THREE ALPHA-PARTICLE SPACE CORRELATIONS IN C12*

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The three-alpha structure of the ground ($J^\pi, T = 0^+, 0$) and first excited ($2^+, 0$) states of the C12 nucleus was investigated in the framework of the shell-model theory. The probability density for the space distribution of the three alphas was obtained by projecting three-alpha states from the intermediate-coupling model wavefunctions of the C12 nucleus. The projection procedure was done with the use of three-alpha coefficients of fractional parentage of the translationally invariant shell model. The main result is that the triangular and linear configurations of the three alphas are present in the intermediate-coupling model wavefunctions of the analyzed states. The intermediate-coupling model privileges a linear oscillations in the ground state, while a triangular oscillations are more probable in the first excited state. The probability of the three-alpha clusterization of C12, predicted with the aid of intermediate-coupling model wavefunctions, is equal to 0.47 and 0.52 for the ground and first excited state, respectively.

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1. Introduction

The alpha-particle model has been used in a description of many properties of light nuclei [1–4]. In particular, there have been many papers [4–18] where the C12 nucleus has been treated as three-alpha bound system. In most of these papers, involving different methods, the attention has been mainly focused on the calculation of observable quantities such as binding energy, energy levels, B(E2) values, electron scattering form factors, the reduced alpha decay widths and their comparison with experimental data.

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Among the papers devoted to the three-alpha structure of C12 the ones reported by Kamimura et al. [5–7] seem to be the most valuable. In Refs. [5–6] the microscopic three--alpha eigen-value problem of C12, based on the resonating group method (RGM), has been solved with total antisymmetrization taken into account. The most successful result of this approach is the proper location of the second 0^+ level (which cannot be satisfactorily predicted by the shell model [19–21]) and negative parity levels as well as explanation of the electric transitions in the ground state band and transitions connecting different bands. On the contrary, in other papers [22–24], the interest has been concentrated on the density distribution of alpha particles, the aim being to connect the space correlations of the alpha--clusters with different modes of excitation of some states of the C12 nucleus.

Nevertheless, in most of these pappers one assumes that C12 nucleus is "a priori" aggregate of the three structure or structureless interacting alpha-particles. So, in this approach, one can notice a trivial fact that the probability of the three-alpha clusterization of C12 is equal to unity for any considered state. On the other hand, looking at the same nucleus from the standpoint of the shell-model theory this probability will be less than one, since except of the three-alpha clusterization any other kinds of clusterization might be preffered by the nuclear interactions. Taking into account this fact it seems reasonable to investigate a content of the three alpha-cluster structures in the shell-model wavefunctions of the C12 nucleus. These investigations become particularly important in the case of lowest states of C12 which reveal a shell-like structure, on the contrary to the excited ($E_x > 7$ MeV) states which, according to the results of Japanese theoreticians [4], possess a clear three-alpha cluster structure.

The present work is along the philosophy of Refs. [22–24] with a special emphasis on the problem of analyzing how much the correlations in the motion of the alpha-clusters are contained in the shell-model wave functions. To this end we will consider the projection of wavefunctions of the lowest states of C12 onto the subspace spanned by the three-alpha states. The wavefunctions will be evaluated in the intermediate-coupling model which has been shown to give a satisfactory description of light nuclei [19–21, 25–30].

In Section 2 the formalism of the projection method is briefly outlined, while in Section 3 we present a method for calculating the three-alpha coefficients of fractional parentage (CFP) of the translationally invariant shell model (TISM) which are required to perform the projection. Some results on the three-alpha structure of the ground (J^π, T = 0^+, 0) and first excited (2^+, 0) states of C12 are shown and discussed in Section 4.

2. *The projection method*

The basic idea of the present method of analyzing the alpha-cluster structure of the lowest states of light nuclei is based on the assumption that, in principle, realistic nuclear wavefunctions describe all possible configurations and space correlations of nucleons in the considered states. In the case of 1p-shell nuclei, the low lying states can be well represented by the intermediate-coupling model wavefunctions [19–21]. Thus, the relevant three-alpha structures in the low lying states of C12 will be described by the projection $\Phi^{\mathrm{P}}_{J,M}$ of the intrinsic part of intermediate-coupling model wavefunctions $\Phi_{J,M}(\mathrm{C12})$ onto the subspace spanned by the three-alpha states. In particular, we will study the three-alpha correlations in the ground (J^π, $T = 0^+, 0$) and first excited ($2^+, 0$) states which are better described in the intermediate-coupling model.

Since, in the present work, shell-model calculations are performed in a restricted harmonic oscillator (h.o.) space i.e., limited to the s^4p^8 configuration or equivalently to the $N = 8$ h.o. quanta, the three-alpha states will be expanded in the h.o. space with the same number of h.o. quanta. A suitable orthonormal set of functions for description of the relative motion of three alphas can be defined as follows

$$f^{\sigma}_{J,M}(\vec{R}_1, \vec{R}_2) = [f_{N_1L_1}(\vec{R}_1)\times f_{N_2L_2}(\vec{R}_2)]^{JM}, \tag{1}$$

where the abbreviated notation $\sigma = N_1L_1N_2L_2$ is used and the h.o. wavefunctions $f_{N_iL_i}(\vec{R}_i)$ ($i = 1, 2$; $N_i = 2n_i+L_i$; $N_1+N_2 = 8$) refer to the Jacobi coordinates

$$\vec{R}_1 = \vec{r}_1-\vec{r}_2 \text{ and } \vec{R}_2 = \vec{r}_3-(\vec{r}_1+\vec{r}_2)/2, \tag{2}$$

and $\vec{r}_i$ ($i = 1, 2, 3$) is the centre of mass (c.m.) coordinate of the i-th alpha particle fixed in the laboratory reference frame. Thus, the wavefunction of the three-alpha system has the form

$$\Phi^{\sigma}_{JM}(\vec{R}_1, \vec{R}_2) = \hat{A}(f^{\sigma}_{JM}(\vec{R}_1, \vec{R}_2)\Phi_{\alpha_1}\Phi_{\alpha_2}\Phi_{\alpha_3}), \tag{3}$$

where

$$\hat{A} = \sqrt{(4!)^3/(3!12!)}\sum_{P}(-)^P P \tag{4}$$

is the antisymmetrization operator with the permutation operator P of nucleons between different alpha paricles, and Φ_{α_i} is the internal wavefunction of the i-th ($i = 1, 2, 3$) alpha--particle. The factor $(\sqrt{3!})^{-1}$ in Eq. (4) takes account of indistinguishability of alpha particles. Although the set states of Eq. (3) does not constitute the orthonormal basis it covers interesting us $N = 8$ three-alpha space. The orthonormal three-alpha basis wavefunctions $F^{\tau}_{J,M}$ are obtained by constructing the matrix

$$\|N^{\sigma\sigma'}\| = \|\langle\Phi^{\sigma}_{JM}(\vec{R}_1, \vec{R}_2) \mid \Phi^{\sigma'}_{JM}(\vec{R}_1, \vec{R}_2)\rangle\| \tag{5}$$

by the functions (3) and by solving the eigen-value problem of this matrix as follows

$$\sum_{\sigma'}\langle\Phi^{\sigma}_{JM}(\vec{R}_1, \vec{R}_2) \mid \Phi^{\sigma'}_{JM}(\vec{R}_1, \vec{R}_2)\rangle C^{\tau\sigma'}_J = \mu_\tau C^{\tau\sigma}. \tag{6}$$

The orthonormal eigen-functions corresponding to the eigen-value $\mu_\tau \neq 0$ take the form[1]

$$F^{\tau}_{JM} = \frac{1}{\sqrt{\mu_\tau}}\sum_{\sigma} C^{\tau\sigma}_J \Phi^{\sigma}_{JM}(\vec{R}_1, \vec{R}_2). \tag{7}$$

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[1] From (7) one obtains $\mu_\tau = |\sum_{\sigma} C^{\tau\sigma}_J\Phi^{\sigma}_{JM}(\vec{R}_1, \vec{R}_2)|^2$. If $\mu_\tau = 0$, then $\sum_{\sigma} C^{\tau\sigma}_J\Phi^{\sigma}_{JM}(\vec{R}_1, \vec{R}_2) = 0$. This defines the linear dependence among the functions $\Phi^{\sigma}_{JM}(\vec{R}_1, \vec{R}_2)$ which is caused by the Pauli principle. Thus, the basis functions of the three-alpha system are defined for $\mu_\tau \neq 0$.

The matrix elements $N^{\sigma\sigma'}$ of Eq. (5), in what follows denoted as norm overlaps, can be immediately obtained with the aid of appropriate RGM norm kernels [31, 32]. Calculations of the RGM norm kernels, in general, are complicated and require advanced algebraic methods (for instance see Refs. [14, 31, 32]). However, in interesting us case, these overlaps can be simply evaluated with the aid of the three-alpha CFP of TISM which are considered in detail in Section 3.

Since states (7) form an orthonormal set, they can be used as basis functions of the three-alpha system and the following projection operator $\hat{P}$ can be constructed

$$\hat{P} = \sum_{\tau} |F^{\tau}_{JM}\rangle \langle F^{\tau}_{JM}| \tag{8}$$

which can serve to select from any wavefunction that part which refers to the three-alpha system. Then, acting on the intrinsic intermediate-coupling model wavefunction $\Phi_{JM}(\mathrm{C12})$ of C12 one obtains

$$|\Phi^{P}_{JM}\rangle = \hat{P}|\Phi_{JM}(C12)\rangle = \sum_{\tau} \langle F^{\tau}_{JM}|\Phi_{JM}(C12)\rangle |F^{\tau}_{JM}\rangle, \tag{9}$$

where $\langle F^{\tau}_{JM}|\Phi_{JM}(12\mathrm{C})\rangle = a([444]L = J)$.
The coefficient $a([444]L = J)$ represents the amplitude of the state $\Phi_J(s^4p^8[444]L = J\ S = T = 0)$ in the intermediate-coupling model wavefunction of C12 which is expanded in the basis of Eq. (16) (for details see Appendix A1). In order to study the geometrical distribution of the three-alpha clusters in C12 the overlap

$$\Gamma(\vec{R}_1, \vec{R}_2) = \langle \hat{A}[\delta(\vec{R}_1 - \vec{\boldsymbol{R}}_1)\delta(\vec{R}_2 - \vec{\boldsymbol{R}}_2)\Phi_{\alpha_1}\Phi_{\alpha_2}\Phi_{\alpha_3}]\ |\Phi^{P}_{JM}\rangle \tag{10}$$

has to be calculated by integrating over all the internal coordinates of the three alphas. Taking into account Eqs. (7) and (9) one can easily point out [32] that this overlap

$$\Gamma(\vec{R}_1, \vec{R}_2) = a([444]L = J\ T = S = 0) \sum_{\tau\sigma} \sqrt{\mu_\tau}\, C^{\tau\sigma}_{J} f^{\sigma}_{JM}(\vec{R}_1, \vec{R}_2). \tag{11}$$

Inserting into (11) Eq. (1), the explicit form of the amplitude $\Gamma(R_1, R_2)$ in the cartesian reference frame (X, Y, Z) of Fig. 1 finally takes the form

$$\Gamma(\vec{R}_1, \vec{R}_2) = a([444]L = J) \sum_{\tau\sigma M_1 M_2} \sqrt{\mu_\tau}\, C^{\tau\sigma}_{J} \langle L_1 M_1 L_2 M_2 | JM\rangle$$

$$\times f_{N_1L_1}(R_1) f_{N_2L_2}(R_2) Y_{L_1M_1}(\vartheta_1, \varphi_1) Y_{L_2M_2}(\vartheta_2, \varphi_2), \tag{12}$$

where $f_{NL}(R)$ is a radial h.o. wavefunction[2], $Y_{LM}(\vartheta, \varphi)$ a spherical harmonic and $\langle L_1M_1L_2M_2|JM\rangle$ a Clebsch-Gordan coefficient. The probability density to find the three alphas at the positions described by the Jacobi coordinates $\vec{R}_1, \vec{R}_2$ of Eq. (2) is equal to

$$P_{\mathrm{D}}(\vec{R}_1, \vec{R}_2) = |\Gamma(\vec{R}_1, \vec{R}_2)|^2. \tag{13}$$

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[2] In the present work the wavefunction $f_{NL}(R)$ corresponds to $f_{nL}(R)$ of Ref. [33] on the condition that $N = 2n + L$.

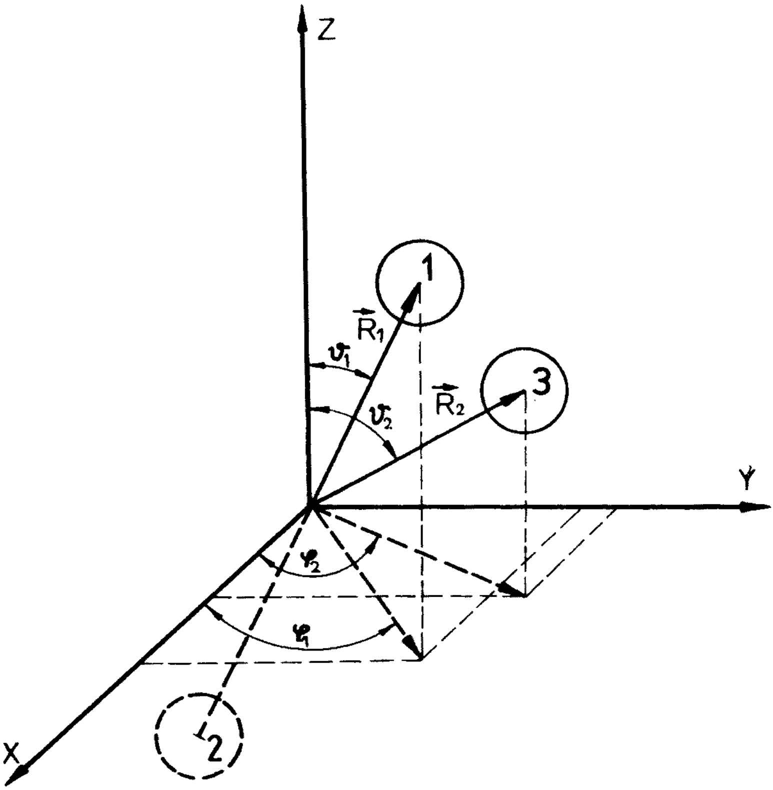


Fig. 1. The reference frame for the three alpha-particle system

Immediately from (13) via (12), one obtains the total probability (i.e., the spectroscopic factor) of the three alpha clusterization of C12. This probability is

$$P = \int |\Gamma(\vec{R}_1, \vec{R}_2)|^2 d\vec{R}_1 d\vec{R}_2 = a^2([444]L = J) \sum_{\tau} \mu_\tau. \tag{14}$$

Eqs. (12) and (13) will be the starting point for study of the behaviour of the three-alpha particles density distribution P_D in the reference frame of Fig. 1.

For the $J = 0$ states the wavefunctions (and their projections) must be isotropic. Thus the P_D will not depend on the orientation of the three-alpha system in reference frame of Fig. 1. On the contrary, in the case of $J \neq 0$ states their wavefunctions are not isotropic (but only symmetric about the Z-axis) and then P_D should depend on the three-alpha system orientation in the considered reference frame (Fig. 1). This property is discussed in detail in Section 4.

3. *The norm overlaps of the three-alpha cluster states*

In this Section a method of calculating the norm overlaps of the three-alpha cluster states of Eq. (3) is shortly presented. The method is based on the use of the three-alpha CFP of TISM for pure s^4p^8 shell-model configuration (see also Ref. [34]).

3.1. Coefficients of fractional parentage of the translationally invariant shell model

The three-alpha CFP of TISM can be defined as follows

$$S(q, \sigma = N_1L_1N_2L_2) = \langle \Phi_J(A = 12, q) \mid \hat{A}\{[f_{N_1L_1}(\vec{R}_1) \times f_{N_2L_2}(\vec{R}_2)]^{L=J} \Phi_{\alpha_1}\Phi_{\alpha_2}\Phi_{\alpha_3}\}\rangle, \tag{15}$$

where $\hat{A}$ is the antisymmetrization operator of Eq. (4), $\Phi_J(A = 12, q)$ represents an internal (translationally invariant) and totally antisymmetric wavefunction of 12 nucleons in the state labelled by the set of quantum numbers q, and Φ_α is the internal wavefunction of alpha particle while the meaning of functions $f_{NL}(\vec{R})$ is the same as in Section 2. The wavefunction $\Phi_J(A = 12, q)$ can be chosen as the one forming internal part of the following h.o. shell--model A-nucleon state

$$\{\Psi(s^4p^{A-4}, q; JT)\} \tag{16}$$

with $q = [f]\beta LSJT$, where $[f]$ is the Young diagram determining the permutational symmetry of the orbital part of the wavefunction and β denotes the set of quantum numbers required for a complete labelling of the states, in addition to the total orbital L and spin S angular momentum, the total spin J and isospin T (the method of constructing wavefunctions (16) is described in Ref. [35]). Thus, the wavefunction $\Phi_J(A = 12, q)$ can be extracted from (16) with the aid of the following relation [36]

$$\Psi(s^4p^{A-4}q; JT) = f_{00}(\vec{R})\Phi_J(s^4p^{A-4}, q), \tag{17}$$

where $f_{00}(\vec{R})$ describes the motion of the c.m. (with zero h.o. quanta) of a nucleus A with respect to the fixed center of the h.o. potential well. In order to calculate the CFP of TISM of Eq. (15) let us define the auxiliary integral (see Fig. 2b)

$$I_1(q, \sigma = N_1L_1N_0L_0; L) = \langle \Psi(s^4p^8, q; JT) \mid \hat{A}\{[f_{N_1L_1}(\vec{R}_1) \times f_{N_0L_0}(\vec{R}_0)]^{L=J} \times \psi_{\alpha_3}\Phi_{\alpha_2}\Phi_{\alpha_1}\}\rangle, \tag{18}$$

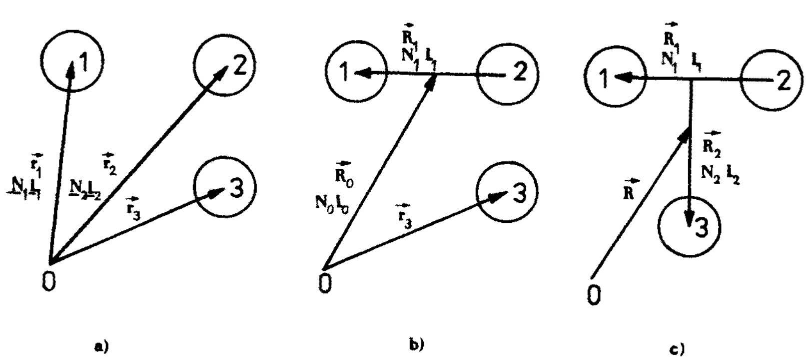


Fig. 2. The sets of coordinates for the three alpha particles

where the h.o. shell-model wavefunction of the alpha-particle is approximated by the $\Psi_\alpha(s^4[4]\, L = S = T = 0)$. Thus, due to the relation (17) Φ_α is directly defined by

$$\psi_\alpha(s^4[4]L = S = T = 0) = f_{00}(\vec{r})\Phi_\alpha(s^4[4]L = S = T = S = 0). \tag{19}$$

Making use of the generalized Talmi-Moshinsky brackets [37] the transformation from the Jacobi coordinates of Fig. 2b to those of Fig. 2c leads to the following relation

$$S(q, \sigma) = \langle 00N_0L_0 : L_0|4/8|00N_2L_2 : L_2\rangle^{-1} I_1(q, \sigma = N_1L_1N_0L_0 : L) \times \delta N_0N_2\delta L_0L_2. \tag{20}$$

In order to evaluate the overlap I_1 one defines a second auxiliary integral I (see Fig. 2a)

$$I(q, \sigma = \underline{N}_1\underline{L}_1\underline{N}_2\underline{L}_2) = \langle \Psi(s^4p^8, qJT) \mid \hat{A}\{[f_{\underline{N}_1\underline{L}_1}(\vec{r}_1)\times f_{\underline{N}_2\underline{L}_2}(\vec{r}_2)]^{L=J} \times \Psi_{\alpha3}\Phi_{\alpha2}\Phi_{\alpha1}\}\rangle = \langle f_{00}(\vec{R})\Phi_J(s^4p^8, q)| |\hat{A}\{[f_{\underline{N}_1\underline{L}_1}(\vec{r}_1)\times f_{\underline{N}_2\underline{L}_2}(\vec{r}_2)]^{L=J} f_{00}(\vec{r}_3)\Phi_{\alpha3}\Phi_{\alpha2}\Phi_{\alpha_1}\}\rangle. \tag{21}$$

Finally, transforming the integral I_1 of Eq. (18) from the coordinates of Fig. 2b to those of Fig. 2a and taking into account Eqs. (20) and (21) one obtains

$$S(q, \sigma = N_1L_1N_2L_2) = \langle 00N_2L_2 : L_2|4/8|00N_2L_2 : L_2\rangle^{-1} \times \sum_{\underline{N}_1\underline{L}_1\underline{N}_2\underline{L}_2} (-)^{L_1+L_2-\underline{L}_1}\langle \underline{N}_1\underline{L}_1\underline{N}_2\underline{L}_2 : L|N_1L_1N_2L_2 : L\rangle I(q, \sigma' = \underline{N}_1\underline{L}_1\underline{N}_2\underline{L}_2). \tag{22}$$

The formula for the integral I of Eq. (21) is derived in Appendix A2.

3.2. The norm overlaps

The norm overlaps $N^{\sigma\sigma'}$ of the states (3) can be written as

$$N^{\sigma\sigma'} = \langle \hat{A}[f^{\sigma}_{JM}(\vec{R}_1, \vec{R}_2)\Phi_{\alpha_1}\Phi_{\alpha_2}\Phi_{\alpha_3}]|\hat{A}[f^{\sigma'}_{JM}(\vec{R}_1, \vec{R}_2)\Phi_{\alpha1}\Phi_{\alpha2}\Phi_{\alpha3}]\rangle, \tag{23}$$

where notation of Section 2 is kept.

Any shell-model, totally antisymmetric wavefunction of A nucleons with defined total spin J and isospin T, corresponding to s^4p^{A-4} configuration can be expanded in the basis of the states of Eq. (16). Thus, due to the relation (17), one can construct in the space restricted to the s^4p^{A-4} shell-model configuration the following unit operator

$$\hat{1} = \sum_{q=[f]\beta LST} |\Phi_J(s^4p^{A-4}, q\rangle \langle \Phi_J(s^4p^{A-4}, q|. \tag{24}$$

Inserting this operator between "bra" and "ket" states of Eq. (23) one obtains that

$$N^{\sigma\sigma'} = \sum_q S(q, \sigma)S(q, \sigma'), \tag{25}$$

where $S(q, \sigma)$ are just the CFP of TISM defined by Eq. (15). Although the present method is applied to the s^4p^8 shell-model configuration, a generalization to more complex cases can be straightforward (on the condition that the appropriate CFP of TISM are known).

4. *Results and discussion*

Calculations of the probability density distribution were performed according to formulas (12) and (13) for the ground (0^+, 0) and first excited (2^+, 0) state, respectively. The norms μ_τ and expansion coefficients $C_J^{\tau\sigma}$ of the three-alpha states of Eq. (7) entering into these formulas were obtained by solving the eigen-value problem of Eq. (6). The solution of Eq. (6) gives only one eigenvalue $\mu_\tau \neq 0$ for $J = 0$ and $J = 2$ (also for $J = 4$) which is equal to 0.593 (see also Ref. [38] where μ_τ was calculated by using the generating function technique). The expansion coefficients $C_J^{\tau\sigma}$ corresponding to this eigenvalue are just the $\langle(40)L_1\ (40)L_2|(04)L = J\rangle$ Clebsch-Gordan coefficients for the SU(3) group[3]. The matrix elements $N^{\sigma\sigma'}$ of Eq. (6) were calculated with the aid of formula (25), while the three-alpha CFP of TISM, entering into (25), were calculated by using formula (22) via (A2.4) and taking the required alpha-particle CFP from Ref. [39]. The CFP of TISM used in the present work are listed in Table I. As expected from the three-alpha structure of C12, the term $\Phi_J(s^4p^8[444]\,L = J\,S = T = 0)$ is dominating in the intermediate-coupling model wavefunction of the C12 nucleus (see Eq. (A1.1)) and appears with the amplitude $a([444]\,L = J)$ equal to 0.89 and -0.937 for the ground ($J = 0$) and first excited ($J = 2$) state, respectively, if the parameter of interactions of Ref. [19] were used in the intermediate-coupling model hamiltonian. The h.o. frequency parameter for the radial wavefunctions of the relative motion of the three-alpha particles was obtained from the formula $\hbar\omega = 41\,A^{-1/3}$ [40].

The results for the ground (0^+, 0) state are orientation independent because its wavefunction $\Phi_{0,0}(\text{C12})$ (as well as $\Phi^P_{0,0}$) is isotropic. In order to present the results, we found more convenient to consider a different reference frame (U, V) obtained from the one of Fig. 1 by translating the origin to the c.m. of the three alphas and by rotating the axes so that the position vector of one particle (say particle 3) is paralell to the V-axis. The

TABLE I

The three-alpha particle coefficients of fractional parentage of the translationally invariant shell model for the s^4p^8 shell-model configuration. For explanation see text

N_1L_1 N_2L_2	4 0 4 0	4 2 4 2	4 4 4 4
$L = 0$	-0.411	0.459	-0.462

N_1L_1 N_2L_2	4 0 4 2	4 2 4 0	4 2 4 2	4 2 4 4	4 4 4 2	4 4 4 4
$L = 2$	0.205	0.205	-0.510	-0.066	-0.066	0.489

[3] Since the three-alpha states are expanded in the h.o. space their classification can be given by the use of SU(3) group. In the $N = 8$ case only $(\lambda\mu) = (04)$ states are allowed [11, 16]. Therefore, the expansion coefficients $C_J^{\tau\sigma}$ of the states (7) (in basis (1)) are just the $\langle(40)L_1\ (40)L_2|(04)L = J\rangle$ coefficients.

distribution of the other two alphas will be represented by the contour plotts of their probability density. Two peaks will show their most probable positions which, together with the already fixed point along the V-axis, give the most probable configuration of the three alphas. In Figs. 3a–d the results for the ground state are shown for some values of the coordinate $v(= 2R_2/3)$. For small distances ($v < 1.3$ fm) particle 3 from the c.m. the dominating mode of motion is a linear oscillation. When particle 3 is placed at $1.3 < v < 1.8$ fm, two configurations; linear and triangular, emerge with a comparable probability. The triangular configuration is represented by an isosceles (almost equilateral) triangle. When $v > 1.8$ fm, the absolute value of the probability is smaller because we are now in a region where the nuclear density is small. In this case we find the triangular shape to be the most probable, but the linear configuration is still present.

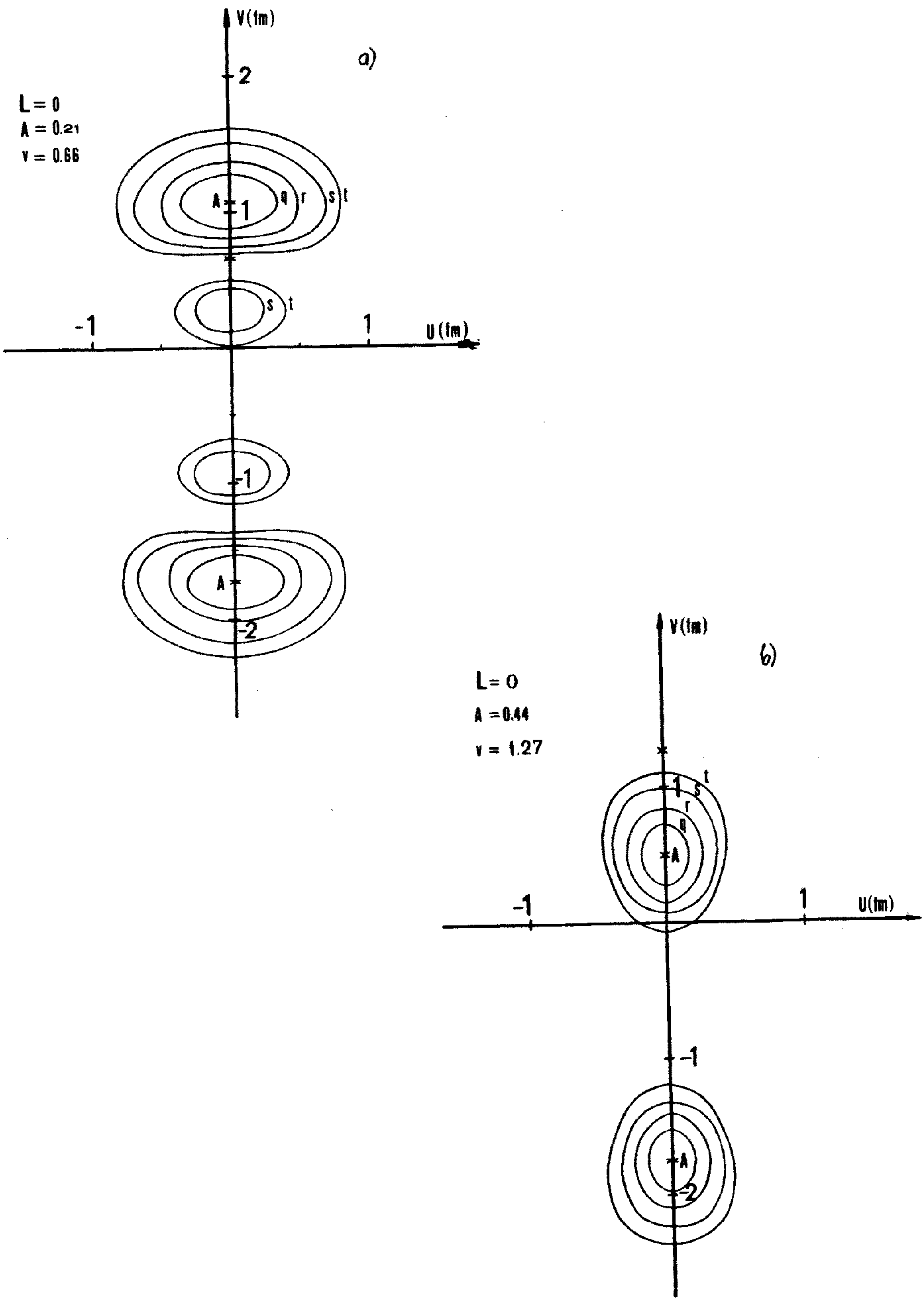
In order to get a closer inspection to the mode of oscillations of the triangular configuration in Fig. 4 the density distribution corresponding to this one is plotted in the R_1, R_2 plane. Looking at Fig. 4, only a single distinct peak is seen at point $(R_1, R_2) = (2.8, 2.4$ fm) which corresponds to an equilateral triangle with the edge equal to 2.8 fm. The distribution of the contours in the neighbourhood of this peak suggests that the most probable motion of this structure is an oscillation around its most probable shape. The second hill, peaked at $(R_1, R_2) = (1.6, 1.4$ fm), allows to suspect the presence of breathing mode in the internal motion of the triangular structure. The two scarcely visible islands around the points $(R_1, R_2) = (3.1, 1.2$ fm) and (1.4, 2.6 fm) may suggest only a small probability of two other kinds of oscillation, i.e. when the height of the triangle varies and its base remains unchanged or vice versa. It is worth noticing that for $R_2 < 1.1$ fm and/or $R_2 < 0.9$ fm the triangular configuration is not present at all.

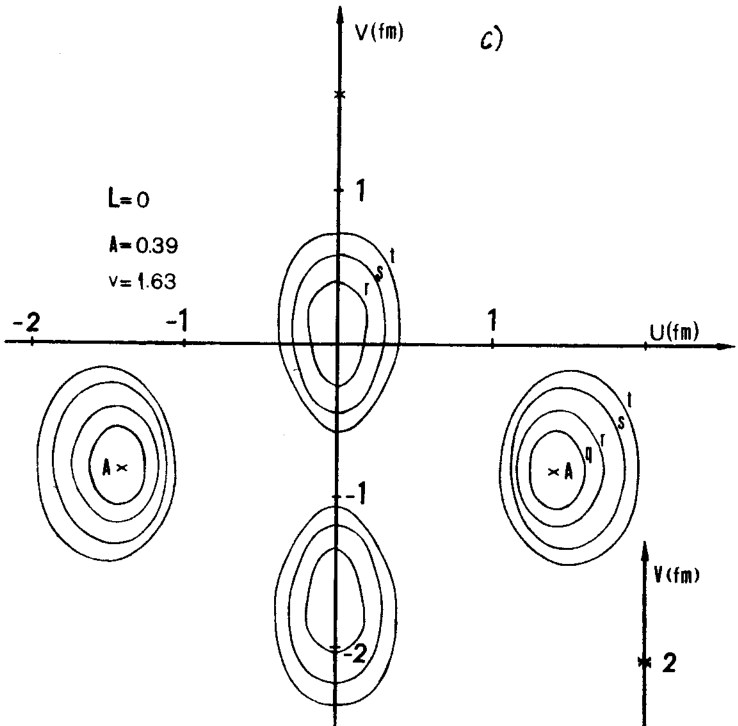
The above results allow to conclude that some competition between the linear and triangular configurations of three alphas is predicted by the intermediate-coupling model in the ground state of C12. Since, as already stressed, for the $J = 0$ states the probability density P_D does not depend on the the three-alpha system orientation (in the reference frame of Fig. 1), thus the quantitative competition between these two configurations can be estimated by calculating the quantity

$$k = \frac{\int |\Gamma(R_1, \vartheta_1 = 0, \varphi_1, R_2, \vartheta_2 = 0, \varphi_2)|^2 dR_1 dR_2 d\varphi_1 d\varphi_2}{\int |\Gamma(R_1, \vartheta_1 = 0, \varphi_1, R_2, \vartheta_2 = \pi/2, \varphi_2)|^2 dR_1 dR_2 d\varphi_1 d\varphi_2} \tag{26}$$

with $\Gamma(\vec{R}_1, \vec{R}_2)$ of Eq. (12). The k-quantity defines the ratio of the probability density (integrated over R_1, R_2, φ_1 and φ_2) for the linear ($\vartheta_1 = 0$, $\vartheta_2 = 0$) configuration to that for the triangular ($\vartheta_1 = 0$, $\vartheta_2 = \pi/2$) one. For the ground state $k = 4.5$.

As already mentioned, in the case of states with $L \neq 0$ the density distribution is orientation-dependent (but symmetric about the Z-axis) and the shape of a system can change with its orientation. This property is visualized for the first excited $(2^+, 0)$ state in Figs. 5–7. From the analysis of the density distribution for some different orientations of the three-alpha system in the reference fram (X, Y, Z) of Fig. 1 and for two alternative orientations of the angular momentum $\vec{L}$, namely, for the $M = 0\,(\vec{L} \perp Z)$ and the $M = L\,(\vec{L} \parallel Z)$ case, it turns out that the most dominating configuration appears when


V(fm)
a)
2
L = 0
A = 0.21
v = 0.66
A
1
q
r
s
t
-1
1
U (fm)
-1
A
-2
V(fm)
b)
L = 0
A = 0.44
v = 1.27
1
t
s
r
q
A
-1
1
U(fm)
-1
A
-2



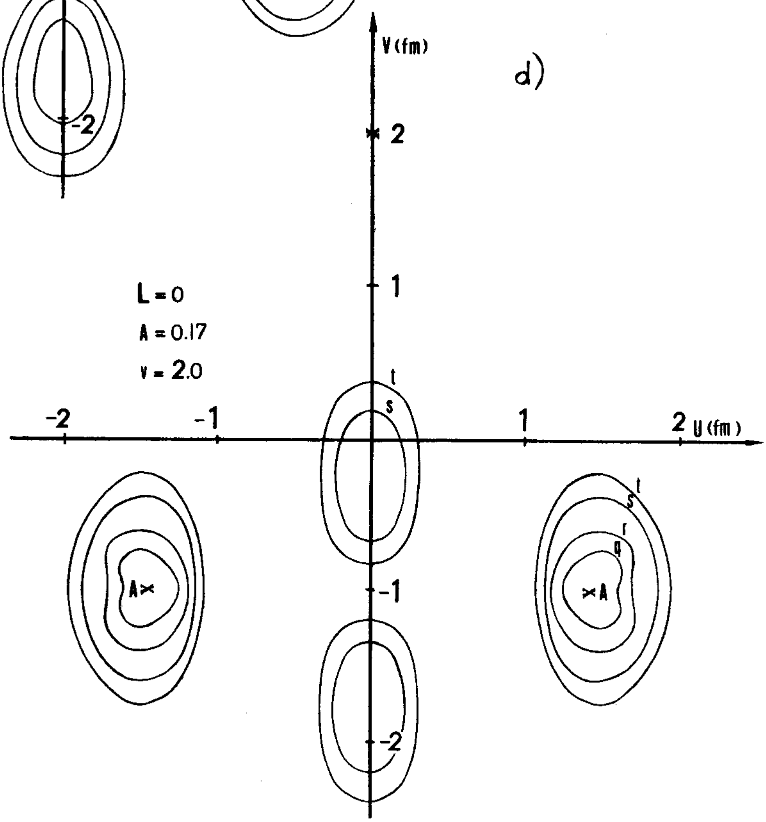


Fig. 3a–d. The probability density P_D distribution for the three alpha particles in the ground (J^π, $T = 0^+, 0$) state of the C12 nucleus plotted in the reference frame (U, V) for different values of the $v = 2R_2/3$ parameter. A-denotes the P_D^{MAX} of a given peak multiplied by the factor 100 (in [fm^{-2}]). In all figures $q = 0.75$, $r = 0.50$, $s = 0.25$, and $t = 0.10$ indicate the fraction of the P_D^{MAX}

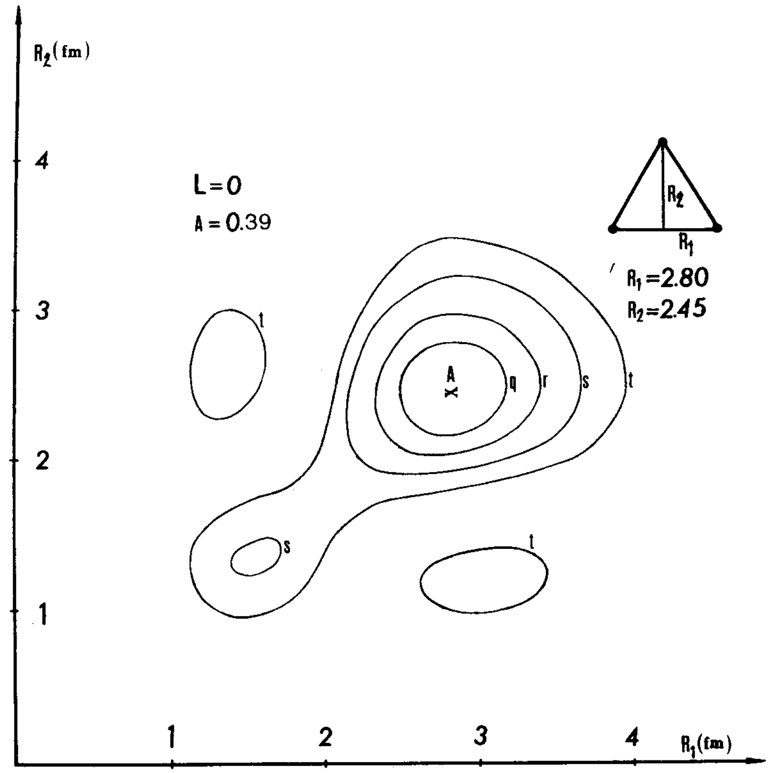
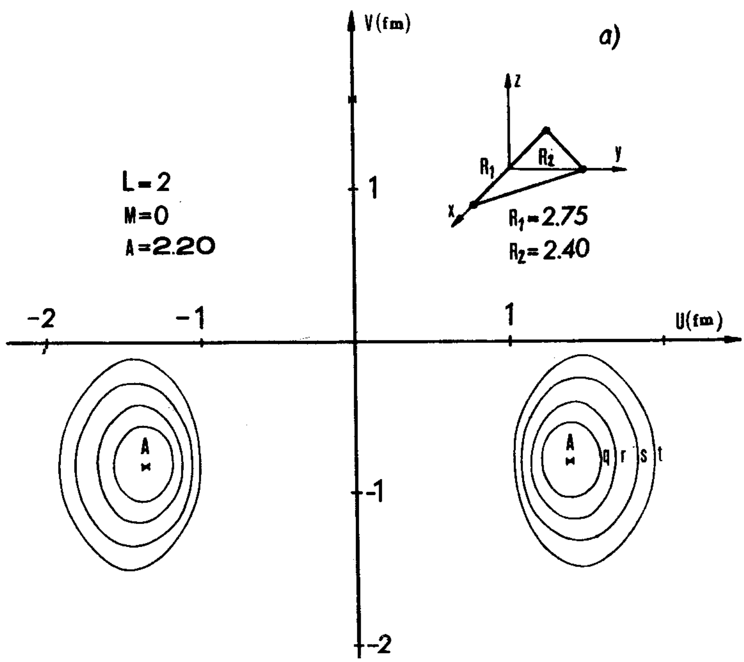


Fig. 4. The P_D for the triangular configuration of the ground $(0^+, 0)$ state of the C12 nucleus plotted as a function R_1, R_2. The most probable triangle is described in the upper right-hand corner. Symbols as in Figs. 3a–d



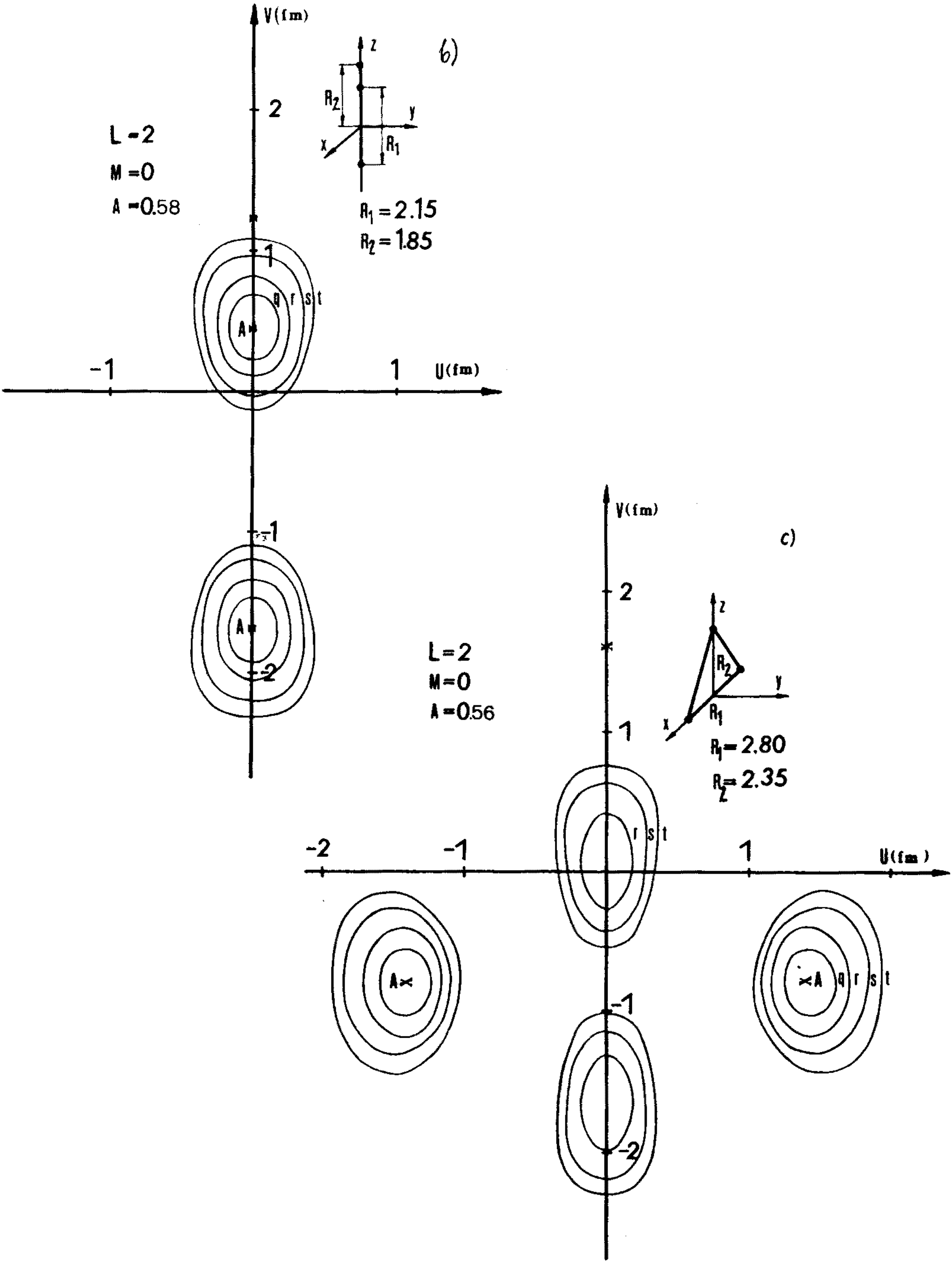


Fig. 5a–c. The P_D for the three alpha particles in the first excited (J^π, M, T + $2^+, 0, 0$) state of the C12 nucleus plotted in the reference frame (U, V). The description of the most probable configuration of the three alphas in the (XYZ) reference frame is given in the upper right-hand corner. Symbols as in Figs. 3a–d

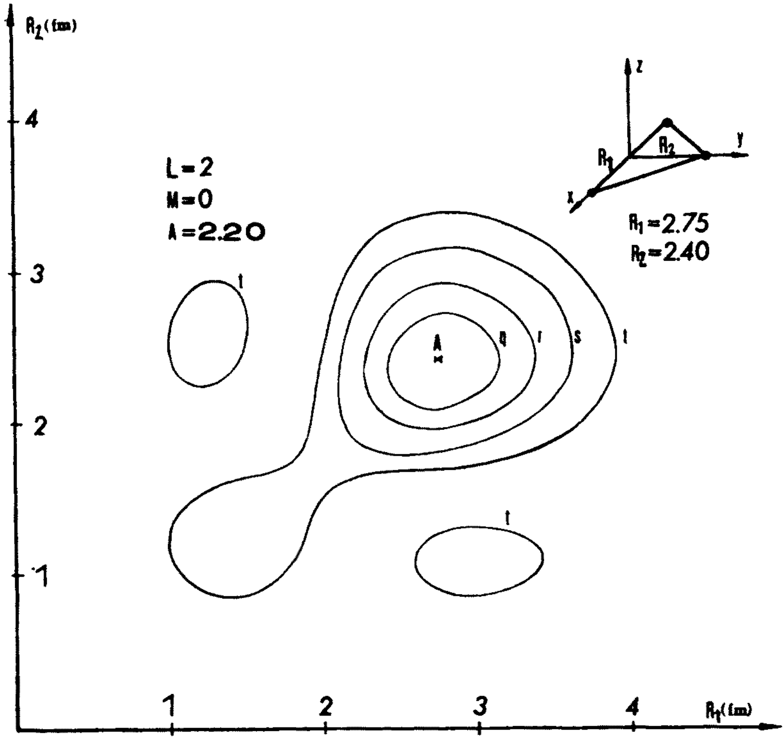


Fig. 6. As Fig. 4 but for the ($J^\pi, M, T = 2^+, 0, 0$) state

$\vec{L} \perp Z$ ($M = 0$). This dominating configuration is triangular one lying in the X-Y plane (Fig. 5a) and its most probable shape is represented by an equilateral triangle with the edge equal to 2.75 fm. If alpha particles are constrain to lie in the X-Y plane a linear configuration is almost absent since in this case $k = 0.15$. We also plot for the triangular configuration lying in this plane the density distribution as function of R_1 and R_2 (Fig. 6). Comparing the shape of the contours plotted in Fig. 6 with those plotted in Fig. 4 it becomes clear that the same modes of motion appear in the triangular configuration of both (0^+, 0) and (2^+, 0) states of C12. When the vector $\vec{R}_2$ is placed along the Z-axis, both the linear and the triangular configurations emerge ($k = 4.5$). But, it depends on the value of the v ($= 2R_2/3$) parameter which configuration is dominating. The most distinct linear configuration emerges at $v = 1.23$ fm (Fig. 5b) and its most probable length $l = 2.93$ fm. When the v-parameter increases the linear configuration disappears while the triangular configuration becomes more and more clear. Its most distinct shape, represented by an equilateral, triangle (with the edge equal to 2.80 fm), appears at $v = 1.57$ fm (Fig. 5c). When $v > 1.6$ fm, only triangular configuration is still present, but with a small probability because we arrived at the border of the nucleus. Similar behaviour of the three-alpha system is observed when the vector $\vec{R}_1$ (instead of $\vec{R}_2$) is placed along the Z-axis of Fig. 1. Considering the $M = 2$ ($\vec{L} \parallel Z$) case, when vector $\vec{R}_1$ is parallel to the Z-axis, the only present configuration is the triangular one ($k = 0$). The most probable triangle is the

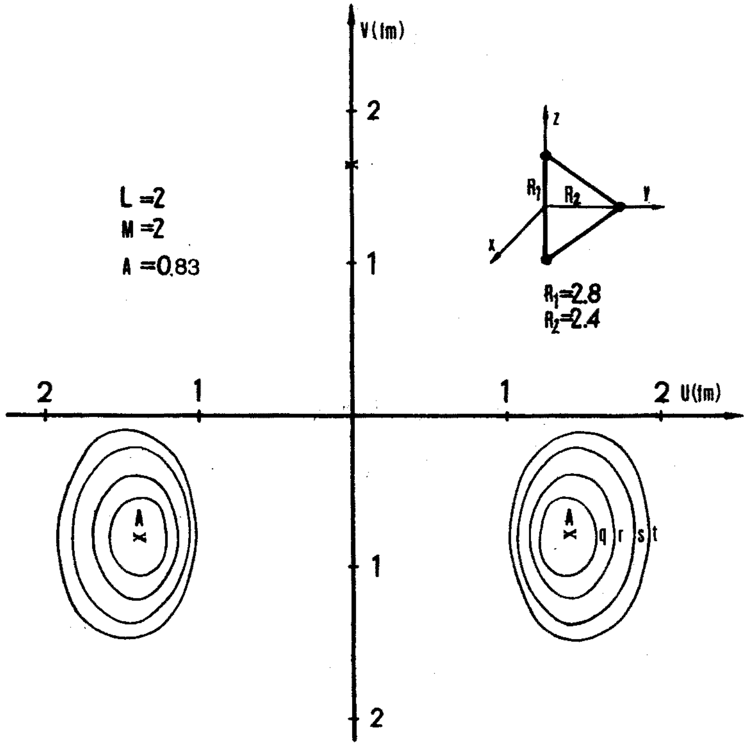


Fig. 7. As Figs. 5 but for $M = 2$

equilateral one with the edge equal to 2.8 fm. This triangular configuration contains the same modes of internal motion as those visualized in Fig. 4 for the ground state. The characteristic density distribution for this case is plotted in Fig. 7 in the (U, V) reference frame.

From the results we have shown, one can conclude that the three-alpha structure of the ground $(0^+, 0)$ and first excited $(2^+, 0)$ states of C12 is contained in the intermediate--coupling model wavefunctions. The probability of the three-alpha clusterization of the ground and first excited state is, according to Eq. (14), equal to 0.47 and 0.52, respectively. The two possible modes of oscillation; the linear (more apparent in the ground state) and the triangular one, are present in both analyzed states. This result is, for the ground state, different from that of Refs. [22–24] where the triangular shape was found to be more probable. Indeed, in Ref. [23] it is remarked that the triangular configuration is prefered when a repulsive core is included in the alpha-alpha interaction. In any case we want to stress that a direct comparison is difficult since the two approaches are completely different. In the approach of Refs. [22–24] the wavefunctions of C12 are expanded in the basis of three alpha particle states and an effective alpha-alpha interaction is used with parameters adjusted to get a reasonable fit to some observable quantities. On the contrary, we start from the wavefunctions obtained in the intermediate-coupling model [19]. As it is well known, these wavefunctions give good results for energy levels, magnetic dipole moments, B(M1) gamma transition strengths, log*ft* for allowed Gamow-Teller beta decays [19], few-particle transfer spectroscopic amplitudes [41] and so on. Then we do not have ad hoc adjustable parameters.

As we said before, our aim was to analyze how much the three-alpha particle structure is contained in the shell-model wavefunctions and which kind of correlated motions of the three alphas emerges. Our main conclusion is that both; the triangular and the linear configurations are present in the intermediate-coupling model wavefunctions of the analyzed ground $(0^+, 0)$ and first excited $(2^+, 0)$ states of C12. The intermediate-coupling model privileges a linear oscillations in the ground state, while a triangular oscillations see mto be more visible in the first excited state.

APPENDIX 1

The overlap $\langle F^{\tau}_{JM} | \Phi_{JM}(\mathrm{C12})\rangle$

The intermediate-coupling model wavefunction of C12 expanded in the basis of Eq. (17) can be written as

$$\Phi_{JM}(\mathrm{C12}) = \sum_q a(q)\Phi_J(s^4p^8, q), \tag{A1.1}$$

where the expansion coefficients $a(q)$ depend on the parameters of interaction used in the intermediate coupling-model hamiltonian [19–21]. Thus, dealing with this wavefunction and with F^{τ}_{JM} expressed by Eq. (7) the above overlap takes the form

$$\langle F^{\tau}_{JM} | \Phi_{JM}(\mathrm{C12})\rangle = \frac{1}{\sqrt{\mu_\tau}} \sum_{q\sigma} a(q) C^{\tau\sigma}_J \langle \Phi^{\sigma}_{JM} | \Phi_J(s^4p^8, q)\rangle, \tag{A1.2}$$

where summation over q reduces to only one term $q = ([444]\, L = J \; T = S = 0)$ because of the already assumed symmetry [4] of the alpha-particles. Since the overlap $\langle \Phi^{\sigma}_{JM} | \Phi_J(s^4p^8, q)\rangle$ is just the CFP of TISM, Eq. (A1.2) can be written as

$$\langle F^{\tau}_{JM} | \Phi_{JM}(\mathrm{C12})\rangle = \frac{1}{\sqrt{\mu_\tau}} a(q = ([444]L = J \;\; T = S = 0)) \sum_{\sigma} C^{\tau\sigma}_J S(q, \sigma). \tag{A1.3}$$

On the other hand, by inserting the unit operator $\hat{1}$ of Eq. (24) between the "bra" and "ket" states of the overlap $\langle F^{\tau}_{JM} | F^{\tau}_{JM}\rangle = 1$ one directly gets that

$$\sum_{\sigma} C^{\tau\sigma}_J S(q = ([444]L = J \;\; S = T = 0), \sigma) = \sqrt{\mu_\tau}. \tag{A1.4}$$

Thus, putting (A1.4) into (A1.3) one finally obtains that

$$\langle F^{\tau}_{JM} | \Phi_{JM}(C12)\rangle = a([444]L = J). \tag{A1.5}$$

APPENDIX 2

The auxiliary integral I

In this Section is derived a formula for calculating the overlap integral I of Eq. (21). This derivation is based on the use of the alpha-particle coefficients of fractional parentage for the $1p$ shell in L-S coupling [39]. If the shell-model alpha-particle wavefunction is given by Eq. (19) the integral I of Eq. (21) becomes

$$I(q, \sigma = N_1L_1N_2L_2) = \langle \Psi(s^4p^8[f_q]\beta_qL_qS_q:JT)|$$

$$\hat{A}\{[f_{N_1L_1}(\vec{r}_1)\times f_{N_2L_2}(\vec{r}_2)]^{L=J}\Psi_{\alpha_3}(s^4[4]L = S = T = 0)\Phi_{\alpha_2}\Phi_{\alpha_1}\}\rangle. \tag{A2.1}$$

By introducing the alpha-particle CFP [39]

$$\langle \Psi(s^4p^8[f_q]\beta_qL_qS_q:JT| = \sum_{L_1L_2} \langle s^4p^8[f_q]\beta_qL_qS_qT|$$

$$|s^4p^4[f_\gamma]\beta_\gamma L_\gamma S_\gamma T,\ p^4[4]L_2S_2 = T_2 = 0\rangle$$

$$\times\langle s^4p^4[f_\gamma]\beta_\gamma L_\gamma S_\gamma T|s^4[4]L_\mu = S_\mu = T_\mu = 0,\ p^4[4]L_1S_1 = T_1 = 0\rangle$$

$$\times\langle(\Psi(s^4[4]L_\mu = S_\mu = T_\mu = 0)\Psi(p^4[4]L_1S_1 = T_1 = 0)\Psi(p^4[4]L_2S_2 = T_2 = 0))L = J|$$

$$\times\,\delta_{s_qs_\gamma}\delta_{s_\gamma 0}\delta_{T0}\delta_{L_\gamma L_1} + Q, \tag{A2.2}$$

where, because of symmetry [4] of alpha-particle, $[f_q] = [444]$ and $[f_\gamma] = [44]$ and Q contains the remaining terms with the Young diagrams of four nucleons different from diagram [4]. Since

$$\langle s^4p^n[f]\beta LST|s^4p^{n-4}[f_1]\beta_1L_1S_1T_1,\ p^4[f_2]\beta_2L_2S_2T_2\rangle = \sqrt{\binom{n}{4}}\Big/\sqrt{\binom{n+4}{4}}$$

$$\times\langle p^n[f_p]\beta LST|p^{n-4}[f_{p_1}]\beta_1L_1S_1T_1,\ p^4[f_2]\beta_2L_2S_2T_2\rangle, \tag{A2.3}$$

where $[f_p]$ and $[f_{p_1}]$ are the counterparts of the diagrams $[f]$ and $[f_1]$ corresponding to $1p$--shell nucleons only, by inserting Eqs. (A2.2) and (A2.3) into Eq. (A2.1) one finally gets

$$I(q, \sigma) = \sqrt{\frac{12!}{3!4!4!4!}}\Big/\sqrt{\binom{12}{4}}\times\langle p^8[44]L00|p^4[4]L_100,\ p^4L_200\rangle$$

$$\times\langle p^4[4]L_1|f_{N_1L_1}(\vec{r}_1)\Phi_{\alpha_1}\rangle\,\langle p^4[4]L_2|f_{N_2L_2}(\vec{r}_2)\Phi_{\alpha_2}\rangle. \tag{A2.4}$$

The overlaps $\langle p^4[4]L|f_{NL}(\vec{r})\Phi_\alpha\rangle$ are the well known overlaps from the theory of the alpha--particle spectroscopic amplitudes for $1p$-shell nuclei, whose value is equal to $\sqrt{3/2}/4$ independently of the quantum numbers NL [42].

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