GENERAL RELATIVISTIC FLUID SPHERES. IV. DIFFERENTIAL EQUATIONS FOR NON-CHARGED SPHERES OF PERFECT FLUID

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(Received January 7, 1971)

Assumptions underlying the search for new exact solutions of Einstein's field equations for space filled with matter are examined. Three assumptions are retained: spherical symmetry, macroscopic neutrality of matter, and the energy-momentum tensor of a perfect fluid. Schwarzschild canonical coordinates have been used in the previous investigations of this series, now other coordinate systems are introduced, and differential equations relating the metric tensor components and their derivatives are given. These equations which result from the conditions of pressure isotropy provide the basis of a systematic search for exact solutions along similar lines as in preceding papers. Such equations are given both for static and non-static matter distributions. Two other possibilities of deriving new exact solutions of the gravitational field equations are based on proposals presented by Buchdahl and by Heintzmann. The possibility of deriving matter distributions with pressure anisotropy is mentioned.

1. *On exact solutions of Einstein's equations*

The first exact solution of Einstein's field equations:

$$R^{\mu\nu}-\tfrac{1}{2}Rg^{\mu\nu}=-8\pi T^{\mu\nu} \tag{1.1}$$

was found by Schwarzschild (1916); this is a quite degenerate static vacuum field, most frequently applied in order to describe the space-time outside static matter distributions of spherical symmetry. In the last fifty years, or so, various different solutions, mainly with high symmetries, have been discovered. In the early days of the relativity theory, investigations on exact solutions were concentrated, however, on static fields but even now new static solutions of Einstein's equations may be found from time to time. So far, only the class of static vacuum fields has been the subject of an exhaustively complete treatment (see the summary by Ehlers and Kundt 1962).

New mathematical techniques (like tetrad formalism, complex vectorial calculus and ray optics) provided powerful tools for dealing with gravitation theory, yet they are applied

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almost exclusively to vacuum fields, especially to algebraically degenerate ones (Debney, Kerr and Schild 1969). Before these techniques are to be applied to give new exact interior solutions, it seems to be advisable to exhaust the possibilities of the older methods which are still of some use in this area of research. A justification for such work may lie in the fact that solutions of Einstein's equations for the internal regions of certain matter bulks are needed by contemporary astrophysics which, beginning several years ago, has to deal with such evidently relativistic objects like quasars and pulsars. In contrast with the situation for empty space, only a surprisingly small number of exact solutions of the gravitational field equations in space filled with matter are known. Some of the available solutions have been given in § 14 of Petrov's monograph (Petrov 1966), other have been referenced by Ehlers and Kundt (1962), in the meantime other ones were derived (*e. g.* Vaidya 1966 and 1968, McVittie 1967, Cahill and McVittie 1970, Nariai 1968, Faulkes 1969), yet no oversupply of simple exact solutions can be noticed. Such solutions, if expressed in terms of known elementary functions, might provide useful models for the internal structure of superdense or supermassive astrophysical configurations which must be described in terms of general relativity. Though model treatment is based on some kind of simplification of the processes which occur in nature, this handicap is compensated by the fact that an exact model of a relativistic star gives us simple, closed formulae for the gravitational potentials, density, pressure *etc.* Every such model would constitute, of course, only an approximation to the real situation which, it may be argued, can be best accounted for with the help of computer calculations. Until we have, however, more reliable knowledge of the unusual physical conditions under which matter exists in the central regions of neutron (perhaps hyperon or even quark?) stars, so that numerical computations of relativistic configurations could compete in their accuracy with recent computations of ordinary stellar models, a use of simplified, exact solutions of Eq. (1.1) may be justified.

Some years ago the author made an attempt to derive in a systematic way new simple and exact solutions of the gravitational field equations for space filled with matter. Several new solutions have been derived in the papers I–III[1].

The following assumptions were made in these papers:

(a) the distribution of matter has spherical symmetry,
(b) matter is non-charged,
(c) the energy-momentum tensor of matter is that of a perfect fluid,
(d) the gravitational field is static,
(e) the canonical Schwarzschild coordinates are used.

Not all these assumptions are necessary to obtain a solution for a general relativistic sphere. It is pointed out in I that in order to obtain further simple solutions of Eq. (1.1) we should abandon some of these assumptions. The fact that we intend to study non-rotating relativistic spheres makes it necessary to preserve our assumption (a). We shall maintain also assumption (b) according to the standard view on macroscopic charge neutrality of matter; the existence of small outward radial electric fields in neutron starts as a result of the displace-

[1] The first three parts of this paper (Kuchowicz 1968a, 1968b and 1970a) are denoted, respectively, by the symbols I, II, and III.

ment of electrons from the core of the star (Bhatia, Bonazzola and Szamosi 1969) seems to be a minor effect, with negligible influence upon the internal structure of the stars.

Assumption (c) will be also preserved in our further study. We would like to add, however, the following remarks concerning the energy-momentum tensor of a perfect fluid:

$$T^{\mu}_{\nu} = (\varrho + p)v^{\mu}v_{\nu} - pg^{\mu}_{\nu}. \tag{1.2}$$

Here ϱ denotes energy density of matter, p- pressure, as measured by a local comoving observer. The 4-velocity v_{μ} of the fluid satisfies the known condition

$$v_{\mu}v^{\mu} = 1. \tag{1.3}$$

A general spherically symmetric space-time with the metric:

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2e^{\sigma}(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \tag{1.4}$$

guarantees the vanishing of v^2 and v^3. The coordinates are identified in the standard way: $(x^1, x^2, x^3, x^4) \equiv (r, \vartheta, \varphi, t)$. If a comoving coordinate system may be used, we have

$$v^1 = 0, \tag{1.5}$$

and hence

$$T^1_4 = 0. \tag{1.6}$$

In this case the non-vanishing components of the tensor T^{μ}_{ν} are:

$$T^1_1 = T^2_2 = T^3_3 = -p,\ T^4_4 = \varrho. \tag{1.7}$$

This special form of the energy-momentum tensor of a perfect fluid is used in papers I–III and will be used in the following, unless otherwise explicitly stated. It should be emphasized here that besides the simple form (1.7) also other forms of the energy-momentum tensor were used by various authors. Namely, Vaidya (1968) considered a perfect fluid with four-dimensional stream lines being normal to the hypersurfaces $\varrho =$ const. He obtained the following formulae:

$$T^1_1 = -\frac{e^{-\lambda}(\varrho')^2}{(\varrho+p)f^2} - p,$$

$$T^1_4 = -\frac{e^{-\lambda}\varrho'\dot{\varrho}}{(\varrho+p)f^2}, \qquad T^2_2 = T^3_3 = -p,$$

$$T^4_4 = \frac{e^{\nu}(\dot{\varrho})^2}{(\varrho+p)f^2} - p, \qquad 1.8)$$

in the Schwarzschild coordinates ($\sigma = 0$ in Eq. (1.4)). The prime denotes here differentiation with respect to the radial variable r, and the dot — a differentiation with respect to t. f is an undetermined function of the density ϱ.

Other specified forms of the energy-momentum tensor (1.2) were given by McVittie (1969), Bondi (1965) and others. We do not discuss them in detail; it should be only emphasized that, in general, the T_4^1 component does not vanish. The latter component proves to be of a certain importance in the case of radiating bodies; in the region surrouding them which is traversed by outflowing radiation it proves to be one of the three non-vanishing components of the energy-momentum tensor (Vaidya 1951). We shall return to this question in a later paper of this series when we shall deal with spheres radiating energy.

The assumptions (d) and (e) from the preceding papers are of no fundamental nature; they have been made only in order to enable us to start the search for new exact solutions. An extension of our search to non-static configurations may provide adequate tools for dealing with relativistic collapse or explosion. In addition, a solution which has a very simple form in a certain coordinate system, may have a very complicated appearance in other coordinates. Thus, a relatively simple solution obtained by Buchdahl (1967) with the general metric (1.4), could not be presented in terms of a finite number of simple elementary functions when transformed to the canonical Schwarzschild coordinates. When we are confined to one kind of coordinates, we may easily overlook such exact solutions which have a complicated appearance in these coordinates (while they might be easily derived in another coordinate system). In a straightforward manner differential equations in various coordinates may be considered, and the investigations of their solutions may go in parallel to those of Eq. (3.1) from I. This paper is devoted to a systematic presentation of all such equations which are of use for our aims; solutions of these equations will be given in subsequent papers.

2. *Static matter distributions with isotropic pressure*

2.1. Canonical Schwarzschild coordinates

They are given by the metric (1.4) with $\sigma = 0$. Einstein equations (1.1) in these coordinates, with the perfect fluid tensor of the simple form (1.7), were given by Eqs (2.4)–(2.7) in I. Pressure isotropy: $T_1^1 = T_2^2$ has led to Eq. (3.1) of paper I. This was reduced either to a homogeneous linear equation of the second order in $y = e^{\nu/2}$, or to an inhomogeneous linear equation of first order in $z = e^{-\lambda}$, or to a system of two differential equations. Solutions of these equations were derived in papers I–III.

2.2. Isotropic coordinates

These are given by the metric (1.4) with $\lambda = \sigma$. Einstein equations in these coordinates follow from Eq. (2.6) when we insert there $\lambda = \sigma$. Since the isotropic pressure is given there by two different expressions, these two expressions may be regarded as equal to each other: $T_1^1 = T_2^2$. Hence the following mathematical relation between the functions λ and ν is obtained:

$$\lambda'' + \nu'' + \frac{1}{2}(\nu'^2 - \lambda'^2) - \lambda'\nu' - \frac{1}{r}(\lambda' + \nu') = 0 \tag{2.1}$$

where primes, as usually, denote differentiation with respect to the radial variable. This equation (Narlikar *et al.* 1943) can be expressed as a homogeneous linear equation of the second order in $z = e^{\nu/2}$:

$$z''-z'\left(\lambda'+\frac{1}{r}\right)+z\left(\frac{\lambda''}{2}-\frac{(\lambda')^2}{4}-\frac{\lambda'}{2r}\right)=0. \tag{2.2}$$

This form of the isotropy equation is especially useful when we wish to obtain an exact expression for the function e^{ν}, with some reasonable assumption about the other function e^{λ}. Eq. (2.1) may be regarded upon also as a differential equation of the first order in $y = \nu'$:

$$y' = -\frac{1}{2}y^2+\left(\lambda'+\frac{1}{r}\right)y+\frac{(\lambda')^2}{2}+\frac{\lambda'}{r}-\lambda''. \tag{2.3}$$

This is the general Riccati equation.

It is obvious, from the form of Eq. (2.1) in which the derivatives of ν and λ appear in a quite symmetric manner, apart from signs, that a change-over to equations in $\tilde{z} = e^{\lambda/2}$, and $\tilde{y} = \lambda'$, is unable to change essentially the mathematical situation. We give nevertheless, these two equations below:

$$\tilde{z}\tilde{z}''-2(\tilde{z}')^2-\tilde{z}\tilde{z}'\left(\nu'+\frac{1}{r}\right)+\tilde{z}^2\left(\frac{\nu''}{2}+\frac{(\nu')^2}{4}-\frac{\nu'}{2r}\right)=0 \tag{2.4}$$

$$\tilde{y}' = \frac{1}{2}\tilde{y}^2+\left(\nu'+\frac{1}{r}\right)\tilde{y}+\frac{\nu'}{r}-\frac{(\nu')^2}{2}-\nu''. \tag{2.5}$$

Eq. (2.5) is of the same type as Eq. (2.3) while Eq. (2.4) is, due to its nonlinearity, much more complicated than Eq. (2.2). Only a few exact solutions have been derived for material spheres in isotropic coordinates (Narlikar, Patwardhan and Vaidya 1943; Buchdahl 1964, Kuchowicz 1971). Perhaps the rather complicated appearance of the external Schwarzschild solution in these coordinates has discouraged physicists from their application.

2.3. General static metric (1.4)

With this general metric, and the perfect fluid tensor of the form (1.7), Einstein's equations become:

$$8\pi T_1^1 = -8\pi p = \frac{e^{-\sigma}}{r^2} - e^{-\lambda}\left[\frac{\nu'}{r}+\frac{\nu'\sigma'}{2}+\left(\frac{1}{r}+\frac{\sigma'}{2}\right)^2\right],$$

$$8\pi T_2^2 = 8\pi T_3^3 = -8\pi p = -e^{-\lambda}\left[\frac{\nu''+\sigma''}{2}+\frac{\nu'^2+\sigma'^2}{4}+\right.$$

$$\left.+\frac{\nu'+2\sigma'-\lambda'}{2r}+\frac{(\sigma'-\lambda')\nu'}{4}-\frac{\lambda'\sigma'}{4}\right],$$

$$8\pi T_4^4 = 8\pi\varrho = \frac{e^{-\sigma}}{r^2} - e^{-\lambda}\left[\sigma''+\frac{3}{4}(\sigma')^2+\frac{3\sigma'}{r}-\frac{\lambda'}{r}-\frac{\lambda'\sigma'}{2}+\frac{1}{r^2}\right]. \tag{2.6}$$

From the condition of pressure isotropy ($T_1^1 = T_2^2$) we get the following general differential equation connecting the functions $e^{-\lambda}$, e^{ν} and e^{σ}:

$$e^{-\lambda}\left[\frac{\sigma''+\nu''}{2} - \frac{\nu'\sigma'}{4} + \frac{(\nu')^2}{4} - \frac{\nu'}{2r} - \frac{1}{r^2}\right] + - \lambda' e^{-\lambda}\left[\frac{\sigma'+\nu'}{4} + \frac{1}{2r}\right] + \frac{e^{-\sigma}}{r^2} = 0. \quad (2.7)$$

This may be regarded as a differential equation for one of the three functions involved, with the remaining two functions given. This is a first order differen.ial equation for the function $z = e^{-\lambda}$, and its formal general solution (Kuchowicz 1969) is given by the following expression:

$$z = e^{-F(r)} \int e^{F(r')} g(r')\, dr' + Ce^{-F(r)}, \quad (2.8)$$

where:

$$h(r) = \frac{\left[\sigma''+\nu''+\frac{(\nu')^2-\sigma'\nu'}{2}\right] r^2 - \nu' r - 2}{r\left[\frac{r}{2}(\sigma'+\nu')+1\right]}$$

$$F(r) = \int h(r')\, dr'$$

$$g(r) = -\frac{2e^{-\sigma}}{r\left[\frac{r}{2}(\sigma'+\nu')+1\right]}. \quad (2.9)$$

These formulae are the generalizations of the respective expressions from Section 8 of paper I. In a similar way, Eq. (3.2) from the latter paper is generalized to:

$$e^{-\lambda} y'' - \left[\left(\frac{\sigma'}{2} + \frac{1}{r}\right) e^{-\lambda} + \frac{\lambda'}{2} e^{-\lambda}\right] y' +$$

$$+ \left[\left(\frac{\sigma''}{2} - \frac{1}{r^2}\right) e^{-\lambda} - \left(\frac{1}{2r} + \frac{\sigma'}{4}\right) \lambda' e^{-\lambda} + \frac{e^{-\sigma}}{r^2}\right] y = 0. \quad (2.10)$$

This homogeneous linear equation enables us to find the exact expressions for the function $y = e^{\nu/2}$, when certain adequately chosen functional forms of $e^{-\lambda}$ and $e^{-\sigma}$ are used. The treatment of this equation does not differ from the treatment of Eq. (3.2) in paper I, and almost all that was said with respect to the latter equation in Section 3 of paper I may be applied also to our Eq. (2.10).

The equation for $u = e^{\sigma}$ is non-linear:

$$\left(u'' - \frac{u'^2}{u}\right) - \frac{\lambda'+\nu'}{2} u' + \left[\nu'' + \frac{(\nu')^2 - \lambda'\nu'}{2} - \frac{\lambda'+\nu'}{r} - \frac{2}{r^2}\right] u + \frac{2e^{\lambda}}{r^2} = 0. \quad (2.11)$$

It should be mentioned here that the possibilities of deriving new exact static solutions are by no means exhausted by taking the general metric form (1.4). Other possibilities are offered by applying systematically two proposals presented by Buchdahl (1959) and by Heintzmann (1969).

2.4. Buchdahl's equation for "mean density"

Buchdahl (1959) exploited the well-known fact that in Schwarzschild canonical coordinates the equation[2] involving T_4^4 may be easily integrated, and the results is:

$$e^{-\lambda} = 1 - \frac{8\pi}{r} \int_0^r \varrho r^2\, dr = 1 - 2r^2 w, \tag{2.12}$$

where

$$w = \frac{4\pi}{r^3} \int_0^r r^2 \varrho dr. \tag{2.13}$$

The quantity $3w$ would be the mean density interior to r provided the calculations are performed in Euclidean space. The latter quantity is frequently used in astrophysics, and therefore its classical name was retained by Buchdahl though it has nothing in common with the true mean density.

Let us introduce the new variable $x = r^2$, and regard w, $e^{-\lambda}$, $y = e^{\nu/2}$, p and ϱ as functions of x. Introducing

$$e^{-\lambda} = 1 - 2xw \tag{2.14}$$

into the isotropy equation (3.2) of paper I we obtain the following differential equation:

$$(1-2xw)y''_{xx} - (xw'_x + w)y'_x - \tfrac{1}{2} w'_x y = 0 \tag{2.15}$$

which is linear in either of the two functions: w or y if the other be prescribed. Solutions of this equation may be easily found, and may be inserted into the formulae for the energy density and pressure:

$$4\pi\varrho = 3w + 2xw'_x,$$

$$4\pi p = 2(1-2xw) \frac{y'_x}{y} - 2w. \tag{2.16}$$

In the formulae (2.15) and (2.16) differentiation was performed with respect to the variable x, and we have denoted this by applying special subscripts in order to distinguish this case from the standard differentiation with respect to r.

Buchdahl's concept cannot be applied to other coordinate systems since its applicability depends upon the integrability of the equation involving T_4^4, and till now, no general integral of the last equation (2.6) with $\sigma \neq 0$ is known to the present author.

2.5. Deriving of new solutions from old ones

It was proposed by Heintzmann (1969) that provided we have the functions w and y which fulfill Eq. (2.15), and w does not contain a free parameter which is already

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[2] This is the Eq. (2.6) of paper I.

present in y, then a new solution is

$$w_N = w + Cw_H \tag{2.17}$$

where

$$w_H = \frac{1}{(2xy'+y)^2}\, e^{4\int y'(2xy'+y)^{-1}dx}. \tag{2.18}$$

This proposal may be extended onto other differential equations which involve two functions and are linear in each of them.

Such an equation is Eq. (8.1) of paper I:

$$z' + \frac{2(r^2y''-ry'-y)}{r(ry'+y)}\, z = -\frac{2y}{r(ry'+y)}.$$

If it is fulfilled by the functions $z = e^{-\lambda}$ and $y = e^{\nu/2}$, and z contains one free parameter less than y, a new solution is

$$z_N = z + Cz_H \tag{2.19}$$

where z_H, the general integral of the homogeneous equation, is given by

$$z_H = \frac{r^2}{(ry'+y)^2}\, e^{4\int\frac{y'dr}{r(ry'+y)}}. \tag{2.20}$$

The same procedure can be applied to Eq. (2.10), regarded upon as a relation between $y = e^{\nu/2}$ and $z = e^{-\lambda}$ only, with a fixed form of e^{σ}.

The new solutions (2.17) and (2.19) contain one parameter more than the old ones and may be thus more useful for astrophysical applications.

3. *Non-static matter distributions with isotropic pressure*

In this section we give from the beginning the formulae only for the general metric form (1.4). They may be specified later to one of the coordinate systems by taking $\sigma = 0$ (canonical coordinates) or by assuming $\lambda = \sigma$ (isotropic coordinates). With the energy-momentum tensor given by Eq. (1.7) we have the following form of gravitational field equations:

$$8\pi T_1^1 = -8\pi p = \left[\ddot{\sigma} + \frac{3\dot{\sigma}^2}{4} - \frac{\dot{\nu}\dot{\sigma}}{2}\right] e^{-\nu} + \frac{e^{-\sigma}}{r^2} - \left[\frac{\nu'}{r} + \frac{\nu'\sigma'}{2} + \left(\frac{1}{r} + \frac{\sigma'}{2}\right)^2\right] e^{-\lambda},$$

$$8\pi T_2^2 = -8\pi p = \left[\frac{\ddot{\lambda}+\ddot{\sigma}}{2} + \frac{(\dot{\sigma})^2+(\dot{\lambda})^2}{4} + \frac{\dot{\sigma}(\dot{\lambda}-\dot{\nu})-\dot{\lambda}\dot{\nu}}{4}\right] e^{-\nu} -$$

$$- \left[\frac{\nu''+\sigma''}{2} + \frac{(\nu')^2+(\sigma')^2}{4} + \frac{\nu'+2\sigma'-\lambda'}{2r} + \frac{(\sigma'-\lambda')\nu'-\lambda'\sigma'}{4}\right] e^{-\lambda},$$

$$8\pi T_4^4 = 8\pi\varrho = \left[\frac{1}{2}\dot{\lambda}\dot{\sigma} + \frac{1}{4}(\dot{\sigma})^2\right] e^{-\nu} - \left[\sigma'' + \frac{3}{4}(\sigma')^2 + \frac{3\sigma'}{r} - \frac{\lambda'}{r} - \frac{\lambda'\sigma'}{2} + \frac{1}{r^2}\right] e^{-\lambda} + \frac{e^{-\sigma}}{r^2}.$$

$$8\pi T_4^1 = 0 = \left[\dot{\sigma}' + \frac{1}{2}\dot{\sigma}\sigma' - \frac{1}{2}\nu'\dot{\sigma} - \frac{1}{2}\dot{\lambda}\sigma' + \frac{\dot{\sigma}-\dot{\lambda}}{r}\right] e^{-\lambda}. \tag{3.1}$$

From the isotropy condition ($T_1^1 = T_2^2$) the following relation is obtained:

$$\left[\ddot{\sigma}-\ddot{\lambda}+(\dot{\sigma})^2+\frac{1}{2}\dot{\lambda}(\dot{\nu}-\dot{\lambda})-\frac{1}{2}\dot{\sigma}(\dot{\nu}+\dot{\lambda})\right]e^{-\nu}+$$

$$+\left[\nu''+\sigma''+\frac{1}{2}\nu'(\nu'-\lambda')-\frac{1}{2}\sigma'(\nu'+\lambda')-\frac{\nu'+\lambda'}{r}-\frac{2}{r^2}\right]e^{-\lambda}+\frac{2}{r^2}e^{-\sigma}=0. \qquad (3.2)$$

In a preliminary not (Kuchowicz 1970b) in which this equation was given first, a minor error slipped in: the term $(-\lambda'/r)$ is missing in the second square bracket in Eq. (2). As usually, differentiation with respect to t is denoted by a dot while differentiation with respect to r is denoted by a prime.

In the non-static case, in addition to Eq. (3.2) another equation must be fulfilled by the respective metric functions. This is the equation (3.1), for T_4^1. Though this may seem to be an additional restriction when compared with the static case, it is not so. Indeed, we have to take into account that if two of the three functions in Eq. (3.2) are given, they are now both the functions of two variables, and even with the condition $T_4^1 = 0$ the arbitrariness of choice is now larger.

Eq. (3.2), though of a very complicated appearance, may admit solutions for one of three functions involved under certain *ad hoc* assumptions which will be studied in detail in subsequent papers. One prospective possibility, connected with the fact that Eq. (3.2) has its left-hand side composed of two terms, one involving time derivatives, and the other — derivatives with respect to r only, is the method of the separation of variables.

Under the substitution $y = e^{\nu/2}$ we obtain a second order equation in partial derivatives which is a generalization of Eq. (2.10) to the non-static case:

$$e^{-\lambda}y''-\left[\left(\frac{\sigma'}{2}+\frac{1}{r}\right)e^{-\lambda}+\frac{\lambda' e^{-\lambda}}{2}\right]y'+\left[\left(\frac{\sigma''}{2}-\frac{1}{r^2}\right)e^{-\lambda}-\left(\frac{1}{2r}+\frac{\sigma'}{4}\right)\lambda' e^{-\lambda}+\right.$$

$$\left.+\frac{e^{-\sigma}}{r^2}\right]y+\left[\frac{\ddot{\sigma}-\ddot{\lambda}}{2}+\frac{\dot{\sigma}^2}{2}-\frac{\dot{\lambda}(\dot{\lambda}+\dot{\sigma})}{4}\right]\cdot\frac{1}{y}+\frac{\dot{\lambda}-\dot{\sigma}}{2}\cdot\frac{\dot{y}}{y^2}=0. \qquad (3.3)$$

In the case of canonical Schwarzschild coordinates the condition $T_4^1 = 0$ reduces to

$$\dot{\lambda} = 0 \qquad (3.4)$$

i. e. λ is a function of r only. Eq. (3.3) is now reduced to Eq. (3.2) of paper I, and the solutions of the latter equation are now applicable, with the two integration constants regarded now as arbitrary functions of t.

The condition $T_4^1 = 0$ is reduced for isotropic coordinates to

$$\dot{\lambda}'-\tfrac{1}{2}\dot{\lambda}\nu' = 0 \qquad (3.5)$$

which gives

$$e^{\nu/2} = \dot{\lambda}F(t) \qquad (3.6)$$

where $F(t)$ is an arbitrary function of t. Eq. (3.3) is reduced now to Eq. (2.2) in $e^{\nu/2}$, with both ν and λ regarded now as functions of r and t. When we substitute $\dot{\lambda}F(t)$ instead of $e^{\nu/2}$

into the latter equation, the following partial differential equation for the function λ is obtained:

$$\dot{\lambda}'' - \left(\lambda' + \frac{1}{r}\right)\dot{\lambda}' + \left(\frac{\lambda''}{2} - \frac{(\lambda')^2}{4} - \frac{\lambda'}{2r}\right)\dot{\lambda} = 0. \tag{3.7}$$

The specific assumption:

$$e^{-\sigma} = r^2 e^{-S(t)} \tag{3.8}$$

leads to a reduction of several terms in Eq. (3.3). The condition $T_4^1 = 0$ now reads:

$$\nu' \dot{S} = 0. \tag{3.9}$$

Under the assumption $\nu' = 0$ the function ν depends only on time but it is possible to make it constant by a suitable change of the time variable. All terms with derivatives in y vanish now in Eq. (3.3) which becomes the following homogeneous linear equation of the second order in $u = e^S$:

$$\ddot{u} - \frac{\dot{\lambda}}{2}\dot{u} + \left(2y^2 - \ddot{\lambda} - \frac{\dot{\lambda}^2}{2}\right) u = 0. \tag{3.10}$$

Here y is constant, while λ is in fact a function of both t and r. Eq. (3.10) can be considered also as a first order equation in $\dot{\lambda}$ when the functional form of $u = u(t)$ is given:

$$\ddot{\lambda} = -\frac{1}{2}\dot{\lambda}^2 - \frac{\dot{u}}{2u}\dot{\lambda} + 2y^2 + \frac{\ddot{u}}{u}. \tag{3.11}$$

This is a Riccati-type equation which may be solved exactly for some substitutions. The time derivatives of λ are in fact partial derivatives which leads to the fact that in the final solution $\lambda = \lambda(r, t)$ integration constants should be replaced by arbitrary functions of time.

The other possibility resulting from Eq. (3.9):

$$\dot{S} = 0, \ i.\ e.\ e^{-S} = \text{const} \tag{3.12}$$

leads to the following form of Eq. (3.3):

$$e^{-\lambda}y'' - \frac{1}{2}\lambda' e^{-\lambda} y' + e^{-S} y - \left[\frac{\ddot{\lambda}}{2} + \frac{(\dot{\lambda})^2}{4}\right]\frac{1}{y} + \frac{\dot{\lambda}}{2}\frac{\dot{y}}{y^2} = 0. \tag{3.13}$$

The general relation (3.3) may be reasonably simplified also under other assumptions concerning the choice of the specific coordinate system, we shall not, however, enter further into these details.

4. *On solutions with anisotropic pressure*

The condition of isotropy does not seem to be necessary in deriving solution for general relativistic matter distributions. If the radial stress T_1^1 is not equal to the transverse stress T_2^2, the isotropy equation (Eq. (2.7), (3.2) and the like) need not be satisfied. Since there are now less restrictions upon the components of the metric tensor, the class of solutions with anisotropic pressure is much larger than that of solutions with isotropic pressure. Few exact solutions of such a type are known, however, presumably because there was not much physical interest in them.

Two groups among such solutions deserve special interest:

I. Those with $T_1^1 = 0$, and $T_2^2 \neq 0$, corresponding to matter moving on concentric spherical surfaces;

II. Those with $T_2^2 = 0$ and $T_1^1 \leqslant 0$; matter in statistical equilibrium is moving to and fro between the centre and the surface.

Solutions describing static spherical distributions of matter of both types were derived by Patwardhan and Vaidya (1943). It seems that solutions of the second type are worth investigating, especially since they may be of some use for the problem of collapsing or bouncing matter distributions.

Various types of solutions of the gravitational field equations, fulfilling isotropy condition as discussed throughout this paper, as well as solutions of the two types mentioned in the last section, will be discussed in forthcoming papers.

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