CYLINDRICAL SYMMETRY IN EINSTEIN'S UNIFIED FIELD THEORY. II

BY A. H. KLOTZ AND G. K. RUSSELL

Department of Applied Mathematics, University of Sydney*

(Received November 24, 1970)

A solution of Einstein's unified field equations corresponding to a static, uniform line charge is derived. It is shown that in the particular case considered the strong and the weak field equations coincide. A possible empirical test of the theory is discussed.

1. Introduction

In the previous article of this series (referred to as I in the sequel) we discuss the general solution of the strong unified field equations of Einstein. There is some uncertainty in the nonsymmetric unified field theories which tensor is to be regarded as the metric tensor. If we write with Einstein' $^*g^{\mu\nu}$ for the inverse tensor of $g_{\mu\nu}$, and let (Schrödinger 1947)

$$a^{\mu\nu} = \frac{\sqrt{-g}}{\sqrt{-a}} \, {}^*g^{\underline{\mu\nu}},$$

where g is the determinant of $g_{\mu\nu}$ and $a = \det(\sqrt{-g}\, {}^*g^{\underline{\mu\nu}})$ then $a_{\mu\nu}$, such that

$$a_{\sigma\mu}a^{\sigma\nu} = \delta^{\nu}_{\mu},$$

can be used as the metric. Alternatively, we can, of course, adopt $g_{\underline{\mu\nu}}$ (a line under the indices indicates the symmetric part; in general, we use the same notation as in I). In either case, we require that for large r, the metric should approach that of a Minkowski space time. This condition is not satisfied by the general solutions of I.

It seems therefore that in the case of cylindrical symmetry only particular solutions can be physically meaningful. We take (as in I) the fundamental tensor in the form

$$g_{\mu\nu} = \begin{pmatrix} -\alpha & & & \\ & -\alpha & E & \\ & -E & -\beta & \\ & & & \gamma \end{pmatrix}$$

the coordinates being $x^1 = r$, $x^2 = z$, $x^3 = \theta$, $x^4 = t$, (cylindrical polar coordinates) and α, β, γ, E being functions of r only (the radial distance from the z axis of symmetry).

* Address: Department of Applied Mathematics, University of Sydney, Sydney N.S.W. 2006, Australia.

If we assume *a priori* that $E = q$ is a constant, the solution will correspond to the case of a static, uniform line charge. The field is

$$\boldsymbol{H} = 0, \quad \boldsymbol{E} = -\frac{q}{r}\,\hat{\boldsymbol{r}},$$

so that q is proportional to the line density of charge.

2. *A static, uniform line charge*

Let us write

$$\alpha = e^{2F}, \;\; \beta = e^{2H}, \;\; \gamma = w^{2K},$$

and define

$$A = \frac{d}{dr}\ln\sqrt{q^2+\alpha\beta}, \qquad B = q\beta(A-H')/(\alpha\beta-q^2). \tag{2.1}$$

The equation

$$R_{44} = 0,$$

then gives immediately

$$K' = \frac{a}{\sqrt{q^2+\alpha\beta}}\,e^{F-K} \tag{2.2}$$

where a is a constant.

Moreover, $R_{\underset{\vee}{23}}$ vanishes when q is zero, and therefore we may write

$$R_{\underset{\vee}{23}} = qb, \tag{2.3}$$

where b is also a constant.

A considerable simplification results if we take

$$B = 0, \tag{2.4}$$

whence

$$\beta = \frac{q^2}{c^2-\alpha}, \tag{2.5}$$

c being another arbitrary constant. Incidentally, we may notice that unless q or b are zero, we are now dealing with the weak field equations which are known to be compatible.

Using (2.5), we now find that

$$A = H' = \frac{\alpha}{c^2-\alpha}\,F', \tag{2.6}$$

and the field equations to be solved reduce to

$$F'H' - H'' - H'^2 + \frac{aA\,e^{F-K}}{\sqrt{\alpha\beta+q^2}} = 0 \tag{2.7a}$$

$$F''+F'H'+F'^2+\frac{aF'e^{F-K}}{\sqrt{\alpha\beta+q^2}}+\frac{2q^2}{\alpha\beta}A^2=0, \tag{2.7b}$$

$$-ab+\frac{aAe^{F-K}}{\sqrt{\alpha\beta+q^2}}+A'+AH'-AF'=0, \tag{2.7c}$$

and

$$F'H'-H''-H'^2-\frac{aH'e^{F-K}}{\sqrt{\alpha\beta+q^2}}=0. \tag{2.7d}$$

From (2.7a) and (2.7d)

$$a(A'+H')e^{F-K}=0,$$

so that

$$a=0, \tag{2.8}$$

since

$$A'+H'=0,$$

leads, together with (2.6) to the trivial case of a constant electric field in the "x" direction for which α and β are also constant.

We now have

$$\gamma=\text{const},$$

and we easily find that

$$b=0,$$

so that the field equations reduce to strong field equations as a consequence of assuming B to be zero. The complete solution now follows readily in the form

$$\alpha=c^2\left[1+\frac{1}{(lr+m)^2}\right]^{-1}, \tag{2.9a}$$

$$\beta=\frac{q^2}{c^2}(lr+m)^2\left[1+\frac{1}{(lr+m)^2}\right]. \tag{2.9b}$$

$$\gamma=\text{const},\qquad E=q\neq 0, \tag{2.9c}$$

c, m, q and γ being arbitrary constants.

For the metric to approach Minkowski's at infinity

$$c=1,\ \ l=\frac{1}{q},\quad \gamma=1 \tag{2.10}$$

(either c or γ can be chosen arbitrarily).

We then have

$$a_{\mu\nu}=\text{diag}\,[-f^{-3/2},\ -f^{-1/2},\ -(r+m')^2f^{1/2}, f^{-1/2}], \tag{2.11}$$

$$q_{\underline{\mu\nu}}=\text{diag}\,[-f^{-1},\ -f^{-1},\ -(r+m')^2f,\ 1], \tag{2.12}$$

where

$$m' = mq$$

$$f = 1 + \frac{q^2}{(r+m')^2}.$$

It seems reasonable to conclude that there is no mass present, since, as $q \to 0$, $m' \to 0$ also, and both $a_{\mu\nu}$ and $q_{\underline{\mu\nu}}$ reduce to the metric of a flat world.

3. *A free trajectory*

A test particle moving in a gravitational field follows, according to General Relativity, a geodesic of the corresponding Riemannian space. In the unified field theory we are discussing the situation is more complicated mainly because of uncertainty of how a "metric" tensor is to be defined. Even if that is settled, it is clear that one cannot obtain the equations of motion of a charged test particle from a minimum time principle of the form

$$\delta \int ds = 0, \qquad ds^2 = a_{\underline{\mu\nu}} dx^\mu dx^\nu. \tag{3.1}$$

The equations

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\underline{\mu\nu}} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \tag{3.2}$$

will not do either. It has been suggested (Stephenson and Kilmister 1953) that one should use

$$ds = -\psi_\mu dx^\mu \pm dw, \qquad dw^2 = a_{\mu\nu} dx^\mu dx^\nu. \tag{3.3}$$

The resulting equations of motion are

$$\frac{d^2x^\alpha}{dw^2} + \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} \frac{dx^\mu}{dw} \frac{dx^\nu}{dw} = a^{\underline{\alpha\mu}} f_{\mu\nu} \frac{dx^\nu}{dw}, \tag{3.4}$$

where $\{^{\alpha}_{\mu\nu}\}$ are the Christoffel brackets found with the help of $a_{\underline{\mu\nu}}$, and

$$f_{\mu\nu} = \psi_{\nu,\mu} - \psi_{\mu,\nu},$$

is identified with the electro-magnetic intensity tensor. If we have

$$f_{\mu\nu} = g_{\underset{\vee}{\mu\nu}},$$

however, equations of the form (3.4), lead to a physically extraordinary conclusion that a uniform line charge along the z axis exerts only a lateral $z-\theta$ force. The alternative

$$f_{\mu\nu} = \frac{1}{2} \sqrt{-g}\ \varepsilon_{\mu\nu\alpha\beta}\, {}^*g^{\underset{\vee}{\alpha\beta}}$$

(${}^*g^{\underset{\vee}{\alpha\beta}}$ being the skew symmetric part of the inverse tensor of $g_{\mu\nu}$) is possible but it leads to equations which do not bear any resemblance to Lorentz equations of motion of an electron in the vicinity of a line charge, to which they ought to reduce in a non-relativistic approximation.

In view of this, we postulate that the equations of motion are given by

$$\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{\underline{\mu\nu}}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}=\frac{1}{2}\varepsilon^{\alpha\lambda\mu\nu}(-g)^{-\frac{1}{2}}g_{\underline{\lambda\varkappa}}g_{\underset{\vee}{\mu\nu}}\frac{dx^\varkappa}{ds}. \tag{3.5}$$

The relevant components of the affine connection are

$$\Gamma^1_{11}=\frac{q^2}{\varrho(\varrho^2+q^2)}=\Gamma^1_{22},\ \Gamma^1_{33}=-\frac{\varrho^2+q^2}{\varrho},$$

$$\Gamma^1_{44}=\Gamma^2_{12}=\Gamma^4_{14}=0,\ \Gamma^3_{13}=\frac{\varrho}{\varrho^2+q^2}, \tag{3.6}$$

where

$$\varrho=r+m'.$$

Also,

$$\sqrt{-g}=\varrho,\quad g_{\underset{\vee}{23}}=q.$$

Hence the equations of motion become

$$\frac{d^2\varrho}{ds^2}+\frac{q^2}{\varrho(\varrho^2+q^2)}\left(\frac{d\varrho}{ds}\right)^2+\frac{q^2}{\varrho(\varrho^2+q^2)}\left(\frac{dz}{ds}\right)^2-$$

$$-\frac{\varrho^2+q^2}{\varrho}\left(\frac{d\theta}{ds}\right)^2=\frac{q}{\varrho}\frac{dt}{ds} \tag{3.7a}$$

$$\frac{d^2z}{ds^2}=0 \tag{3.7b}$$

$$\frac{d^2\theta}{ds^2}+\frac{2\varrho}{\varrho^2+q^2}\frac{d\varrho}{ds}\frac{d\theta}{ds}=0 \tag{3.7c}$$

$$\frac{d^2t}{ds^2}=\frac{q\varrho}{\varrho^2+q^2}\frac{d\varrho}{ds}, \tag{3.7d}$$

where we have written $t=x^4$.

Consider now an electron (charge e, mass μ) projected parallel to the line charge of density λ, with a speed V. (3.7b), (3.7c) integrate immediately to give

$$\frac{dz}{ds}=b,\ \frac{d\theta}{ds}=\frac{a}{\varrho^2+q^2},\ a,\ b \text{ const.} \tag{3.8}$$

In the non-relativistic case, the quadratic terms in (3.7a) disappear, and with $x^4=ct=s$, (c speed of light in vacuum) (3.7a) reduces to the classical equation of motion of the electron, providing we identify q by

$$q=\frac{2\lambda e}{c^2\mu}. \tag{3.9}$$

In the present theory, we can still adjust the scale so that, without loss of generality,

$$q^2b^2 = a^2,$$

and (3.7a) becomes

$$\frac{d^2\varrho}{ds^2} \frac{q^2}{\varrho(\varrho^2+q^2)} \left(\frac{d\varrho}{ds}\right)^2 = \frac{q}{\varrho} \frac{dt}{ds} \tag{3.7a}$$

(3.7d) gives

$$\frac{dt}{ds} = \frac{q}{2} \ln (q^2+\varrho^2), \tag{3.10}$$

again without loss of generality, and (3.7a) can therefore be written in the form

$$\frac{d}{ds} \frac{\varrho^2}{\varrho^2+q^2} \left(\frac{d\varrho}{ds}\right)^2 = \frac{q^2}{\varrho^2+q^2} \ln (\varrho^2+q^2) \frac{d\varrho}{ds},$$

or

$$\frac{\varrho^2}{\varrho^2+q^2} \left(\frac{d\varrho}{ds}\right)^2 = h + \frac{q^4}{4} (\ln (\varrho^2+q^2))^2, \tag{3.11}$$

h being a constant of integration. (3.8), (3.10) and (3.11) now define the trajectory since a solution can be obtained by quadratures. There is little point in writting it out here unless we were considering a concrete situation.

If we identify s with the proper time $c\tau$ then, we must take

$$b = V/c.$$

In principle, these results should be experimentally verifiable.

4. *Discussion*

The solution of Einstein's unified field equations derived in this article is a particular solution only. We have seen in I that general solutions of the cylindrically symmetric case with only the electric field present, do not seem to have any obvious physical interpretation. This must enhance the meaningfulness of our solution and of the simplifying assumptions made in its derivation. Incidentally, it is also a solution of Schrödinger's strong field equations. As far as the proposed test of the theory is concerned, we must observe that the experiment will be difficult in practice. The q given by (3.9) is necessarily very small. We found it necessary to postulate a somewhat strange form of the equations of motion. This may be justified on two grounds. One is that for our solution they reduce in a non-relativistic approximation to the classical equation of motion of an electron. Secondly, it is known (Callaway 1953) that one cannot obtain Lorentz equations of motion from Einstein's field equations in any case. However, it is significant that Callaway obtains his result on the assumption that charged particles are singularities of the field. It is not known how equations of motion are to be derived if only everywhere nonsingular solutions are allowed as Einstein himself maintained. Hence it is not quite so reprehensible to introduce additional postulates in writing them down. It should be remembered all the same that a negative outcome of our experiment may mean simply that this postulate is wrong and not Einstein's theory. On the other hand, a positive result would be a clear confirmation of both. It would then follow from (3.5) that Maxwell's theory may have to be amended.

At present we are working on the case when both electric and magnetic fields are present. It should be possible to extend this investigation without undue difficulty to a theory of steady linear currents.

REFERENCES

Callaway, J., *Phys. Rev.*, **92**, 6, 1567 (1953).
Klotz, A. H., Russell, G. K., *Acta Phys. Polon.*, **B2**, 623 (1971).
Schrödinger, E., *Proc. Roy. Ir. Acad.*, **51**, 147, 163, 205 (1947).