QUASIRELATIVISTIC EQUATIONS IN QUANTUM MECHANICS

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(Received May 13, 1971)

Two known methods of deducing from quantum electrodynamics the approximately relativistic quantum-mechanical equations for n charged fermions in an external electromagnetic field have been discussed in connection with the problem (not completely solved, as yet) of a consistent quantum-mechanical interpretation of such systems in terms of relativistic particle-observables (in the sense of the "mean" observables of Foldy and Wouthuysen). The important role of a "quasirelativistic" wave equation describing positive as well as negative energy states has been stressed. A new form of such an equation has been proposed and the idea of a new effective method of its reduction to the subspace of positive energy states shortly outlined.

The well known difficulties of the relativistic quantum mechanics are caused by: 1) the distinguished role of the time variable, as compared to that played by the position observables, 2) the approximate character of all quantum-mechanical schemes describing interactions between particles. Many attempts towards overcoming the former difficulty have been given in the last years. Some of them introduce the time (or the proper time) operators, while others try to avoid this assumption, as provoking some too far reaching modifications in the standard formalism and interpretation of the non-relativistic quantum mechanics. Following the latter path, a general scheme of a covariant Hamiltonian formalism of quantum mechanics has been proposed by one of us[1]. In particular, the case of the homogeneous Dirac equation shows that its usual quantum-mechanical interpretation in one specified Lorentz frame remains consistent with the relativity postulates. A further extension of this covariant formalism to the problem of n (identical or different) free Dirac particles described in the "covariant configuration space" is straightforward. As usually, the Hilbert space of states of the whole system becomes then the direct product of the spaces belonging to the single particles (or its subspace of antisymmetrized states in the case of identical particles).

On the other hand, it is obvious that no strictly covariant formulation can persist, if interactions between the particles are to be taken into account. However, an approximately relativistic, quantum-mechanical description is still possible in this case and it consists in eliminating the field observables, so that only the particle observables appear in the final

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[1] Hanus [1]; other references may be found there.

formulae expressing these interactions. It is well known that for electromagnetic interactions (discussed in this note) such a simplified formulation remains valid to the second order of approximation obtained by an expansion in a power series of $\alpha = \frac{e^2}{\hbar c} = (137)^{-1}$. The terms expressing the influence of a given external electromagnetic field can also be taken into account. The so described procedure leads to an approximate "quasi-relativistic" wave equation, still including positive and negative energy states of the Dirac particles. Its subsequent reduction to the subspace of the positive energy states must lead to the known Schrödinger equation for n charged spin-one-half particles, with the "effective" positive definite Hamiltonian containing relativistic corrections of the first and second orders. Hence, the final result as well as the starting point of considerations (from the respective formulae of quantum electrodynamics) are well established. Nevertheless, there are two, rather controversial, view-points of the method of deducing that result: either the (previously mentioned) "quasirelativistic quantum-mechanical equation can be used in the role similar to that of the Dirac equation in its Hamiltonian formulation, or alternatively, this intermediary step of calculations can be completely circumvented, if the elimination of the negative frequency part of the spinor field precedes the reduction to the approximate, quantum-mechanical description of the electromagnetic interactions. Both methods are found in literature, with a variety of further modifications. Let us call them shortly the "intermediary-equation method" or the "direct-reduction method", respectively.

The "d-method" has been used, *i. e.* in the papers of Berestecki and Landau [2], Itoh [3], and Weber [4]. This is the simplest way of descending from the general formulae of quantum electrodynamics to the explicit form of the final effective Hamiltonian. However, this method leaves aside the important problem of other relativistic observables of particles described by the so established Hamiltonian. The known transformation of Foldy and Wouthuysen [5] has thrown some new light onto this problem in the simple case of the one Dirac particle. In particular, the definitions of the "mean-position" and "mean-spin" operators have contributed to the explanation of some ambiguities having their source in the "trembing motion" of the electron. For the case of two (or, generally, n) Dirac particles this problem remains still open[2] but it is obvious that only the "i-method" allows one to expect a satisfactory solution. Therefore, we limit our further considerations to this method only. It has been used in a large number of publications and is based on the well known quasirelativistic equation given many years ago by Breit [7]. For two Dirac particles it has the form

$$i\hbar \frac{\partial \Psi}{\partial t} = \mathscr{H}_B \Psi, \quad \mathscr{H}_B = \mathscr{H}_{\mathrm{I}} + \mathscr{H}_{\mathrm{II}} + V \tag{1}$$

where

$$\mathscr{H}_K = m_K c^2 \varrho_{3,K} + c \varrho_{1,K} Z_K + e_K \Phi_K^{\mathrm{ex}}, \tag{2}$$

$$Z_K = \boldsymbol{\sigma}_K \cdot \left(\boldsymbol{p}_K - \frac{e_K}{c} \boldsymbol{A}_K^{\mathrm{ex}} \right), \tag{3}$$

[2] For an interesting special case dealing with the relativistic "relative position operator" defined in connection with the scattering problems see Mc Donald [6].

$$V = e_{\mathrm{I}} \cdot e_{\mathrm{II}} \left(\frac{1}{r} - \frac{1}{2} \varrho_{1,\mathrm{I}} \varrho_{1,\mathrm{II}} J \right), \tag{4}$$

$$J = \frac{\boldsymbol{\sigma}_{\mathrm{I}} \cdot \boldsymbol{\sigma}_{\mathrm{II}}}{r} + \frac{(\boldsymbol{\sigma}_{\mathrm{I}} \cdot \boldsymbol{r})\,(\boldsymbol{\sigma}_{\mathrm{II}} \cdot \boldsymbol{r})}{r^3}, \tag{5}$$

$$\boldsymbol{r} = \boldsymbol{r}_{\mathrm{I}} - \boldsymbol{r}_{\mathrm{II}}, \; r = |\boldsymbol{r}|. \tag{6}$$

In (1)–(6) quantities describing each of the two particles are labelled by the subscripts $K = \mathrm{I}, \mathrm{II}$ so that m_K, e_K, $\boldsymbol{r}_K$, and $\boldsymbol{p}_K$ denote the mass, charge, position, and momentum of the K-th particle, respectively. $\boldsymbol{A}_K^{\mathrm{ex}} \equiv \boldsymbol{A}^{\mathrm{ex}}(\boldsymbol{r}_K)$, $\Phi_K^{\mathrm{ex}} \equiv \Phi^{\mathrm{ex}}(\boldsymbol{r}_K)$ express the potentials of the stationary external electromagnetic field in the position of the K-th particle while $\Psi(\boldsymbol{r}_{\mathrm{I}}, \boldsymbol{r}_{\mathrm{II}}, t)$ is the wave function (in the six-dimensional configuration space and in the sixteen-dimensional spinor space). The symbols $\varrho_K, \boldsymbol{\sigma}_K$ denote the well known Dirac operators[3] (extended in the usual way to this direct-product spinor space).

During a long period of time, the Breit equation (deduced after some approximations from quantum electrodynamics) was treated as the only possible form of the quansirelativistic two-fermion equation. In this role it was widely used despite of its known defficiencies (for details see *e. g.* Bethe and Salpeter [8] p. 256). Its reduction to the Pauli formalism has been performed in a close analogy to the case of the Dirac equation (although by means of much more complicated calculations and in the presence of some specific difficulties not encountered in the former case). After the elementary method of large components (used, in particular, by Breit, himself) a generalization of the more correct method of Foldy and Wouthuysen [5] has been displayed in a series of papers by Chraplyvy ([9], [10]), Barker and Glover [11], Chraplyvy and Glover [12] (see also Chraplyvy and Glover [13], Chraplyvy [14]). The formal consistency of these two methods[4] has been verified by the identity of the final result (*i. e.* of the effective Hamiltonian).

The use of the explicit form of the 16×16 matrices defined as direct products of the Dirac matrices seems to prevent this method of reduction from being generalized to an arbitrary number of interacting particles.

An interesting new idea has been introduced in [11] (see also [14] and [13]) where an equation different form that of Breit has been proposed (and reduced to the identical final result), however only for the special case

$$\boldsymbol{A}^{\mathrm{ex}} = 0, \; \Phi^{\mathrm{ex}} = 0. \tag{7}$$

That new equation, deducible, after some approximations, from the Bethe-Salpeter equation ([18], [19]), can be expressed in the form (see also Eriksen [20])

$$i\hbar \frac{\partial \Psi^0}{\partial t} = \mathcal{H}_{BS}^0 \Psi^0, \; \mathcal{H}_{BS}^0 = \mathcal{H}_{\mathrm{I}}^0 + \mathcal{H}_{\mathrm{II}}^0 + W^0, \tag{8}$$

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[3] The use of $\boldsymbol{\rho}$, $\boldsymbol{\sigma}$ instead of $\boldsymbol{\alpha} = \varrho_1 \boldsymbol{\sigma}$, $\beta = \varrho_3$ is more advantageous in problems dealing with the two signs of energy, but leaving aside all questions concerning the spin.

[4] The question of their strict equivalence (to an arbitrary order of approximation) was widely discussed for the Dirac equation, until it has been solved (in the positive sense) by de Vries and Jonker ([15], [16], [17]). For many-particle problems this question becomes rather immaterial as their formulation is, from the beginning, approximate only.

$$W^0 = \frac{1}{4}\,[\lambda_{\mathrm{I}}^0 + \lambda_{\mathrm{II}}^0,\ V]_+, \tag{9}$$

$$\lambda_K^0 = \mathscr{H}_K^0\,[(\mathscr{H}_K^0)^2]^{-1/2},\ \mathscr{H}_K^0 = m_K c^2\,\varrho_{3,K} + c\varrho_{1,K}\,\boldsymbol{\sigma}_K \cdot \boldsymbol{p}_K, \tag{10}$$

where $[\ ,]_+$ denotes the anticommutator. Hence, in the case (7) there are two different, at the first sight equally justified, quasirelativistic equations (2) or (8). The detailed discussion given in [11] strongly implies the superiority of (8), as compared to (2): as the former equation automatically includes the hole theory, it is free of some peculiar difficulties encounterned in the interpretation and reduction of the latter. On the other hand, (8) cannot be applied, as yet, in the presence of an external electromagnetic field.

We propose such a generalization which must be treated as an independent assumption, rather than a conclusion from the Bethe-Salpeter equation. We can put, namely

$$i\hbar\,\frac{\partial \Psi}{\partial t} = \mathscr{H}_{BS}\Psi,\ \mathscr{H}_{BS} = \mathscr{H}_{\mathrm{I}} + \mathscr{H}_{\mathrm{II}} + W, \tag{8'}$$

$$W = \frac{1}{4}\,[\lambda_{\mathrm{I}} + \lambda_{\mathrm{II}},\ V]_+, \tag{9'}$$

$$\lambda_K = T_K (T_K^2)^{-1/2},\ T_K = m_K c^2 \varrho_{3,K} + c\varrho_{1,K} Z_K, \tag{10'}$$

instead of (8)–(10), respectively. The verification of the proposed formulae (8')–(10') may be summarized as follows: in accordance with the point of view proposed in [1], we can base our considerations on the quantum-mechanical formulation for two free Dirac particles, *i. e.* on the equation

$$i\hbar\,\frac{\partial \Phi^0}{\partial t} = (\mathscr{H}_{\mathrm{I}}^0 + \mathscr{H}_{\mathrm{II}}^0)\Phi^0, \tag{11}$$

for their wave function $\Phi^0(\boldsymbol{r}_{\mathrm{I}}, \boldsymbol{r}_{\mathrm{II}}, t)$ (instead of referring these considerations at first to the respective formulae of quantum electrodynamics containing the spinor field function). The equation (11) is — at least formally — strict (and if necessary, it can even be put into the manifestly covariant form). All limitations to the approximate description are brought about by taking into account interactions of two kinds: 1) with an external electromagnetic field, following the standard method of introducing the potentials $\boldsymbol{A}^{\mathrm{ex}}$, Φ^{ex}, 2) of the two particles with each other. Here the necessary recurrence to the known approximate results of quantum electrodynamics gives, alternatively, V or W^0, the way of deducing the latter term being more satisfactory. However, while the Breit term V can be added to the Hamiltonian (11) independently of the correction related to the external field, the introduction of W^0 containing the momenta[5] $\boldsymbol{p}_K$ is not so simple, unless an additional postulate — that of the gauge invariance — leads unambiguously to the postulated formulae (8')–(10'). The consistency of the so established new quasirelativistic two-fermion equation (8') within the scheme of the relativistic quantum mechanics must be verified by a detailed comparison of its formal properties (as compared to those of the Breit equation) and by the inspection of

[5] One can say that, in contradistinction to V, W^0 does not possess the character of an "effective potential energy" depending merely on the position coordinates.

conclusions obtained, in particular during its reduction to the Pauli formalism. This can be done on the basis of a new method of reduction, simultaneously proposed by us, different from Chraplyvy's method, although also related to the idea of Foldy and Wouthuysen. The idea of our method may be summarized as follows: an extension of the FW-transformation for a free Dirac particle has been given by Case [21] for $A^{ex} \neq 0$, $\Phi^{ex} = 0$. It has been shown, subsequently, by one of us (Garszczyński, Hanus [22]) that the same transformation used in the general case $A^{ex} \neq 0$, $\Phi^{ex} \neq 0$ gives a much more simple procedure of reducing the Dirac equation than the original iterative procedure used in [5]. Simultaneously, the so obtained modified scheme of describing the Dirac equation gives some new insight into the general problem of separating the positive and negative energy states. All these results can be immediately extended to the case of two Dirac particles, despite of the higher degree of complexity of the problem. The so constructed new method of reduction can be applied to the equations (1) and (8'), successively. A comparison of them shows that only the latter equation can be unambiguously reduced to the Pauli formalism, without introducing any additional assumption. The result holds for different as well as for identical particles. Hence, a striking contrast exists between this simple and consistent result for the equation (8') proposed by us and many controversial aspects accompanying the reduction of the Breit equation (by our method as well as by that of Chraplyvy). Our method seems to be simpler. Moreover, its extension from two to an arbitrary number of particles is almost trivial as no use need be made of the explicit form of the Dirac matrices and of their direct products. Explicit, rather lenghty, calculations and a detailed discussion of the new scheme of describing the n-fermion quantum-mechanical systems in the "generalized Case representation" will be given separately by one of us (Janyszek [23]).

We conclude by characterizing the role played by the covariant Bethe-Salpeter equation [4] in deducing the approximately relativistic, quantum-mechanical wave equations in the configuration space expressed by (8) or (8'), respectively. The connection is more immediate for (8) whose deduction has started from the BS-equation, although it must not be forgotten that — independently of the approximations made during this deduction — an additional assumption has been introduced by Barker and Glover, in order to arrive to the Hermitian Hamiltonian H^0_{BS}. Our generalization (leading to the "generalized BS-Hamiltonian" H_{BS}) refers to the equation (8) so that only indirectly it is related to the original BS-equation, preserving, however, the most important physical content of the latter (as conclusions related to the hole theory appear in our final results, in contrast with those based of the Breit equation). Both, (8) and (8') represent an approach essentially different from that of the BS-equation, not only in its original, many-time formulation, but also in the covariant one-time scheme proposed by Królikowski and Rzewuski [24], (the difference becoming, however, less striking in the latter case). The cause is that (8) and (8') are, from the beginning, expressed in the Hilbert space of the first quantization, in terms of the Hermitian Hamiltonian, the formulation being, by definition, restrained to the second-order of approximation, imposed by the way of introducing the interaction between particles. This formulation seems to give the better starting point for defining other relativistic particle-observables, including the position observable, treated as different from the geometrical space-time coordinate.

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