $g_{\mu\nu}$ equation (2.4) gives

$$g_{\mu\nu,3} = 0, \tag{2.7}$$

$g_{\mu\nu}$ being thus independent of ψ, we are left with sixteen functions of three variables each which can be specialised further by two more coordinate conditions. Since $g_{\underline{\mu\nu}}$ and $g_{\check{\mu\nu}}$ transform independently of each other, a particularly simple form results when $g_{\underline{\mu\nu}}$ is assumed to be the "metric" tensor in the following sense.

Invariance under the transformations

$$\psi = -\psi, \qquad t = -t, \tag{2.8}$$

gives

$$g_{\underline{\mu\nu}} dx^\mu dx^\nu = g_{11} dx^{12} + 2g_{\underline{12}} dx^1 dx^2 + g_{22} dx^2 + g_{33} dx^{32} + g_{44} dt^2,$$

and the remaining coordinate conditions enable us to take

$$g_{11} = g_{22}, \qquad g_{\underline{12}} = 0.$$

We then have the so-called isothermal coordinates.

As far as the skew symmetric $g_{\mu\nu}$ is concerned, it is simplest to identify it (Russell 1970) with the electromagnetic field tensor but, following a suggestion of Einstein, with the electric vector $\boldsymbol{E}$ interchanged with the magnetic vector $\boldsymbol{H}$. Thus we take

$$g_{ij} = E_K; \qquad i, j, k = \text{cyclic } 1, 2, 3; \qquad g_{i4} = -H_i. \tag{2.9}$$

In the sequel, we consider the problem of solving the unified field equations corresponding to a purely electric field

$$\boldsymbol{E} = (E(r), 0, 0), \qquad \boldsymbol{H} = 0, \tag{2.10}$$

in cylindrical polar coordinates

$$x^1 = r, \qquad x^2 = z, \qquad x^3 = \theta, \qquad x^4 = t,$$

for which the axis of symmetry is

$$r = 0,$$

and r is the outward radial distance.

In view of the above discussion, we can take $g_{\mu\nu}$ in the form

$$g_{\mu\nu} = \begin{Vmatrix} -\alpha & \cdot & \cdot & \cdot \\ \cdot & -\alpha & E & \cdot \\ \cdot & -E & -\beta & \cdot \\ \cdot & \cdot & \cdot & \gamma \end{Vmatrix} \tag{2.11}$$

where α, β, γ and E are functions of r only.

3. *The field equations*

The Ricci tensor is defined by

$$R_{\mu\nu} = \Gamma^{\sigma}_{\mu\nu,\sigma} - \Gamma^{\sigma}_{\mu\sigma,\nu} - \Gamma^{\varrho}_{\mu\sigma}\Gamma^{\sigma}_{\varrho\nu} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\varrho}_{\sigma\varrho}. \tag{3.1}$$

As mentioned before, we can consider concurrently the strong unified field equations

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda} = 0, \quad R_{\mu\nu} = 0, \quad \Gamma_{\mu}(\equiv \Gamma^{\sigma}_{\underset{\vee}{\mu\sigma}}) = 0 \tag{3.2}$$

and the weak ones

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda} = 0, \quad R_{\underline{\mu\nu}} = 0, \quad R_{\underset{\cdot\vee\cdot}{\mu\nu,\lambda}} = 0, \quad \Gamma_{\mu} = 0, \tag{3.3}$$

where, as for $g_{\mu\nu}$, a hook denotes skew part (and underlining of a pair of suffixes, the symmetric part) of a quantity and the dots indicate a cyclic sum. The difference between (3.2) and (3.3) will be seen in the case of $g_{\mu\nu}$ given by (2.11), to reduce to the vanishing or otherwise of a constant. The first equation of either set (2.5) can be solved for $\Gamma^{\lambda}_{\mu\nu}$ to give

$$\Gamma^{1}_{11} = \frac{1}{2a}\frac{da}{dr},$$

$$\Gamma^{1}_{22} = \frac{D}{a}\left\{-\alpha(\alpha\beta - E^2)\frac{dE^2}{dr} + 2\alpha^2E^2\frac{d\beta}{dr} - \frac{d\alpha}{dr}(\alpha^2\beta^2 + E^4)\right\},$$

$$\Gamma^{1}_{33} = \frac{D}{a}\left\{-\beta(\alpha\beta - E^2)\frac{dE^2}{dr} + 2\beta^2E^2\frac{d\alpha}{dr} - \frac{d\beta}{dr}(\alpha^2\beta^2 + E^4)\right\},$$

$$\Gamma^{1}_{44} = \frac{1}{2a}\frac{d\gamma}{dr},$$

$$\Gamma^{1}_{\underset{\vee}{23}} = \frac{D}{a}(\alpha\beta - E^2)\left\{-(\alpha\beta - E^2)\frac{dE}{dr} + E\frac{d(\alpha\beta)}{dr}\right\},$$

$$\Gamma^{2}_{\underline{12}} = D\left\{\frac{1}{2}(\alpha\beta - E^2)\frac{dE^2}{dr} - \alpha E^2\frac{d\beta}{dr} + \alpha\beta^2\frac{d\alpha}{dr}\right\},$$

$$\Gamma^{2}_{\underset{\vee}{13}} = D\left\{-\beta(\alpha\beta - E^2)\frac{dE}{dr} - E^3\frac{d\beta}{dr} + E\beta^2\frac{d\alpha}{dr}\right\},$$

$$\Gamma^{3}_{\underset{\vee}{12}} = D\left\{\alpha(\alpha\beta - E^2)\frac{dE}{dr} + E^3\frac{d\alpha}{dr} - E\alpha^2\frac{d\beta}{dr}\right\},$$

$$\Gamma^{3}_{\underline{13}} = D\left\{\frac{1}{2}(\alpha\beta - E^2)\frac{dE}{dr} - \beta E^2\frac{d\alpha}{dr} + \beta\alpha^2\frac{d\beta}{dr}\right\},$$

$$\Gamma^{4}_{\underline{14}} = \frac{1}{2\gamma}\frac{d\gamma}{dr}, \tag{3.4}$$

where

$$D^{-1} = 2(\alpha^2\beta^2 - E^4),$$

all other components being zero. The equation (2.5) which determines the above components of the affine connection is written out in full in the Appendix.

The only components of the Ricci tensor which do not vanish identically are now

$$R_{11} = -\frac{d}{dr}\{\Gamma^2_{12}+\Gamma^3_{13}+\Gamma^4_{14}\}-(\Gamma^2_{12})^2-(\Gamma^3_{13})^2-(\Gamma^4_{14})^2-$$

$$-2\Gamma^2_{\underset{\vee}{31}}\Gamma^3_{\underset{\vee}{12}}+\Gamma^1_{11}\{\Gamma^2_{12}+\Gamma^3_{13}+\Gamma^4_{14}\},$$

$$R_{22} = \frac{d\Gamma^1_{22}}{dr}+\Gamma^1_{22}\{\Gamma^1_{11}-\Gamma^2_{12}+\Gamma^3_{13}+\Gamma^4_{14}\}-2\Gamma^1_{\underset{\vee}{23}}\Gamma^3_{\underset{\vee}{12}},$$

$$R_{\underset{\vee}{23}} = \frac{d\Gamma^1_{\underset{\vee}{23}}}{dr}-\Gamma^1_{33}\Gamma^3_{\underset{\vee}{21}}-\Gamma^1_{22}\Gamma^2_{\underset{\vee}{13}}+\Gamma^1_{\underset{\vee}{23}}(\Gamma^1_{11}+\Gamma^4_{14}),$$

$$R_{33} = \frac{d\Gamma^1_{33}}{dr}+\Gamma^1_{33}\{\Gamma^1_{11}+\Gamma^2_{12}-\Gamma^3_{13}+\Gamma^4_{14}\}-2\Gamma^1_{\underset{\vee}{23}}\Gamma^2_{\underset{\vee}{31}},$$

$$R_{44} = \frac{d\Gamma^1_{44}}{dr}+\Gamma^1_{44}\{\Gamma^1_{11}+\Gamma^2_{12}+\Gamma^3_{13}-\Gamma^4_{14}\}. \tag{3.5}$$

In view of (3.4), the last of the field equations:

$$\Gamma_\mu = 0$$

(the condition that $R_{\mu\nu}$ should be Hermitian symmetric) is identically satisfied. Hence the weak field equations become

$$R_{11} = 0, \quad R_{22} = 0, \quad R_{33} = 0, \quad R_{44} = 0 \tag{3.6}$$

and

$$\frac{dR_{\underset{\vee}{23}}}{dr} = 0. \tag{3.7}$$

From (3.7)

$$R_{\underset{\vee}{23}} = K, \text{ a constant},$$

and when $K = 0$ we obtain the strong field equations.

4. The general solutions of the strong field equations

The form of the components $\Gamma^\lambda_{\mu\nu}$ given by (3.4) suggests that we write

$$u = \alpha\beta, \quad v = \frac{\alpha}{\beta}, \quad e = E^2. \tag{4.1}$$

Then, dashes signifying differentiation with respect to r,

$$\Gamma^1_{11} = \frac{1}{4}\left(\frac{u'}{u} + \frac{v'}{v}\right), \qquad \Gamma^1_{44} = \frac{1}{2\sqrt{uv}}\,\gamma', \qquad \Gamma^4_{14} = \frac{1}{2}\frac{\gamma'}{\gamma},$$

$$\Gamma^1_{22} = -\frac{1}{2}\frac{u'+e'}{u+e} + \frac{1}{4}\frac{u'}{u} - \frac{1}{4}\frac{u+e}{u-e}\frac{v'}{v} = v\,\Gamma^1_{33} - \frac{1}{2}\frac{u+e}{u-e}\frac{v'}{v},$$

$$\Gamma^2_{12} = \frac{1}{4}\frac{u'+e'}{u+e} + \frac{1}{4}\frac{u}{u-e}\frac{v'}{v} = \Gamma^3_{13} + \frac{1}{2}\frac{u}{u-e}\frac{v'}{v},$$

$$\Gamma^1_{\check{2}3} = \frac{e}{2q}\left[\frac{u'+e'}{u+e} - \frac{1}{2}\frac{e'}{e}\right],$$

$$\Gamma^2_{\check{3}1} = \frac{1}{4}\frac{u}{q}\left[\frac{u'+e'}{u+e} - \frac{u'}{u} - \frac{e}{u-e}\frac{v'}{v}\right], \quad \Gamma^3_{\check{1}2} = \frac{1}{4}\frac{q}{e}\left[\frac{u'+e'}{u+e} - \frac{u'}{u} + \frac{e}{u-e}\frac{v'}{v}\right], \tag{4.2}$$

where

$$q = \sqrt{uve}.$$

The equation

$$R_{44} = 0$$

integrates immediately once, to give

$$\frac{\gamma'}{\sqrt{\gamma}}(uv)^{-\frac{1}{4}}(u+e)^{\frac{1}{2}} = k_1, \text{ a constant.} \tag{4.3}$$

Also, from

$$R_{22} = 0 = R_{33},$$

we find readily that

$$\left(\frac{u}{v}\right)^{\frac{1}{4}}\gamma^{\frac{1}{2}}\frac{v'}{v}\left(\frac{u}{u+e}\right)^{\frac{1}{2}}\frac{u+e}{u-e} = k_2, \text{ a constant,} \tag{4.4}$$

so that, from (4.3) and (4.4)

$$\frac{\gamma'}{\gamma} = \frac{k_1}{k_2}\frac{u}{u-e}\frac{v'}{v}.$$

Clearly, we can impose one (and only one) scale condition on the metric. Choosing therefore

$$k_1 = k_2,$$

we obtain the result

$$\frac{\gamma'}{\gamma} = \frac{u}{u-e}\frac{v'}{v}, \tag{4.5}$$

which enables us to proceed.

Let us write

$$\Gamma^1_{11} = U, \quad \Gamma^2_{12} = V, \quad \Gamma^4_{14} = W, \; p = \frac{u}{e}. \tag{4.6}$$

The field equations now become

$$-V' - (1+p)V^2 - pU^2 - W^2 + UV(1+2p) + W(2V-U) = 0, \tag{4.7}$$

$$U' - 2V' + (1+2p)U^2 + 2(p-1)V^2 - 4pUV = 0, \tag{4.8}$$

$$pU' - (p-1)V' + p(1+2p)U^2 + 2(1+p^2)V^2 - (1+p+4p^2)UV = \frac{qK}{e} \tag{4.9}$$

and (using (4.8))

$$W' + W(2V-U) = 0. \tag{4.10}$$

Let us now take

$$K = 0 \tag{4.11}$$

(strong case).
We obtain one first integral of the equations (4.7)-(4.10), by eliminating U', V' between them:

$$(1-p)V^2 - pU^2 + 2pUV - W^2 + W(2V-U) = 0. \tag{4.12}$$

Moreover, eliminating U' between (4.8) and (4.9), we find that

$$V' + V(2V-U) = 0,$$

so that, from (4.10)

V is proportional to W,

or

$$\gamma = B(u+e)^a, \tag{4.13}$$

where B and a are constants.

If we write now

$$z = ln(u+e),$$

and

$$a^2 b = (1+4a-a^2),$$

equation (4.12) gives

$$a^2(bp+1)z'^2 = (p+1)^{-1}p'^2 \tag{4.14}$$

and a straight forward though tedious calculation shows that the equations (4.7)-(4.10) are compatible only if

$$p' = 0. \tag{4.15}$$

Assuming that $p = \text{const} \; (\neq -1)$ in (4.2), gives

$$U = \frac{1}{4}\frac{u'}{u} + \frac{1}{4}\frac{v'}{v}, \quad V = \frac{1}{4}\frac{v'}{v} 1 + \frac{p}{p-1}\frac{v'}{v}, \quad W = \frac{1}{2}\frac{p}{p-1}\frac{v'}{v}$$

and

$$V = CW, \qquad \text{or } \frac{u'}{u} = \frac{2C-1}{4} \frac{p}{p-1} \frac{v'}{v}.$$

Hence $\quad V = mU, \; W = nU,$

$$m = \frac{2Ck}{(k+1)(2C-1)}, \qquad n = \frac{2k}{(k+1)(2C-1)},$$

$$\frac{k}{p} = \frac{2C-1}{p-1}.$$

Equations (4.7)–(4.10) then give as a compatibility condition,

$$(1-m)^2(1+p) = 0. \tag{4.16}$$

The case $p = -1$ must be investigated separately and we easily convince ourselves that $m = 1$ is impossible.

When $p = -1$, D^{-1} in equations (3.4) vanishes. Nevertheless, equations

$$g_{\underset{+}{\mu}\,\underset{-}{\nu};\lambda} = 0,$$

still have a solution for $\Gamma^{\lambda}_{\mu\nu}$, indeed a much simpler one. As before

$$\Gamma^1_{11} = \frac{1}{2}\frac{\alpha'}{\alpha}, \qquad \Gamma^1_{44} = \frac{1}{2}\frac{\gamma'}{\alpha}, \qquad \Gamma^4_{\underline{14}} = \frac{1}{2}\frac{\gamma'}{\gamma},$$

but, since now

$$\beta = -\frac{E^2}{\alpha}, \tag{4.17}$$

$$\Gamma^1_{22} = -\frac{z'}{z}, \quad \Gamma^1_{33} = zz', \quad \Gamma^2_{\underline{12}} = \frac{1}{2}\frac{z'}{z}, \quad \Gamma^3_{\underline{13}} = \frac{1}{2}\frac{\alpha'}{\alpha},$$

$$\Gamma^1_{23} = \frac{z'}{z^2}, \quad \Gamma^2_{31} = \frac{1}{2z}\left(\frac{1}{2}\frac{\alpha'}{\alpha} + \frac{z'}{z}\right), \quad \Gamma^3_{12} = \frac{1}{2}z' \tag{4.18}$$

where we have written

$$z = \frac{\alpha}{E}.$$

The field equations become

$$-\left(\frac{1}{2}\frac{z'}{z} + \frac{1}{2}\frac{\gamma'}{\gamma} + \frac{1}{2}\frac{\alpha'}{\alpha}\right)' - \frac{1}{4}\left(\frac{z'}{z}\right)^2 - \frac{z'}{z}\left(\frac{z'}{z} + \frac{1}{2}\frac{\alpha'}{\alpha}\right) -$$

$$-\frac{1}{4}\left(\frac{\alpha'}{\alpha}\right)^2 - \frac{1}{4}\left(\frac{\gamma'}{\gamma}\right)^2 + \frac{1}{2}\frac{\alpha'}{\alpha}\left(\frac{1}{2}\frac{z'}{z} + \frac{1}{2}\frac{\alpha'}{\alpha} + \frac{1}{2}\frac{\gamma'}{\gamma}\right) = 0, \tag{4.19a}$$

$$\left(\frac{z'}{z}\right)' + \frac{z'}{z}\left(\frac{\alpha'}{\alpha} - \frac{1}{2}\frac{z'}{z} + \frac{1}{2}\frac{\gamma'}{\gamma}\right) + \left(\frac{z'}{z}\right)^2 = 0, \tag{4.19b}$$

$$\left(\frac{z'}{z^2}\right)' + \frac{1}{2}\frac{z'^2}{z^3} - \frac{z'}{z^2}\left(\frac{z'}{z} + \frac{1}{2}\frac{\alpha'}{\alpha}\right) + \frac{z'}{z^2}\left(\frac{1}{2}\frac{\alpha'}{\alpha} + \frac{1}{2}\frac{\gamma'}{\gamma}\right) = K = 0, \tag{4.19c}$$

$$\left(\frac{z'}{z^3}\right)' - \frac{2z'}{z^3}\left(\frac{z'}{z} + \frac{1}{2}\frac{\alpha'}{\alpha}\right) + \frac{z'}{z^3}\left(\frac{1}{2}\frac{z'}{z} + \frac{1}{2}\frac{\gamma'}{\gamma}\right) = 0, \tag{4.19d}$$

and

$$\left(\frac{\gamma'}{\alpha}\right)' + \frac{\gamma'}{\alpha}\left(\frac{\alpha'}{\alpha} + \frac{1}{2}\frac{z'}{z}, \; - \frac{1}{2}\frac{\gamma'}{\gamma}\right) = 0. \tag{4.19e}$$

Equations (b), (d) and (e) integrate immediately (providing neither z' nor γ' are zero) to give respectively

$$z' = C\alpha^{-1}\sqrt{z/\gamma}, \; z' = Az^4\alpha\sqrt{z/\gamma} \quad \text{and} \quad \gamma' = B\sqrt{\gamma/z}, \; A, \; B, \; C$$

being constants, whence

$$\alpha^2 = \frac{C}{A}z^{-4}, \quad \frac{\alpha'}{\alpha} = -2\frac{z'}{z}, \quad \frac{\gamma'}{\gamma} = \pm\frac{B}{\sqrt{AC}}\frac{\gamma}{z^3} = S\frac{z'}{z^3} \text{ say.}$$

Therefore, α and γ can be eliminated from (4.19a, c) and it is easily seen that these equations are compatible only if

$$\text{either } z' = 0, \quad \text{or} \quad S = 0.$$

The latter case implies $\gamma' = 0$, and both when $z = \text{const}$ (K) (or $\alpha = KE$) and $\gamma = \text{const}$ (ε), we have to go back to (4.19) to obtain the corresponding solutions.

1. When $z' = 0$, we get from the equations (4.19a, c) which alone survive,

$$\alpha = -h^2 \exp(m_1 r^2 + f_1 r) = KE,$$

$$\gamma = \frac{1}{4}(m_2 r + f_2)^2,$$

$$\beta = \frac{h^2}{K^2}\exp(m_1 r^2 + f_1 r), \tag{4.20}$$

where k, m_1, f_1, m_2, f_2 are arbitrary constants related by

$$2m_1 f_2 = m_2 f_1.$$

2. When $\gamma' = 0, z' \neq 0$,

$$\alpha = (Br + C)^{4/3},$$

$$\alpha^3 = AE^2,$$

$$\beta = -\frac{1}{A}\alpha^2$$

$$\gamma = \varepsilon \tag{4.21}$$

with A, B and C arbitrary constants.

3. Finally, when both z and γ are constants, equations (4.19ɪ) give

$$\alpha = KE = B \exp(Ar),$$

$$\beta = -\frac{E}{K}. \tag{4.22}$$

These solutions cover all the possible forms of the general case. Particular solutions can be obtained by taking *a priori* any of, say u, v and e variables as constant.

5. *Discussion*

Up to the point when we found it necessary to postulate equation (4.17), the functions α, β and γ were regarded as possitive. For a real E this is no longer possible. However, analogy with General Relativity shows that the signature of $g_{\underline{\mu\nu}}$ should be ± 2. Hence, if β is, for example, positive, the parts played by the coordinates θ and t must be reversed. The same happens if β is negative, the sign of γ being in each case the same as that of β.

The above, however, is only of academic interest since a glance at the solutions (4.20), (4.21) and (4.22) shows that their physical meaning as global solutions is doubtful. In particular they are singular at infinity. Thus, they can have at most local significance. This conclusion may be used as evidence that the strong field theory is untenable except that in the second article we find a particular solution which is perfectly meaningful and which is in addition singularity free in the sense of Einstein.

APPENDIX

The equations determining $\Gamma^{\lambda}_{\mu\nu}$

$$\frac{da}{dr} - 2a\Gamma^1_{11} = 0$$

$$-\alpha\Gamma^1_{12} - \alpha\Gamma^1_{21} = 0$$

$$-\alpha\Gamma^1_{13} - \alpha\Gamma^1_{31} = 0$$

$$-\alpha\Gamma^1_{14} - \alpha\Gamma^1_{41} = 0$$

$$-\alpha\Gamma^2_{11} - E\Gamma^3_{11} - \alpha\Gamma^1_{12} = 0$$

$$-\alpha\Gamma^2_{12} - E\Gamma^3_{12} - \alpha\Gamma^1_{22} = 0$$

$$-\alpha\Gamma^2_{13} - E\Gamma^3_{13} - \alpha\Gamma^1_{22} = 0$$

$$-\alpha\Gamma^2_{14} - E\Gamma^3_{14} - \alpha\Gamma^1_{42} = 0$$

$$E\Gamma^2_{11}-\beta\Gamma^3_{11}-\alpha\Gamma^1_{13}=0$$

$$E\Gamma^2_{12}-\beta\Gamma^3_{12}-\alpha\Gamma^1_{23}=0$$

$$E\Gamma^2_{13}-\beta\Gamma^3_{13}-\alpha\Gamma^1_{33}=0$$

$$E\Gamma^2_{14}-\beta\Gamma^3_{14}-\alpha\Gamma^1_{43}=0$$

$$\gamma\Gamma^4_{11}-\alpha\Gamma^1_{14}=0$$

$$\gamma\Gamma^4_{12}-\alpha\Gamma^1_{24}=0$$

$$\gamma\Gamma^4_{13}-\alpha\Gamma^1_{34}=0$$

$$\gamma\Gamma^4_{14}-\alpha\Gamma^1_{44}=0$$

$$-\alpha\Gamma^1_{21}-\alpha\Gamma^2_{11}+E\Gamma^3_{11}=0$$

$$-\alpha\Gamma^1_{22}-\alpha\Gamma^2_{21}+E\Gamma^3_{21}=0$$

$$-\alpha\Gamma^1_{23}-\alpha\Gamma^2_{31}+E\Gamma^3_{31}=0$$

$$-\alpha\Gamma^1_{24}-\alpha\Gamma^2_{41}+E\Gamma^3_{41}=0$$

$$\frac{da}{dr}-\alpha\Gamma^2_{21}-E\Gamma^3_{21}-\alpha\Gamma^2_{12}+E\Gamma^3_{12}=0$$

$$-\alpha\Gamma^2_{22}-E\Gamma^3_{22}-\alpha\Gamma^2_{22}+E\Gamma^3_{22}=0$$

$$-\alpha\Gamma^2_{23}-E\Gamma^3_{23}-\alpha\Gamma^2_{32}+E\Gamma^3_{32}=0$$

$$-\alpha\Gamma^2_{24}-E\Gamma^3_{24}-\alpha\Gamma^2_{42}+E\Gamma^3_{42}=0$$

$$-\frac{dE}{dr}+E\Gamma^2_{21}-\beta\Gamma^3_{21}-\alpha\Gamma^2_{13}+E\Gamma^3_{13}=0$$

$$E\Gamma^2_{22}-\beta\Gamma^3_{22}-\alpha\Gamma^2_{23}+E\Gamma^3_{23}=0$$

$$E\Gamma^2_{23}-\beta\Gamma^3_{23}-\alpha\Gamma^2_{33}+E\Gamma^3_{33}=0$$

$$E\Gamma^2_{24}-\beta\Gamma^3_{24}-\alpha\Gamma^2_{43}+E\Gamma^3_{43}=0$$

$$\gamma\Gamma^4_{21}-\alpha\Gamma^2_{14}+E\Gamma^3_{14}=0$$

$$\gamma\Gamma^4_{22}-\alpha\Gamma^2_{24}+E\Gamma^3_{24}=0$$

$$\gamma\Gamma^4_{23}-\alpha\Gamma^2_{34}+E\Gamma^3_{34}=0$$

$$\gamma\Gamma^4_{24}-\alpha\Gamma^2_{44}+E\Gamma^3_{44}=0$$

$$-\alpha\Gamma^1_{31}-E\Gamma^2_{11}-\beta\Gamma^3_{11}=0$$

$$-\alpha\Gamma^1_{32}-E\Gamma^2_{21}-\beta\Gamma^3_{21}=0$$

$$-\alpha\Gamma^1_{33}-E\Gamma^2_{31}-\beta\Gamma^3_{31}=0$$

$$-\alpha\Gamma^1_{34}-E\Gamma^2_{41}-\beta\Gamma^3_{41}=0$$

$$\frac{dE}{dr}-\alpha\Gamma^2_{31}-E\Gamma^3_{31}-E\Gamma^2_{12}-\beta\Gamma^3_{12}=0$$

$$-\alpha\Gamma^2_{32}-E\Gamma^3_{32}-E\Gamma^2_{22}-\beta\Gamma^3_{22}=0$$

$$-\alpha\Gamma^2_{33}-E\Gamma^3_{33}-E\Gamma^2_{32}-\beta\Gamma^3_{32}=0$$

$$-\alpha\Gamma^2_{34}-E\Gamma^3_{34}-E\Gamma^2_{42}-\beta\Gamma^3_{42}=0$$

$$\frac{d\beta}{dr}+E\Gamma^2_{31}-\beta\Gamma^3_{31}-E\Gamma^2_{13}-\beta\Gamma^3_{13}=0$$

$$E\Gamma^2_{32}-\beta\Gamma^3_{32}-E\Gamma^2_{23}-\beta\Gamma^3_{23}=0$$

$$E\Gamma^2_{33}-\beta\Gamma^3_{33}-E\Gamma^2_{33}-\beta\Gamma^3_{33}=0$$

$$E\Gamma^2_{34}-\beta\Gamma^3_{34}-E\Gamma^2_{43}-\beta\Gamma^3_{43}=0$$

$$\gamma\Gamma^4_{31}-E\Gamma^2_{14}-\beta\Gamma^3_{14}=0$$

$$\gamma\Gamma^4_{32}-E\Gamma^2_{24}-\beta\Gamma^3_{24}=0$$

$$\gamma\Gamma^4_{33}-E\Gamma^2_{34}-\beta\Gamma^3_{34}=0$$

$$\gamma\Gamma^4_{34}-E\Gamma^2_{44}-\beta\Gamma^3_{44}=0$$

$$-\alpha\Gamma^1_{41}+\gamma\Gamma^4_{11}=0$$

$$-\alpha\Gamma^1_{42}+\gamma\Gamma^4_{21}=0$$

$$-\alpha\Gamma^1_{43}+\gamma\Gamma^4_{31}=0$$

$$-\alpha\Gamma^1_{44}+\gamma\Gamma^4_{41}=0$$

$$-\alpha\Gamma^2_{41}-E\Gamma^3_{41}+\gamma\Gamma^4_{12}=0$$

$$-\alpha\Gamma^2_{42}-E\Gamma^3_{42}+\gamma\Gamma^4_{22}=0$$

$$-\alpha\Gamma^2_{43}-E\Gamma^3_{43}+\gamma\Gamma^4_{32}=0$$

$$-\alpha\Gamma^2_{44}-E\Gamma^3_{44}+\gamma\Gamma^4_{42}=0$$

$$E\Gamma^2_{41}-\beta\Gamma^3_{41}+\gamma\Gamma^4_{13}=0$$

$$E\Gamma^2_{42}-\beta\Gamma^3_{42}+\gamma\Gamma^4_{23}=0$$

$$E\Gamma^2_{43}-\beta\Gamma^3_{43}+\gamma\Gamma^4_{33}=0$$

$$E\Gamma^2_{44}-\beta\Gamma^3_{44}+\gamma\Gamma^4_{43}=0$$

$$-\frac{d\gamma}{r}+\gamma\Gamma^4_{41}+\gamma\Gamma^4_{14}=0$$

$$\gamma\Gamma^4_{42}+\gamma\Gamma^4_{24}=0$$

$$\gamma\Gamma^4_{43}+\gamma\Gamma^4_{34}=0$$

$$\gamma\Gamma^4_{44}+\gamma\Gamma^4_{44}=0$$

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