Vol. B2 (1971) ACTA PHYSICA POLONICA Fasc. 5

ON THE MOTION OF DYONS

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(Received January 12, 1971)

The paper deals with the problem of motion of two dyons (particles with electric and magnetic charges) in relativistic classical mechanics. It is shown that the Bohr-Sommerfeld quantization rules lead to an exact formula for energy levels.

Introduction

A recent article by Schwinger [1] renewed interest in the dynamics of particles with both electric and magnetic charges. Following Schwinger we call them "dyons". The quantization of electric and magnetic charges and a small value of the electric charge leads to a very large value of the magnetic charge [2]. As a result the "fine" structure constant α of the magnetic Coulomb interaction is no longer small (~ 137 rather than $\sim \frac{1}{137}$) and a relativistic treatment of the problem of motion becomes a necessity.

In Section I we present the solution ot the nonrelativistic classical problem of two dyons. In Section II the relativistic classical problem is discused and its canonical description is given. This allows us to use the Bohr-Sommerfeld quantization rules to find the energy levels of the corresponding quantum system. The resulting formula turns out to be exact. The estimated mass of the dyon is roughly the same as that given by Schwinger [1].

1. Nonrelativistic classical motion of two dyons

The Newton equation for the relative motion of two dual charged particles, called dyons, with the charges e_1, g_1 and e_2, g_2 respectively, has the form:

$$m\frac{d\boldsymbol{v}}{dt} = -\alpha\frac{\boldsymbol{r}}{r^3} + \beta\frac{\boldsymbol{v}\times\boldsymbol{r}}{r^3} \tag{1}$$

where[1] m is the reduced mass, $\alpha = -(e_1e_2+g_1g_2)$ and $\beta = (e_1g_2-e_2g_1)$.

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[1] We omit $\hbar$ and c in all formulas.

Equation (1) has four standard constants of motion: the angular momentum vector $\boldsymbol{J}$ corrected by the term representing the angular momentum of the electromagnetic field [3], [4],

$$\boldsymbol{J} = m\boldsymbol{r} \times \boldsymbol{v} - \beta \frac{\boldsymbol{r}}{r} \tag{2}$$

and the energy

$$E = \frac{mv^2}{2} - \frac{a}{r}. \tag{3}$$

Since the projection of $\boldsymbol{J}$ on $\frac{\boldsymbol{r}}{r}$ is also a constant of motion we can choose the z-axis of the spherical coordinates in the direction of $\boldsymbol{J}$. This gives

$$\cos\vartheta = -\frac{\beta}{J} = \text{const.} \tag{4}$$

This means that the trajectory lies on the surface of a cone. The motion is given by the equations

$$mr^2\dot{\varphi} = J \tag{5}$$

$$\frac{m}{2}\left[\dot{r}^2 + \frac{J^2-\beta^2}{m^2 r^2}\right] - \frac{a}{r} = E. \tag{6}$$

Exactly as in the case of the pure Coulomb force it is possible to find the trajectory. To this end we use the relation

$$\dot{r} = -\frac{J}{m}\frac{d}{d\varphi}\left(\frac{1}{r}\right)$$

to eliminate the time variable from the equation (6). The trajectory is given by the formula:

$$\frac{1}{r} = \sqrt{\frac{2Em}{J^2-\beta^2} + \frac{a^2m^2}{(J^2-\beta^2)^2}}\cos\left[\sqrt{\frac{J^2-\beta^2}{J^2}}(\varphi-\varphi_0)\right] + \frac{am}{J^2-\beta^2}. \tag{7}$$

With the exception of circular orbits, all trajectories have the form of rosettes.

To investigate the scattering we introduce a unit vector in the direction of the incoming particle:

$$\boldsymbol{i} = (\sin\vartheta\cos\Phi, \sin\vartheta\sin\Phi, \cos\vartheta)$$

and the outgoing particle,

$$\boldsymbol{f} = (\sin\vartheta\cos\Phi, -\sin\vartheta\sin\vartheta, \cos\vartheta),$$

where Φ is the smallest positive solution of the equation

$$\cos\left(\sqrt{\frac{J^2-\beta^2}{J^2}}\,\Phi\right) = \frac{-a}{\sqrt{a^2 + \frac{2E}{m}(J^2-\beta^2)}}. \tag{8}$$

The scattering angle Θ is introduced through the relation

$$\cos(\pi-\Theta) = \boldsymbol{i}\cdot\boldsymbol{f} = \sin^2\vartheta\cos 2\Phi + \cos^2\vartheta. \tag{9}$$

Conservation of J at infinity gives

$$J^2 = (mbV_\infty)^2 + \beta^2 \tag{10}$$

where b is an impact parameter and V_∞ is the velocity of the particle at infinity. Using (4), (8), (9) and (10) we find the relation between the scattering angle Θ and the impact parameter b in the form

$$\cos\Theta = \frac{-m^2V_\infty^2b^2}{m^2V_\infty^2b^2+\beta^2}\cos\left[2\sqrt{\frac{m^2V_\infty^2b^2+\beta^2}{m^2V_\infty^2b^2}}\arccos\left(\frac{-\alpha}{\sqrt{\alpha^2+m^2V_\infty^4b^2}}\right)\right] - \frac{\beta^2}{m^2V_\infty^2b^2+\beta^2}.$$

To reduce the results of this section to those given in [4], Section 3.1, we should set $\alpha = 0$.

2. *Relativistic classical motion and the quantization*

The relativistic equation of motion has the form

$$\frac{d}{dt}\frac{m\boldsymbol{v}}{\sqrt{1-v^2}} = -\alpha\frac{\boldsymbol{r}}{r} + \beta\frac{\boldsymbol{v}\times\boldsymbol{r}}{r^3} \tag{11}$$

and should be understood as the description of the motion of a light dyon of mass m in the field produced by a heavy dyon located at the origin of the coordinate system. Constants (2) and (3) are replaced by

$$\boldsymbol{J} = \frac{m\boldsymbol{r}\times\boldsymbol{v}}{\sqrt{1-v^2}} - \beta\frac{\boldsymbol{r}}{r}$$

$$E = \frac{m}{\sqrt{1-v^2}} - \frac{\alpha}{r}.$$

The motion is again on the surface of a cone. This allows us to handle the problem as if it were two dimensional. Similarity of the nonrelativistic motion to the Coulomb case suggests the following choice of canonical momenta conjugate to φ and r:

$$p_\varphi = \frac{mr^2\dot\varphi}{\sqrt{1-v^2}} \qquad p_r = \frac{m\dot r}{\sqrt{1-v^2}}.$$

The energy of the system expressed through the canonical variables can be called the Hamiltonian

$$H = \sqrt{m^2+p_r^2+\frac{p_\varphi^2-\beta^2}{r^2}} - \frac{\alpha}{r}$$

because the equation of motion can be obtained as canonical equations, *viz.*,

$$\dot\varphi = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{r^2\sqrt{m^2+p_r^2+\dfrac{p_\varphi^2-\beta^2}{r^2}}}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{\sqrt{m^2 + p_r^2 + \frac{p_\varphi^2 - \beta^2}{r^2}}}$$

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\varphi^2 - \beta^2}{m} \cdot \frac{1}{r^3} - \frac{a}{r^2}.$$

The last equation is the radial component of (11). It is possible to give a three dimensional canonical formulation of the problem [5]. The essential feature of this formulation is that the canonical coordinates cannot be the physical particle's coordinates.

Now, one can make use of the Bohr-Sommerfeld quantization rules to determine the energy levels of the bound states of the corresponding quantum system

$$\frac{1}{2\pi} \oint p_\varphi \, d\varphi = j + \frac{1}{2}$$

$$\frac{1}{2\pi} \oint p_r \, dr = n_r + \frac{1}{2}.$$

The resulting formula reads:

$$E_{n_r, j} = m \left[1 + \frac{a^2}{(n_r + \frac{1}{2} + \sqrt{(j+\frac{1}{2})^2 - a^2 - \beta^2})^2} \right]^{-1/2}. \tag{12}$$

It coincides with the exact formula derived by Białynicki-Birula [6] with the help of the relativistic hydrodynamic formulation of the quantum problem. The radii of the circular orbits are equal to

$$r_j = \frac{\sqrt{[(j+\frac{1}{2})^2 - \beta^2]\,[(j+\frac{1}{2})^2 - \beta^2 - a^2]}}{ma}. \tag{13}$$

Results for the circular orbits were obtained for the first time in [5].

Both (12) and (13) show that there exists a limitation on the quantum number j similar to that which exists in hydrogenlike atoms whose nuclei have a charge number greater than 137. In fact for a system composed of two dyons of magnetic charges $g^2 = 4.137$ and $\beta = 2$ (compare [1]), the lowest state could be realized for $n_r = 0$ and $j = 548$ (!). Applying formula (12) to the system of two[2] spinless dyons of equal masses and equating the mass of the ground state to the mass of the charged π-meson, we estimate the mass of dyon to be

$$m_D = 6.5 \text{ GeV},$$

in agreement with Schwinger's predictions.

[2] This implies an instant interaction and it therefore undermines the relativistic character of the problem.

If the electrically neutral meson is built in an analogous way as the charged one, the corresponding case is obtained by setting $\beta = 0$. The electromagnetic mass difference is

$$\frac{m_0 - m_\pm}{m_0} \sim 1\%.$$

The order of magnitude is reasonable, but it is alvays positive.

I am indebted to Professor I. Białynicki-Birula for bringing the problem considered to my attention and for valuable conversations.

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